The objective is to clarify the representation of vectors in two of the most popular coordinate systems such as the cylindrical and the spherical coordinate systems.
An important concept to keep in mind when dealing with the Cylindrical and Spherical coordinate systems is that the representations of a position vector $\vec{\boldsymbol{r}}$ and a general oriented vector $\vec{\boldsymbol{A}}$ in space are somewhat different. The difference is that the components of the position vector $\vec{\boldsymbol{r}}$ in the directions of the basic angular unit vectors $\widehat{\boldsymbol{\theta}}$, $\widehat{\emptyset}$ will vanish in the Cylindrical and Spherical coordinate systems because the position vector $\vec{\boldsymbol{r}}$ will always be perpendicular to the basic angular unit vectors $\widehat{\boldsymbol{\theta}}$, $\widehat{\emptyset}$. However, the latter is not the case for more general vectors $\vec{\boldsymbol{A}}$ that do not cross the origin of the coordinate system and are not orthogonal to the basic unit vectors $\widehat{\boldsymbol{\theta}}$, $\widehat{\emptyset}$**.**

Note: In order to simplify the work and expedite this paper, the figures used in this article were copied from papers and articles found in the internet.

Figures 1, 2, and 3 represent three of the most popular coordinate systems, Cartesian, Cylindrical, and Spherical, respectively.

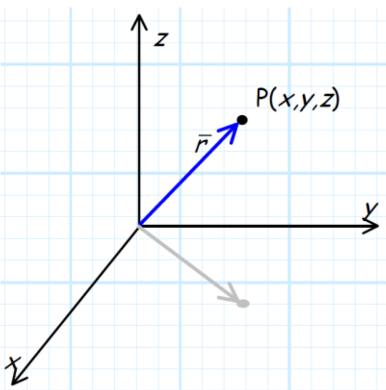

Figure 1

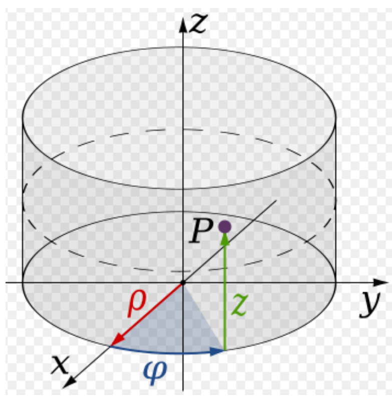

Figure 2

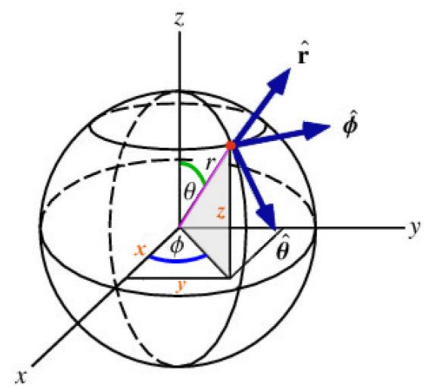

Figure 3

In these coordinates systems,
(a) a point P is specified as:

Cartesian:	(x, y, z)	(1)
Cylindrical:	(ρ, ϕ, z)	(2)
Spherical:	(r, ϕ, θ)	(3)

(b) a position vector $\vec{\boldsymbol{r}}$ is specified as:

Cartesian: $\vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}$ (4)

Cylindrical: $\vec{\boldsymbol{r}} = \rho\,\hat{\boldsymbol{\rho}} + \emptyset\,\widehat{\emptyset} + z\,\hat{\boldsymbol{z}} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\boldsymbol{z}}$ (5)

Spherical: $\vec{\boldsymbol{r}} = r\,\hat{\boldsymbol{r}} + \theta\,\widehat{\boldsymbol{\theta}} + \emptyset\,\widehat{\emptyset} = r\,\hat{\boldsymbol{r}}$ (6)

(c) a general oriented vectors $\vec{\boldsymbol{A}}$ is specified as:

Cartesian: $\vec{\boldsymbol{A}} = A_x\,\hat{\boldsymbol{x}} + A_y\,\hat{\boldsymbol{y}} + A_z\,\hat{\boldsymbol{z}}$ (7)

Cylindrical: $\vec{\boldsymbol{A}} = A_\rho\,\hat{\boldsymbol{\rho}} + A_\emptyset\,\widehat{\emptyset} + A_z\,\hat{\boldsymbol{z}}$ (8)

Spherical: $\vec{\boldsymbol{A}} = A_r\,\hat{\boldsymbol{r}} + A_\theta\,\widehat{\boldsymbol{\theta}} + A_\emptyset\,\widehat{\emptyset}$ (9)

Where A_i represents the component of the vector $\vec{\boldsymbol{A}}$ in the *'I'* unit vector direction. Note that $\hat{\boldsymbol{x}} = \hat{\boldsymbol{\imath}}$; $\hat{\boldsymbol{y}} = \hat{\boldsymbol{\jmath}}$; $\hat{\boldsymbol{z}} = \widehat{\boldsymbol{k}}$.

Notice the difference in the definitions of a general oriented vector $\vec{\boldsymbol{A}}$ in equations (8), (9) and the position vector $\vec{\boldsymbol{r}}$ equations (5), (6). In the Cylindrical and Spherical coordinates, the components related to the angles θ, ϕ have vanished. Next, the proofs of why the angular components of a position vector vanish will be provided.
Figures 4, 5, and 6 show a point in space with the associated basic unit vectors in Cartesian, Cylindrical, and Spherical coordinates, respectively.

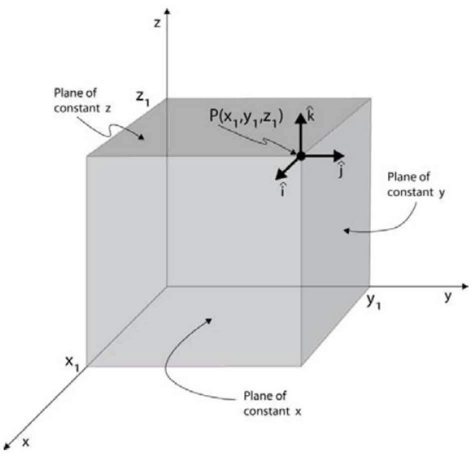

Figure 4

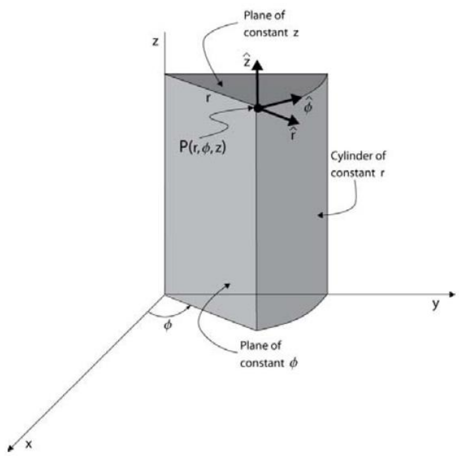

Figure 5

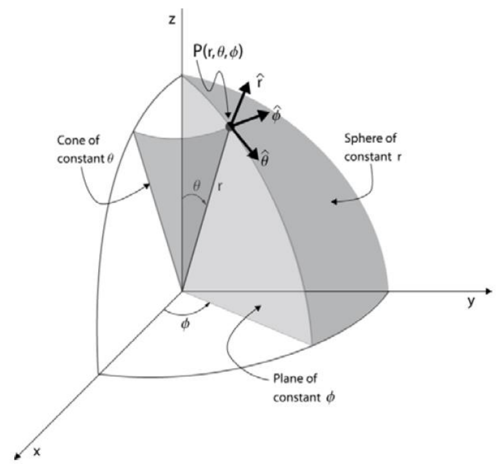

Figure 6

CYLINDRICAL COORDINATES:

The objective is to derive the Cylindrical position vector (5) from the Cartesian position vector (4). The derivation will require finding the components (x, y, z) of the Cartesian position vector $\vec{\boldsymbol{r}}$ as functions of the Cylindrical components (ρ, $\emptyset$, z) and the Cartesian unit vectors ($\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$) as a function of the Cylindrical unit vectors ($\hat{\boldsymbol{\rho}}$, $\hat{\emptyset}$, $\hat{\boldsymbol{z}}$).
For the derivation of the proofs, refer to the Figures 7 and 8 for the basic unit vectors in the Cylindrical coordinate system, respectively.

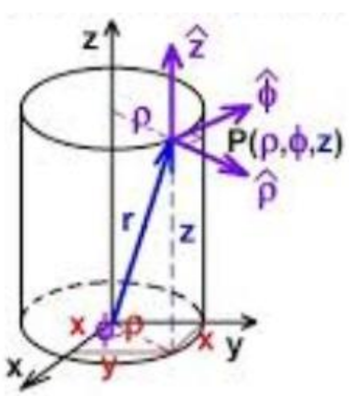

Figure 7

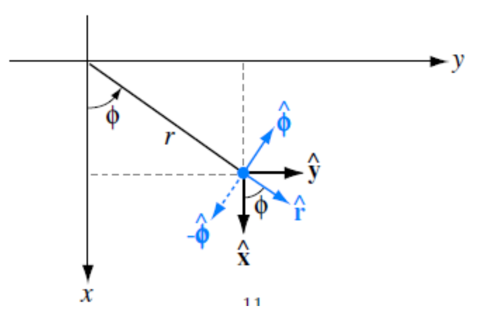
Figure 8

From the above figures, the Cartesian-Cylindrical coordinate transforms for the components in the axes of the coordinate systems are:
(d) from Cylindrical to Cartesian

$x = \rho\cos(\emptyset)$ (10)

$y = \rho\sin(\emptyset)$ (11)

$z = z$ (12)

(e) from Cartesian Cylindrical

$\rho = \sqrt{x^2 + y^2}$ (13)

$\emptyset = \tan^{-1}(^{y}/_{x})$ (14)

$z = z$ (15)

Determine the Cartesian basic unit vectors ($\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$) as a function of the Cylindrical basic unit vectors ($\hat{\boldsymbol{\rho}}$, $\hat{\emptyset}$, $\hat{\boldsymbol{z}}$):

For this analysis, I will refer to Figures 7 and 8.

In general form, the Cartesian basic unit vectors can be expressed as

$\hat{\boldsymbol{x}} = m_1\hat{\boldsymbol{\rho}} + n_1\hat{\emptyset} + q_1\hat{\boldsymbol{z}}$ (16)

$\hat{\boldsymbol{y}} = m_2\hat{\boldsymbol{\rho}} + n_2\hat{\emptyset} + q_2\hat{\boldsymbol{z}}$ (17)

$\hat{\boldsymbol{z}} = m_3\hat{\boldsymbol{\rho}} + n_3\hat{\emptyset} + q_3\hat{\boldsymbol{z}}$ (18)

Each component m, n, q is found by performing the dot product vector operation for the corresponding vector component:

$m_1 = \hat{\boldsymbol{x}}.\hat{\boldsymbol{\rho}} = |\hat{\boldsymbol{x}}||\hat{\boldsymbol{\rho}}|\, cos(\emptyset) = cos(\emptyset)$ (19)

$n_1 = \hat{\boldsymbol{x}}.\hat{\emptyset} = |\hat{\boldsymbol{x}}||\hat{\emptyset}|\, cos\left(\frac{\pi}{2} + \emptyset\right) = -sin(\emptyset)$ (20)

$q_1 = \hat{\boldsymbol{x}}.\hat{\boldsymbol{a}}_z = |\hat{\boldsymbol{x}}||\hat{\boldsymbol{z}}|\, cos\left(\frac{\pi}{2}\right) = 0$ (21)

As expected, there should be no x-component in the z-axis.

$m_2 = \hat{\boldsymbol{y}}.\hat{\boldsymbol{\rho}} = |\hat{\boldsymbol{y}}||\hat{\boldsymbol{\rho}}|\, cos(\frac{\pi}{2} - \emptyset) = sin(\emptyset)$ (22)

$n_2 = \hat{\boldsymbol{y}}.\hat{\emptyset} = |\hat{\boldsymbol{y}}||\hat{\emptyset}|\, cos(\emptyset) = cos(\emptyset)$ (23)

$q_2 = \hat{\boldsymbol{y}}.\hat{\boldsymbol{z}} = |\hat{\boldsymbol{y}}||\hat{\boldsymbol{z}}|\, cos\left(\frac{\pi}{2}\right) = 0$ (24)

As expected, there should be no y-component in the z-axis.

$m_3 = \hat{\boldsymbol{z}}.\hat{\boldsymbol{\rho}} = |\hat{\boldsymbol{z}}||\hat{\boldsymbol{\rho}}|\, cos(\frac{\pi}{2}) = 0$ (25)

$n_3 = \hat{\boldsymbol{z}}.\hat{\emptyset} = |\hat{\boldsymbol{z}}||\hat{\emptyset}|\, cos(\frac{\pi}{2}) = 0$ (26)

$q_3 = \hat{\boldsymbol{z}}.\hat{\boldsymbol{z}} = |\hat{\boldsymbol{z}}||\hat{\boldsymbol{z}}|\cos(0) = 1$ (27)

As expected, there should be no z-component in the ρ or ϕ directions.

<u>Substituting (19) through (27) into (16) through (18):</u>

$\hat{\boldsymbol{x}} = \cos(\emptyset)\ \hat{\boldsymbol{\rho}} - \sin(\emptyset)\hat{\emptyset}$ (28)

$\hat{\boldsymbol{y}} = \sin(\emptyset)\ \hat{\boldsymbol{\rho}} + \cos(\emptyset)\,\hat{\emptyset}$ (29)

$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$ (30)

At this point we have the components that are needed to perform the conversion of a position vector (4) in Cartesian coordinates to a position vector in Cylindrical coordinates. Substituting the Cartesian-Cylindrical coordinate transform equations (10), (11), (12) for the components and the Cartesian transform equations (28, (29), (30) for the basic unit vectors into the Cartesian position vector (4):

$\overrightarrow{\boldsymbol{r}} = (\rho\cos(\emptyset))\left(\hat{\boldsymbol{\rho}} - \sin(\emptyset)\hat{\emptyset}\right) + (\rho\sin(\emptyset))\left(\sin(\emptyset)\,\hat{\boldsymbol{\rho}} + \cos(\emptyset)\hat{\emptyset}\right) + z\,\hat{\boldsymbol{z}}$ (31)

Multiplying the terms:

$\overrightarrow{\boldsymbol{r}} = (\rho\cos^2(\emptyset))\,\hat{\boldsymbol{\rho}} - \left(\rho\cos(\emptyset)\sin(\emptyset)\hat{\emptyset}\right) + (\rho\ \sin^2(\emptyset)\,\hat{\boldsymbol{\rho}}) + (\rho\, sin(\emptyset)\cos(\emptyset))\hat{\emptyset} + z\,\hat{\boldsymbol{z}}$ (32)

Grouping similar unit vector terms and cancelling the $\hat{\emptyset}$ components will result in the above Cylindrical position vector (5):

$\overrightarrow{\boldsymbol{r}} = \rho\,\hat{\boldsymbol{\rho}} + z\ \hat{\boldsymbol{z}}$ (5)

The angle ϕ-component will always cancel out in a position vector expressed in Cylindrical coordinates. Even though the angle ϕ-component is not explicitly shown in (5), the information of the angle ϕ is implicitly included in the position vector $\vec{\boldsymbol{r}}$ since the component $\rho(\emptyset)$ and $\widehat{\boldsymbol{\rho}}(\emptyset)$ are both functions of the angle ϕ.

SPHERICAL COORDINATES:

The objective is to derive the Spherical position vector (6) from the Cartesian position vector (4). For the derivation of the proofs, refer to the figure 3 for the basic unit vectors in the Spherical coordinate system and to figures 9 and 10 for the position vector components. Figure 11 shows the directional angles α, β, and γ.

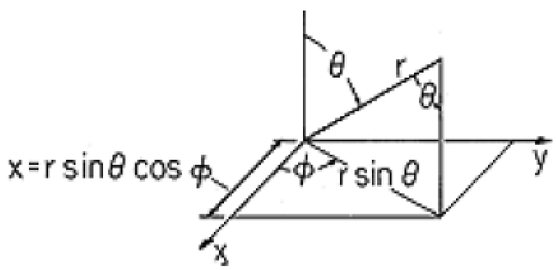

Figure 9

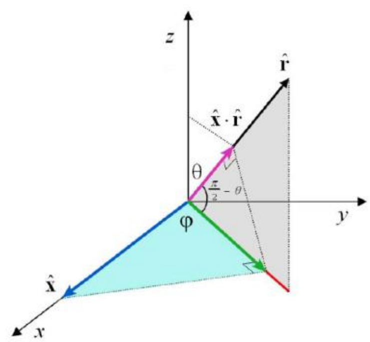

Figure 10

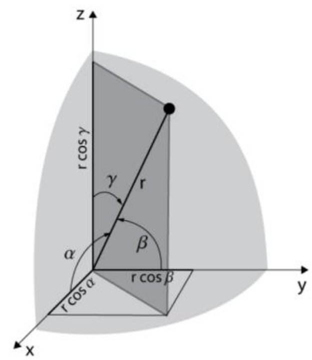

Figure 11

From the above figures, the Cartesian-Spherical coordinate transforms for the components in the axes of the coordinate systems are:

(f) from Cylindrical to Cartesian

$x = r\sin(\theta)\cos(\emptyset)$ (33)

$y = r\ \sin(\theta)\sin(\emptyset)$ (34)

$z = r\cos(\theta)$ (35)

(g) from Cartesian Cylindrical

$r = \sqrt{x^2 + y^2 + z^2}$ (36)

$\theta = \tan^{-1}(\sqrt{x^2 + y^2}/_z)$ (37)

$\emptyset = \tan^{-1}(^y/_x)$ (38)

Assume:

$\delta = angle\ between\ \widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{x}}$

$\sigma = angle\ between\ \widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{y}}$

Determine the Spherical unit vectors as a function of the Cartesian unit vectors and vice versa.

In general form, the Spherical basic unit vectors can be expressed as

$\widehat{\boldsymbol{r}} = m_1\widehat{\boldsymbol{x}} + n_1\widehat{\boldsymbol{y}} + q_1\widehat{\boldsymbol{z}}$ (39)

$\widehat{\boldsymbol{\theta}} = m_2\widehat{\boldsymbol{x}} + n_2\widehat{\boldsymbol{y}} + q_2\widehat{\boldsymbol{z}}$ (40)

$\widehat{\emptyset} = m_3\widehat{\boldsymbol{x}} + n_3\widehat{\boldsymbol{y}} + q_3\widehat{\boldsymbol{z}}$ (41)

Each component m, n, q is found by performing the dot product vector operation for the corresponding vector component:

$m_1 = \widehat{\boldsymbol{r}}.\widehat{\boldsymbol{x}} = |\widehat{\boldsymbol{r}}||\widehat{\boldsymbol{x}}|\ cos(\propto) = sin(\theta)\ cos(\emptyset)$ (42)

$n_1 = \widehat{\boldsymbol{r}}.\widehat{\boldsymbol{y}} = |\widehat{\boldsymbol{r}}||\widehat{\boldsymbol{y}}|\ cos(\beta) = sin(\theta)\ sin(\emptyset)$ (43)

$q_1 = \widehat{\boldsymbol{r}}.\widehat{\boldsymbol{z}} = |\widehat{\boldsymbol{r}}||\widehat{\boldsymbol{z}}|\ cos(\gamma) = cos(\theta)$ (44)

Notice that there is an r-component in each of the Cartesian basic unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$, which is to be expected since, in general, the $\hat{\boldsymbol{r}}$ is not orthogonal to $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, or $\hat{\boldsymbol{z}}$.

$$m_2 = \hat{\boldsymbol{\theta}}.\hat{\boldsymbol{x}} = |\hat{\boldsymbol{\theta}}||\hat{\boldsymbol{x}}|\ cos(\delta) = cos(\theta)\ cos(\emptyset) \quad (45)$$
$$n_2 = \hat{\boldsymbol{\theta}}.\hat{\boldsymbol{y}} = |\hat{\boldsymbol{\theta}}||\hat{\boldsymbol{y}}|\ \cos(\sigma) = cos(\theta)\ sin(\emptyset) \quad (46)$$
$$q_2 = \hat{\boldsymbol{\theta}}.\hat{\boldsymbol{z}} = |\hat{\boldsymbol{\theta}}||\hat{\boldsymbol{z}}|\ cos\left(\frac{\pi}{2} + \theta\right) = -sin(\theta) \quad (47)$$

There is a Θ-component in each of the Cartesian basic unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$, which is to be expected since, in general, the $\hat{\boldsymbol{\theta}}$ is not orthogonal to $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, or $\hat{\boldsymbol{z}}$.

$$m_3 = \hat{\emptyset}.\hat{\boldsymbol{x}} = |\hat{\emptyset}||\hat{\boldsymbol{x}}|\ cos\left(\frac{\pi}{2} - \theta\right) = -\ sin(\emptyset) \quad (48)$$
$$n_3 = \hat{\emptyset}.\hat{\boldsymbol{y}} = |\hat{\emptyset}||\hat{\boldsymbol{y}}|\ cos(\emptyset) = cos(\emptyset) \quad (49)$$
$$q_3 = \hat{\emptyset}.\hat{\boldsymbol{z}} = |\hat{\emptyset}||\ \hat{\boldsymbol{z}}|\ cos\left(\frac{\pi}{2}\right) = 0 \quad (50)$$

Notice that there is no ϕ-component in the Cartesian basic unit vectors $\hat{\boldsymbol{z}}$, which is to be expected since, in general, the $\hat{\emptyset}$ is orthogonal to $\hat{\boldsymbol{z}}$.

<u>Substituting (42) through (50) into (39) through (41):</u>

$$\hat{\boldsymbol{r}} = sin(\theta)\ cos(\emptyset)\ \hat{\boldsymbol{x}} + sin(\theta)\ sin(\emptyset)\ \hat{\boldsymbol{y}} + cos(\theta)\ \hat{\boldsymbol{z}} \quad (51)$$
$$\hat{\boldsymbol{\theta}} = cos(\theta)\ cos(\emptyset)\ \hat{\boldsymbol{x}} + cos(\theta)\ sin(\emptyset)\ \hat{\boldsymbol{y}} - sin(\theta)\ \hat{\boldsymbol{z}} \quad (52)$$
$$\hat{\emptyset} = -\ sin(\emptyset)\ \hat{\boldsymbol{x}} + cos(\emptyset)\ \hat{\boldsymbol{y}} \quad (53)$$

Determine the Cartesian unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ as a function of the Spherical unit vectors $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\emptyset}$.
In general form, the Cartesian basic unit vectors can be expressed as

$$\hat{\boldsymbol{x}} = m_1\hat{\boldsymbol{r}} + n_1\hat{\boldsymbol{\theta}} + q_1\hat{\emptyset} \quad (54)$$
$$\hat{\boldsymbol{y}} = m_2\hat{\boldsymbol{r}} + n_2\hat{\boldsymbol{a}}_\theta + q_2\hat{\emptyset} \quad (55)$$
$$\hat{\boldsymbol{z}} = m_3\hat{\boldsymbol{r}} + n_3\hat{\boldsymbol{\theta}} + q_3\hat{\emptyset} \quad (56)$$

Each component m, n, q is found by performing the dot product vector operation for the corresponding vector component:

$$m_1 = \hat{\boldsymbol{x}}.\hat{\boldsymbol{r}} = |\ \hat{\boldsymbol{x}}||\hat{\boldsymbol{r}}|\ cos(\propto) = sin(\theta)\ cos(\emptyset) \quad (57)$$
$$n_1 = \hat{\boldsymbol{x}}.\hat{\boldsymbol{\theta}} = |\ \hat{\boldsymbol{x}}||\hat{\boldsymbol{\theta}}|\ cos(\delta) = cos(\theta)\ cos(\emptyset) \quad (58)$$
$$q_1 = \hat{\boldsymbol{x}}.\hat{\emptyset} = |\ \hat{\boldsymbol{x}}||\hat{\emptyset}|\ cos\left(\frac{\pi}{2} - \theta\right) = -\ sin(\emptyset) \quad (59)$$

Notice that there is an x-component in each of the Spherical basic unit vectors $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\emptyset}$, which is to be expected since, usually, the $\hat{\boldsymbol{x}}$ is not orthogonal to $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, or $\hat{\emptyset}$.

$$m_2 = \hat{\boldsymbol{y}}.\hat{\boldsymbol{r}} = |\hat{\boldsymbol{y}}||\hat{\boldsymbol{r}}|\ cos(\beta) = sin(\theta)\ sin(\emptyset) \quad (60)$$
$$n_2 = \hat{\boldsymbol{y}}.\hat{\boldsymbol{\theta}} = |\hat{\boldsymbol{y}}||\hat{\boldsymbol{\theta}}|\ \cos(\sigma) = cos(\theta)\ sin(\emptyset) \quad (61)$$
$$q_2 = \hat{\boldsymbol{y}}.\hat{\emptyset} = |\hat{\boldsymbol{y}}||\hat{\emptyset}|\ cos(\emptyset) = cos(\emptyset) \quad (62)$$

There is a y-component in each of the Spherical basic unit vectors $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\emptyset}$, which is to be expected since, usually, the $\hat{\boldsymbol{y}}$, is not orthogonal to $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, or $\hat{\emptyset}$.

$$m_3 = \hat{\boldsymbol{z}}.\hat{\boldsymbol{r}} = |\hat{\boldsymbol{z}}||\hat{\boldsymbol{r}}|\ cos(\gamma) = \cos(\theta) \quad (63)$$
$$n_3 = \hat{\boldsymbol{z}}.\hat{\boldsymbol{\theta}} = |\hat{\boldsymbol{z}}||\hat{\boldsymbol{\theta}}|\ cos\left(\frac{\pi}{2} + \theta\right) = -sin(\theta) \quad (64)$$
$$q_3 = \hat{\boldsymbol{z}}.\hat{\emptyset} = |\hat{\boldsymbol{z}}||\hat{\emptyset}|\ cos\left(\frac{\pi}{2}\right) = 0 \quad (65)$$

There is no z-component in the Spherical basic unit vector $\hat{\emptyset}$ which is to be expected since the $\hat{\boldsymbol{z}}$ is orthogonal to $\hat{\emptyset}$.

Substituting (57) through (65) into (54) through (56):

$$\hat{\boldsymbol{x}} = \sin(\theta)\cos(\emptyset)\,\hat{\boldsymbol{r}} + \cos(\theta)\cos(\emptyset)\,\hat{\boldsymbol{\theta}} - \sin(\emptyset)\,\hat{\emptyset} \quad (66)$$

$$\hat{\boldsymbol{y}} = \sin(\theta)\sin(\emptyset)\,\hat{\boldsymbol{r}} + \cos(\theta)\sin(\emptyset)\hat{\boldsymbol{\theta}} + \cos(\emptyset)\,\hat{\emptyset} \quad (67)$$

$$\hat{\boldsymbol{z}} = \cos(\theta)\,\hat{\boldsymbol{r}} - \sin(\theta)\,\hat{\boldsymbol{\theta}} \quad (68)$$

At this point we have the all the components that are needed to perform the conversion of a position vector (4) in Cartesian coordinates to a position vector (6) in Spherical coordinates. Substituting the Cartesian coordinate transform equations (33), (34), (35) and the Cartesian basic unit vector equations (66), (67), (68) into the Cartesian position vector (4) $\vec{\boldsymbol{r}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}$ result in the following:

$$\begin{aligned}\vec{\boldsymbol{r}} = &(r\sin(\theta)\cos(\emptyset))\big(\sin(\theta)\cos(\emptyset)\,\hat{\boldsymbol{r}} + \cos(\theta)\cos(\emptyset)\,\hat{\boldsymbol{\theta}} - \sin(\emptyset)\,\hat{\emptyset}\big) + \\ &(r\sin(\theta)\sin(\emptyset))\big(\sin(\theta)\sin(\emptyset)\,\hat{\boldsymbol{r}} + \cos(\theta)\sin(\emptyset)\hat{\boldsymbol{\theta}} + \cos(\emptyset)\,\hat{\emptyset}\big) + \\ &(r\cos(\theta))(\cos(\theta)\,\hat{\boldsymbol{r}} - \sin(\theta)\,\hat{\boldsymbol{\theta}})\end{aligned} \quad (69)$$

Grouping Similar unit vector terms:

$$\begin{aligned}\vec{\boldsymbol{r}} = &r\sin^2(\theta)\cos^2(\emptyset)\,\hat{\boldsymbol{r}} + r\sin(\theta)\cos(\theta)\cos^2(\emptyset)\,\hat{\boldsymbol{\theta}} - r\sin(\theta)\cos(\emptyset)\sin(\emptyset)\,\hat{\emptyset} + \\ &r\sin^2(\theta)\sin^2(\emptyset)\,\hat{\boldsymbol{r}} + r\sin(\theta)\sin^2(\emptyset)\,\hat{\boldsymbol{\theta}} + r\sin(\theta)\sin(\emptyset)\cos(\emptyset)\,\hat{\emptyset} + \\ &r\cos^2(\theta)\,\hat{\boldsymbol{r}} - r\sin(\theta)\cos(\theta)\,\hat{\boldsymbol{\theta}}\end{aligned} \quad (70)$$

The $\hat{\emptyset}$ components cancels out and grouping $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$ components considering that $\sin^2(\theta) + \cos^2(\emptyset) = 1$,

$$\begin{aligned}\vec{\boldsymbol{r}} = &r\sin^2(\theta)\,\hat{\boldsymbol{r}} + r\sin(\theta)\cos(\theta)\,\hat{\boldsymbol{\theta}} \\ &r\cos^2(\theta)\,\hat{\boldsymbol{r}} - r\sin(\theta)\cos(\theta)\,\hat{\boldsymbol{\theta}}\end{aligned} \quad (71)$$

The $\hat{\boldsymbol{a}}_\theta$ components cancels out and grouping $\hat{\boldsymbol{r}}$ components result in equation (6),

$$\vec{\boldsymbol{r}} = r\,\hat{\boldsymbol{r}} \quad (6)$$

The angle components $\theta, \emptyset$ will always cancel out in a position vector expressed in Cylindrical coordinates. Even though the angle components are not explicitly shown in (6), the information of the angles $\theta, \emptyset$ is implicitly included in the position vector $\vec{\boldsymbol{r}}$ since the radius component $r(\theta, \emptyset)$ and the unit vector $\hat{\boldsymbol{r}}(\theta, \emptyset)$ are both functions of the angles $\theta, \emptyset$.

In conclusion,
The conversions from a position vector in Cartesian coordinate system (4) to both position vectors (5) and (6) in Cylindrical and Spherical coordinates, respectively, will always result in the cancellation of the angular components due to the fact that the position vector will always be perpendicular to these angular unit vectors. However, a vector other than a position vector may have components in the angular unit vectors as shown in equations (8) and (9).

Note: figures from papers published by Jim Stiles, Michael Rosenthal, and other unknown authors.