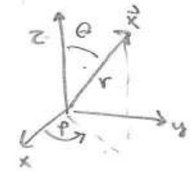
Laplace equation in spherical coordinates

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\phi\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \quad -||- \quad = 0$$



↳ separation of variables

$$\phi = \frac{U(r)}{r}\,P(\theta)\,Q(\varphi)$$

$$\Rightarrow \quad \underbrace{r^2\sin^2\theta\left[\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{Pr^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right)\right]}_{\text{depends on } r,\theta} + \underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{\text{depends on } \varphi \text{ only}} = 0$$

$\Rightarrow \quad \frac{1}{Q}\frac{d^2Q}{d\varphi^2} = -m^2$ is a constant

$$Q = e^{\pm im\varphi}$$

↳ if the region is covering full azimuthal range $\varphi \in [0,2\pi)$

$$Q\big|_{\varphi=0} = Q\big|_{\varphi=2\pi} \Rightarrow m \text{ is integer}$$

↳ similarly

$$\frac{d^2U}{dr^2} - \frac{l(l+1)}{r^2}U = 0$$

here $l(l+1)$ is a real constant

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dP}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]P = 0$$

⇒ the solution for U

$$U = A\,r^{l+1} + B\,r^{-l}$$

with l an unknown constant

↳ the equation for $\cos\theta = x$

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]P = 0$$

generalized Legendre equation

⇒ solutions are $P_l^m(x)$ associated Legendre function for $\underline{\underline{m=0}}$ $\Rightarrow$ ordinary Legendre differential equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + l(l+1)P = 0 \qquad \Rightarrow \qquad P_l(x)$$

for instance $\quad P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2-1) \qquad P_3(x) = \frac{1}{2}(5x^3-3x)$

↳ some properties of Legendre polynomials of order l

$$P_l(1) = 1$$

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$

↳ they form a complete orthogonal set of functions on the interval $-1 \leq x \leq 1$

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\, dx = \frac{2}{2l+1}\delta_{ll'}$$

$\Rightarrow$ thus the orthonormal functions are

$$U_l(x) = \sqrt{\frac{2l+1}{2}}\, P_l(x)$$

↳ they are even (odd) for l even (odd)

$$P_l(-x) = (-1)^l P_l(x)$$

↳ satisfy recurrence formulas

$$(l+1)P_{l+1} + (2l+1)xP_l + lP_{l-1} = 0$$

$$\frac{dP_{l+1}}{dx} - x\frac{dP_l}{dx} - (l+1)P_l = 0$$

$$(x^2-1)\frac{dP_l}{dx} - lxP_l + lP_{l-1} = 0$$

↳ some integrals

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\,x\,dx = \begin{cases} \dfrac{2(l+1)}{(2l+1)(2l+3)} & ; \ l' = l+1 \\[2ex] \dfrac{2l}{(2l-1)(2l+1)} & ; \ l' = l-1 \end{cases}$$

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\,x^2\,dx = \begin{cases} \dfrac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)} & ; \ l' = l+2 \\[2ex] \dfrac{2(2l^2+2l-1)}{(2l-1)(2l+1)(2l+3)} & ; \ l' = l \end{cases}$$

BOUNDARY VALUE PROBLEM WITH AZIMUTHAL SYMMETRY

↳ if there is no φ dependence ⇒ $m = 0$ as then $Q(\varphi) = 1$, i.e. just a constant

$$\phi = \frac{U(r)}{r} P(\theta) \underbrace{Q(\varphi)}_{1}$$

$$U(r) = A r^{l+1} + B r^{-l}$$

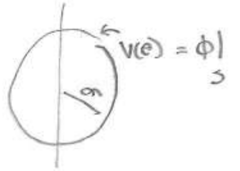
$$P(\theta) = P_l(\cos\theta)$$

⇒ we thus have for problems with azimuthal symmetry

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta)$$

↳ A_l, B_l from boundary conditions

- for instance, consider that potential is specified on the surface of the sphere



what is the potential inside the sphere?

↳ no charges inside ⇒ $B_l = 0$

↳ A_l from the b.c.

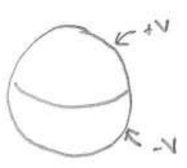
$$V(\theta) = \sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta)$$

from $\int_{-1}^{1} P_{l'}(x) P_l(x)\, dx = \frac{2}{2l+1} \delta_{ll'}$ we can get

$$\int_{-1}^{1} V(\theta) P_{l'}(\cos\theta)\, d\cos\theta = \sum_{l=0}^{\infty} A_l a^l \int_{-1}^{1} P_{l'}(\cos\theta) P_l(\cos\theta)\, d\cos\theta = A_{l'} a^{l'} \frac{2}{2l+1}$$

$$\Rightarrow \quad A_l = \frac{2l+1}{2a^l} \int_0^{\pi} V(\theta) P_l(\cos\theta) \sin\theta\, d\theta$$

↳ for instance $V(\theta)$ is



$$V(\theta) = \begin{cases} +V \ ; & 0 \le \theta \le \frac{\pi}{2} \\ -V \ ; & \frac{\pi}{2} \le \theta \le \pi \end{cases}$$

$$\Rightarrow \quad \phi(r,\theta) = V \left[\frac{3}{2} \frac{r}{a} P_1(\cos\theta) - \frac{7}{8} \left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16} \left(\frac{r}{a}\right)^5 P_5(\cos\theta) + \cdots \right]$$

↳ outside the sphere

$r \to \infty$ $\quad \phi$ should be zero $\quad \Rightarrow A_\ell = 0$

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell \, r^{-(\ell+1)} P_\ell(\cos\theta)$$

↳ from b.c. one obtains B_ℓ

$$\phi(a,\theta) = \sum_{\ell=0}^{\infty} B_\ell \, a^{-(\ell+1)} P_\ell(\cos\theta) = V(\theta)$$

$\Rightarrow$ apart from $a^{-(\ell+1)}$ instead of a^ℓ before the same integrals

Thus, outside:

$$\phi(r,\theta) = V\left[\frac{3}{2}\left(\frac{a}{r}\right)^2 - \frac{7}{8}\left(\frac{a}{r}\right)^4 P_3(\cos\theta) + \frac{11}{16}\left(\frac{a}{r}\right)^5 P_5 \cos\theta + \cdots \right]$$

↳ note, if one knows

$$\phi(z=r) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-(\ell+1)} \right]$$

this already fixes A_ℓ, B_ℓ, and thus $\phi(r,\theta)$

↳ for instance for the problem

we found a solution for z-axis before:

$$\phi(z=r) = V\left[1 - \frac{r^2-a^2}{r\sqrt{r^2+a^2}} \right]$$

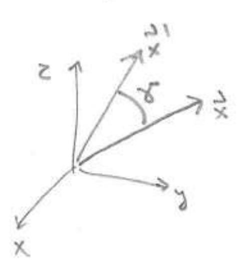
– expanding in powers of a^2/r^2 (i.e. outside sphere)

$$\phi(z=r) = \frac{V}{\sqrt{\pi}} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{(2j-\frac{1}{2})\Gamma(j-\frac{1}{2})}{j!} \left(\frac{a}{r}\right)^{\overbrace{2j}^{\ell+1}}$$

$$\phi(r,\theta) = \quad -\|- \quad \cdot P_{\underset{\uparrow \ell}{2j-1}}(\cos\theta)$$

this agrees with our previous solution

↳ another example: expansion of the potential at $\vec{x}$ due to point charge at $\vec{x}'$



$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

$r_< = \min\{|\vec{x}|, |\vec{x}'|\}$

$r_> = \max\{|\vec{x}|, |\vec{x}'|\}$

γ – angle between $\vec{x}$ and $\vec{x}'$

↳ this expansion can be obtained by:

- rotating z-axis along $\vec{x}'$
- the potential is then axially symmetric and can be expanded as

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + B_l r^{-(l+1)} \right] P_l(\cos\theta)$$

↑ $\theta = \gamma$ in our case

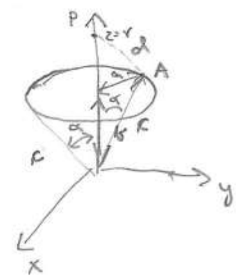
$$\Rightarrow \quad \frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos\gamma)$$

↳ if $\vec{x}$ is on z-axis:

$$\frac{1}{|\vec{x}-\vec{x}'|} \rightarrow \frac{1}{|r-r'|} = \frac{1}{r_>} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>} \right)^l \qquad \text{note: } \frac{1}{|r-r'|} = \frac{1}{(r^2+r'^2-2rr'\cos\gamma)}$$

to obtain full γ dependence, one only needs to multiply each term with $P_l(\cos\gamma)$ as we argued before (z-expansion gives coeffs. of $P_l(\cos\theta)$)

<u>example</u>: potential due to a total charge q uniformly distributed around a circular ring of radius a, z-axis is its symmetry axis, center at $z=b$



↳ the potential at $z=r$:

$$\phi(z=r) = \frac{q}{4\pi\varepsilon_0} \frac{1}{d} = \frac{q}{4\pi\varepsilon_0} \frac{1}{(r^2+c^2-2rc\cos\alpha)^{1/2}}$$

$$c^2 = a^2+b^2 \qquad \tan\alpha = \frac{a}{b}$$

↳ for $r>c$ one can expand in c/r

$$\phi(z=r) = \frac{q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{c^l}{r^{l+1}} P_l(\cos\alpha)$$

↳ for $r<c$ expand in r/c

$$\phi(z=r) = \frac{q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{r^l}{c^{l+1}} P_l(\cos\alpha)$$

↳ the potential at any point in space is thus

$$\phi(r,\theta) = \frac{q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\alpha) P_l(\cos\theta)$$

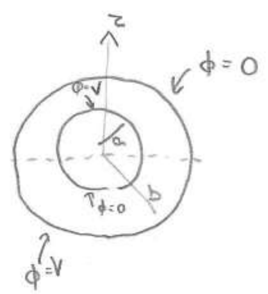
$$r_< = \min\{c, r\}$$
$$r_> = \max\{c, r\}$$

Jackson Problem 3.1

Two concentric spheres have radii a, b ($b > a$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential. Determine the potential in the region $a \leq r \leq b$ as a series in Legendre polynomials. Include terms at least up to $l = 4$. Check your solution against known results in the limiting case $b \to \infty$, and $a \to 0$

Solution:

axialy symmetric problem



boundary conditions:

$$\phi\Big|_{r=a} = \begin{cases} V ; & 0 \leq \Theta \leq \frac{\pi}{2} \\ 0 ; & \frac{\pi}{2} \leq \Theta \leq \pi \end{cases}$$

$$\phi\Big|_{r=b} = \begin{cases} 0 ; & 0 \leq \Theta \leq \frac{\pi}{2} \\ V ; & \frac{\pi}{2} \leq \Theta \leq \pi \end{cases}$$

$$\nabla^2 \phi = 0 \qquad \text{for } m = 0 \Rightarrow \quad \phi(r, \theta, \varphi) = \sum_\ell \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta)$$

↳ A_ℓ and B_ℓ are determined from boundary conditions

$$\phi\Big|_{r=a} = \sum_\ell \left[A_\ell a^\ell + B_\ell a^{-(\ell+1)}\right] P_\ell(\cos\theta)$$

The coefficient from

$$\int_{-1}^{1} \phi\Big|_{r=a} P_\ell(\cos\theta)\, d\cos\theta = \int_{-1}^{1} \sum_{\ell'} [\quad] P_{\ell'}(\cos\theta) P_\ell(\cos\theta)\, d\cos\theta = \sum_{\ell'} [\quad] \frac{2}{2\ell+1} \delta_{\ell\ell'}$$

$$= \left[A_\ell a^\ell + B_\ell \ell^{-(\ell+1)}\right] \frac{2}{2\ell+1}$$

on the other hand also

$$\int_{-1}^{1} \phi\Big|_{r=a} P_\ell(\cos\theta)\, d\cos\theta = \int_0^1 V P_\ell(\cos\theta)\, d\cos\theta$$

$$\Rightarrow \quad \frac{2}{2\ell+1}\left[A_\ell a^\ell + B_\ell \ell^{-(\ell+1)}\right] = V \int_0^1 P_\ell(\cos\theta)\, d\cos\theta \equiv C_\ell$$

$$C_\ell = \int_0^1 P_\ell(x)\,dx = \frac{\sqrt{\pi}}{2} \frac{1}{\Gamma\left(1-\frac{\ell}{2}\right)\Gamma\left(\frac{3+\ell}{2}\right)} = \begin{cases} 0 ; & \ell = 2k \\ (-1)^k \frac{1}{2^{k+1}} \frac{(2k-1)!!}{(k+1)!} ; & \ell = 2k+1 \end{cases}$$

↳ the other boundary condition

$$\phi\Big|_{r=b} = \sum_\ell \left[A_\ell b^\ell + B_\ell a^{-(\ell+1)}\right] P_\ell(\cos\theta) = \begin{cases} 0 \ ; & 0 \le \theta \le \frac{\pi}{2} \\ V \ ; & \frac{\pi}{2} \le \theta \le \pi \end{cases}$$

$$\int_{-1}^{1} \phi\Big|_{r=b} P_\ell(\cos\theta)\, d\cos\theta = \sum_{\ell'} [\quad] \int_{-1}^{1} P_{\ell'}(\cos\theta) P_\ell(\cos\theta)\, d\cos\theta = \sum_{\ell'} [\quad] \frac{2}{2\ell+1} \delta_{\ell'\ell} =$$

$$= \left[A_\ell b^\ell + B_\ell b^{-(\ell+1)}\right] \frac{2}{2\ell+1}$$

$$= \int_{-1}^{0} V\, P_\ell(\cos\theta)\, d\cos\theta = -\int_{0}^{1} V P_\ell(x)\, dx = -C_\ell$$

can use $\int_{-1}^{1} P_\ell(x)\, dx = 0$

↳ we thus have a set of eqs.

$$\left[A_\ell b^\ell + B_\ell b^{-(\ell+1)}\right] = -C_\ell$$
$$\left[A_\ell a^\ell + B_\ell a^{-(\ell+1)}\right] = C_\ell$$

$\ell = 2k \quad C_\ell = 0 \Rightarrow \begin{matrix} A_\ell a^\ell = -B_\ell a^{-(\ell+1)} \\ A_\ell b^\ell = -B_\ell b^{-(\ell+1)} \end{matrix} \Rightarrow A_\ell = B_\ell = 0$ for $\ell = 2k$

$\ell = 2k+1 \quad C_\ell \neq 0 \Rightarrow \begin{matrix} A_\ell b^{2\ell+1} + B_\ell = -C_\ell b^{\ell+1} \\ A_\ell a^{2\ell+1} + B_\ell = C_\ell a^{\ell+1} \end{matrix} \Rightarrow \begin{matrix} A_\ell (a^{2\ell+1} - b^{2\ell+1}) = C_\ell (a^{\ell+1} + b^{\ell+1}) \\ A_\ell = C_\ell \cdot \dfrac{a^{\ell+1} + b^{\ell+1}}{a^{2\ell+1} - b^{2\ell+1}} \end{matrix}$

$$\begin{matrix} A_\ell + B_\ell b^{-(2\ell+1)} = -C_\ell b^{-\ell} \\ A_\ell + B_\ell a^{-(2\ell+1)} = C_\ell a^{-\ell} \end{matrix} \Rightarrow \begin{matrix} B_\ell \left[b^{-(2\ell+1)} - a^{-(2\ell+1)}\right] = -C_\ell (b^{-\ell} + a^{-\ell}) \\ B_\ell = C_\ell \dfrac{a^{-\ell} + b^{-\ell}}{a^{-(2\ell+1)} - b^{-(2\ell+1)}} \end{matrix}$$

thus finally

$$\phi(r,\theta,\varphi) = \sum_{\ell = \text{odd}} C_\ell \left[r^\ell \frac{a^{\ell+1} + b^{\ell+1}}{a^{2\ell+1} - b^{2\ell+1}} + r^{-(\ell+1)} \frac{a^{-\ell} + b^{-\ell}}{a^{-(2\ell+1)} - b^{-(2\ell+1)}} \right] P_\ell(\cos\theta)$$

where C_ℓ on previous page

↳ general potential problem, $m \neq 0$

- the solution to the generalized Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2}\right]P = 0 \qquad x = \cos\theta$$

are associated Legendre functions $P_\ell^m(x)$, $m = -\ell, \dots, \ell$

↳ they satisfy

$$\int_{-1}^{1} P_{\ell'}^m(x)\, P_\ell^m(x)\, dx = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\,\delta_{\ell'\ell}$$

↳ the solution of Laplace equation was decomposed in r, θ, φ

$$\nabla^2\phi \;\rightarrow\; \phi = \frac{U(r)}{r}P(\theta)\,Q(\varphi) \qquad \begin{cases} \dfrac{1}{Q}\dfrac{d^2Q}{d\varphi^2} = -m^2 & \rightarrow\; Q = e^{\pm im\varphi} \\ \dfrac{d^2U}{dr^2} - \dfrac{\ell(\ell+1)}{r^2}U = 0 & \rightarrow\; U = Ar^{\ell+1} + Br^{-\ell} \\ \dfrac{d}{dx}\left[(1-x^2)\dfrac{dP}{dx}\right] + \left[\ell(\ell+1) - \dfrac{m^2}{1-x^2}\right]P = 0 & \rightarrow\; P_\ell^m(x) \qquad x = \cos\theta \end{cases}$$

↳ the angular variables can be combined in a set of orthonormal functions

spherical harmonics

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\, e^{im\varphi} \qquad m = -\ell, \dots, \ell$$

the angles: $0 \le \varphi \le 2\pi$

$-1 \le \cos\theta \le 1$ or $0 \le \theta \le \pi$

↳ properties of spherical harmonics

$$Y_{\ell,-m}(\theta,\varphi) = (-1)^m\, Y_{\ell m}^*(\theta,\varphi)$$

orthonormality conditions

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\; Y_{\ell' m'}^*(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

completeness relation

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi) = \delta(\varphi-\varphi')\,\delta(\cos\theta - \cos\theta')$$

↳ some explicit spherical harmonics

$l=0$ $$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$l=1$ $$\begin{cases} Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta \, e^{i\varphi} \\ Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta \\ Y_{1-1} = -1 Y_{11}^* \end{cases}$$

$l=2$ $$\begin{cases} Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \sin^2\theta \, e^{2i\varphi} \\ Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta \, e^{i\varphi} \\ Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \\ Y_{2-1} = -Y_{21}^* \\ Y_{2-2} = Y_{22}^* \end{cases}$$

↳ for $m=0$

$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} \, P_l(\cos\theta)$$

↳ an arbitrary function of θ,φ can be expanded in spherical harmonics

$$g(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta,\varphi)$$

with the coefficients

$$A_{lm} = \int d\Omega \, Y_{lm}^*(\theta,\varphi) \, g(\theta,\varphi)$$

↳ at $\theta=0$ one has

$$g(\theta,\varphi)\Big|_{\theta=0} = \sum_{l=0}^{\infty} \sqrt{\frac{2l+1}{4\pi}} \, A_{l0} \qquad \text{since} \quad Y_{lm}(\theta,\varphi)\Big|_{\theta=0} = \begin{cases} 0 & ; m \neq 0 \\ \sqrt{\frac{2l+1}{4\pi}} \underbrace{P_l(\cos\theta)\Big|_{\theta=1}}_{1} = \sqrt{\frac{2l+1}{4\pi}} & ; m=0 \end{cases}$$

here

$$A_{l0} = \sqrt{\frac{2l+1}{4\pi}} \int d\Omega \, P_l(\cos\theta) \, g(\theta,\varphi)$$

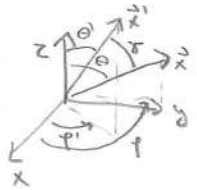
↳ the general solution to a boundary value problem can be written as

$$\boxed{\phi(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[A_{lm} r^l + B_{lm} r^{-(l+1)} \right] Y_{lm}(\theta,\varphi)}$$

useful especially if the potential is specified on a spherical surface

↳ addition theorem for spherical harmonics

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \varphi') \, Y_{\ell m}(\theta, \varphi)$$



γ is angle between $\vec{x}$ and $\vec{x}'$

$$\cos\gamma = \cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\varphi - \varphi')$$

↳ using this it immediately follows

$$\sum_{m=-\ell}^{\ell} |Y_{\ell m}(\theta, \varphi)|^2 = \frac{2\ell+1}{4\pi}$$

↳ the addition theorem applied to

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

gives

$$\frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y^*_{\ell m}(\theta', \varphi') \, Y_{\ell m}(\theta, \varphi)$$

$$r_< = \min\{|\vec{x}|, |\vec{x}'|\}$$
$$r_> = \max\{|\vec{x}|, |\vec{x}'|\}$$

↳ this form is useful when integrating over charge densities, e.g. $\vec{x}'$, since θ', φ' will be integration variables, θ, φ the position of the point where we want the potential evaluated

Problem 3.4

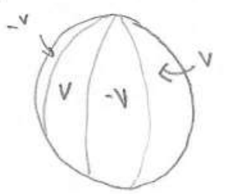
The surface of a hollow conducting sphere of inner radius a is divided into an even number of equal segments by a set of planes; their common line of intersection is the z axis and they are distributed uniformly in the angle ϕ (The segments are like the skin on wedges of an apple, or the earth's surface between successive meridians of longitude) The segments are kept at fixed potentials $\pm V$, alternately.

a) Set up a series representation for the potential inside the sphere for the general case of 2m segments, and carry the calculation of the coefficients in the series far enough to determine exactly which coefficients are different from zero. For the nonvanishing terms, exhibit the coefficients as an integral over $\cos\theta$.

b) For the special case m=1 (two hemispheres) determine explicitly the potential up to and including all terms with $l=3$. By a coordinate transformation verify that this reduces to result (3.36) of Section 3.3

Solution

a)



2m segments

↳ we have Laplace eq. + Dirichlet boundary condition

$$\nabla^2 \phi = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\phi\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0$$

⇒ the solution is expressible in series

$$\phi(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^l + B_{lm} r^{-(l+1)}\right) Y_{lm}(\theta,\varphi)$$

– no charges at r=0 ⇒ $B_{lm} = 0$

the solution thus

$$\phi(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{lm} r^l \, Y_{lm}(\theta,\varphi)$$

↳ boundary condition

$$\phi(a,\theta,\varphi) = \begin{cases} V \;; & \frac{2\pi}{2m} 2k \leq \varphi < \frac{\pi}{m}(2k+1) \\ -V \;; & \frac{\pi}{m}(2k+1) \leq \varphi \leq \frac{\pi}{m}(2k+2) \end{cases} \qquad k=0,\dots,m-1$$

↳ we get to the $A_{\ell m}$

$$\int \phi(a,\theta,\varphi)\, Y^*_{\ell m}\, d\Omega = \sum_{\ell,m} a^{\ell} \int A_{\ell' m'} \underbrace{Y^*_{\ell' m'} Y_{\ell m}\, d\Omega}_{\delta_{\ell\ell'}\delta_{mm'}} = A_{\ell m}\, a^{\ell}$$

$$A_{\ell m}\, a^{\ell} = \int \phi(a,\theta,\varphi)\, Y^*_{\ell m}\, d\Omega = \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi\; \phi(a,\theta,\varphi)\, Y^*_{\ell m}(\theta,\varphi) =$$

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\; P_\ell^m(\cos\theta)\, e^{im\varphi}$$

$$= \underbrace{\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}}_{C} \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi\, P_\ell^m(\cos\theta)\, e^{-im\varphi}\, \phi(a,\theta,\varphi) = C\int_{-1}^{1} d\cos\theta\, P_\ell^m(\cos\theta)\Big[V\sum_{k=0}^{n-1} \int_{\frac{\pi}{n}(2k)}^{\frac{\pi}{n}(2k+1)} d\varphi\, e^{-im\varphi} -$$

$$- V\sum_{k=0}^{n-1} \int_{\frac{\pi}{n}(2k+1)}^{\frac{\pi}{n}(2k+2)} d\varphi\, e^{-im\varphi}\Big] =*$$

$$[\quad] = V\Big\{ \sum_{k=0}^{n-1}\Big(\int_{\frac{\pi}{n}2k}^{\frac{\pi}{n}(2k+1)} d\varphi\, e^{-im\varphi} - \int_{\frac{\pi}{n}(2k+1)}^{\frac{\pi}{n}(2k+2)} d\varphi\, e^{-im\varphi}\Big) \overset{m\neq 0}{=} V\sum_{k=0}^{n-1}\Big(\frac{1}{-im}\Big)\Big(e^{-im\varphi}\Big|_{\frac{\pi}{n}2k}^{\frac{\pi}{n}(2k+1)} - e^{-im\varphi}\Big|_{\frac{\pi}{n}(2k+1)}^{\frac{\pi}{n}(2k+2)}\Big)$$

$$= \frac{Vi}{m}\sum_{k=0}^{n-1}\Big(2e^{-im(\frac{\pi}{n}(2k+1))} - e^{-im\frac{\pi}{n}2k} - e^{-im\frac{\pi}{n}(2k+2)}\Big) = \frac{Vi}{m}\sum_{k=0}^{n-1} e^{-i\frac{m\pi}{n}2k}\underbrace{\Big(2e^{-i\frac{m\pi}{n}} - 1 - e^{-i\frac{2m\pi}{n}}\Big)}_{-\left(e^{-i\frac{m\pi}{n}}-1\right)^2}$$

use $\displaystyle\sum_{k=0}^{n-1} a^k = \frac{1-a^n}{1-a} \Rightarrow \sum_{k=0}^{n-1}\Big(e^{-i\frac{2\pi}{n}m}\Big)^k = \frac{1-e^{-i\frac{2\pi m n}{n}}}{1-e^{-i\frac{2\pi m}{n}}}$

↑ for $e^{-i\frac{2\pi m}{n}} \neq 1$ i.e. $\frac{m}{n}$ not integer

$$[\quad] = -\frac{Vi}{m}\;\frac{1-e^{-i\frac{2\pi m n}{n}}}{\underbrace{1-e^{-i\frac{2\pi m}{n}}}_{(1-e^{-i\frac{\pi m}{n}})(1+e^{-i\frac{\pi m}{n}})}}\Big(e^{-i\frac{\pi}{n}m}-1\Big)^2 = -\frac{Vi}{m}\big(1-e^{-i\frac{2\pi m n}{n}}\big)\frac{\big(1-e^{-i\frac{\pi m}{n}}\big)}{\big(1+e^{i\frac{\pi m}{n}}\big)} \quad \text{for } \frac{m}{n} \text{ not integer}$$

for $\frac{m}{n} = N$ integer

$$[\quad] = -\frac{Vi}{m}\underbrace{\sum_{k=0}^{n-1}\underbrace{e^{-i2\pi k N}}_{1}}_{n}\big(e^{-i\pi N}-1\big)^2 = -Vi\,\big((-1)^N-1\big)^2 = \begin{cases} -4Vi/m & N=\text{odd} \\ 0 & N=\text{even}\end{cases}$$

$$\Rightarrow A_{lm} = \frac{1}{a^l} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} \int_{-1}^{1} dx\, P_l^m(x) \cdot D_{lm}$$

$D_{lm} = [\ \ ]$ on the previous side

$$\Rightarrow \quad \phi = \sum_{l,m} \left(\frac{r}{a}\right)^l \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} \int_{-1}^{1} dx\, P_l(x)\, D_{lm}\, Y_{lm}(\theta,\varphi)$$

b) for $n=1 \Rightarrow \frac{m}{M} = m$ is integer and thus $D_{lm} = -\frac{4V_i}{m}$ for m odd

$$\phi = \sum_{m=-1,1} A_{1m}\, r\, Y_{lm}(\theta,\varphi) + \sum_{m=-1,1} A_{2m}\, r^2 Y_{2m}(\theta,\varphi) + \sum_{m=-3,-1,1,3} A_{3m}\, r^3 Y_{3m}(\theta,\varphi)$$

$$A_{1m} = -\frac{4V_i}{m} \frac{1}{a} \sqrt{\frac{3}{4\pi}} \sqrt{\frac{(3-m)!}{(3+m)!}} \int_{-1}^{1} dx\, P_1^m(x)$$

$$A_{1-1} = V i \sqrt{\frac{3\pi}{2}} \frac{1}{a} \qquad A_{11} = A_{1-1}$$

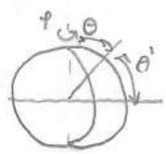
$$A_{2,-1} = A_{2,1} = 0$$

$$A_{3,-3} = A_{33} = iV\sqrt{\frac{35\pi}{256}} \frac{1}{a^3}$$

$$A_{3,-1} = A_{3,1} = iV\sqrt{\frac{21\pi}{256}} \frac{1}{a^3}$$

$$\Rightarrow \phi = \frac{3}{2} V \left(\frac{r}{a}\right) \sin\theta \cos\theta + \left(\frac{r}{a}\right)^3 V \left[\frac{35}{64} \sin^3\theta \sin 3\varphi + \frac{21}{64} \sin\theta \left(5\cos^2\theta - 1\right) \sin\varphi\right] + \dots$$

$\hookrightarrow$ transformation



symmetric around this axis

$$\cos\theta' = \sin\theta \sin\varphi$$

in new variable

$$\phi = \frac{3}{2} V \left(\frac{r}{a}\right) \cos\theta' - \frac{7}{8} \left(\frac{r}{a}\right)^3 V \left[\frac{5}{2} \cos^3\theta' - \frac{3}{2} \cos\theta'\right] + \dots$$

$$= \frac{3}{2} V \left(\frac{r}{a}\right) P_1(\cos\theta') - \frac{7}{8} \left(\frac{r}{a}\right)^3 V P_3(\cos\theta') + \dots$$

3.7 CYLINDRICAL COORDINATES

$$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

separation of variables

$$\phi = R(\rho)\,Q(\varphi)\,Z(z)$$

↳ from here

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2 \quad\Rightarrow\quad Z(z) = e^{\pm kz}$$

$$\frac{1}{Q}\frac{d^2Q}{d\varphi^2} = -\nu^2 \quad\Rightarrow\quad Q(\varphi) = e^{\pm i\nu\varphi} \qquad \nu = m \text{ integer if } \varphi \in [0, 2\pi]$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho} + \left(k^2 - \frac{\nu^2}{\rho^2}\right)R = 0$$

⇓

this is equivalent to $(x = \rho k)$

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0 \qquad \text{Bessel equation}$$

↳ the solutions are Bessel functions of the first kind of the order $\pm\nu$

$$J_\nu(x) \qquad J_{-\nu}(x) \qquad x = k\rho$$

- if ν not integer $\Rightarrow$ $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent
- if ν is an integer, they are not

$$J_{-m}(x) = (-)^m J_m(x)$$

$\Rightarrow$ then the second solution can be taken as Bessel function of the second kind (or the Neumann function)

$$N_\nu(x) = \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi}$$

↳ useful to define Hankel Functions (Bessel functions of the 3rd kind)

$$H_\nu^{(1)}(x) = J_\nu(x) + i N_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - i N_\nu(x)$$

↳ they also form a set of indep. solutions to Bessel eq.

↳ for small x

$x \ll 1$ $\quad J_\nu(x) \to \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu$

$$N_\nu(x) \to \begin{cases} \frac{2}{\pi}\left(\ln\left(\frac{x}{2}\right) + 0.5772 + \cdots\right) & \nu = 0 \\ -\frac{\Gamma(\nu)}{\pi}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases}$$

↳ for large x

$x \gg 1, \nu$

$$J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

⇒ each Bessel function has an infinite number of roots

↳ normalization of Bessel functions is such that

$$\int_0^a d\rho\, \rho\, J_\nu\left(x_{\nu m'} \frac{\rho}{a}\right) J_\nu\left(x_{\nu m} \frac{\rho}{a}\right) = \frac{a^2}{2}\left[J_{\nu+1}(x_{\nu m})\right]^2 \delta_{mm'}$$

where $J_\nu(x_{\nu m}) = 0$

$x_{\nu m}$ is the m-th root of $J_\nu(x)$

↳ for instance

$\nu = 0 \quad \Rightarrow \quad x_{0m} = 2.405,\ 5.520, \ldots$

$\nu = 1 \quad \Rightarrow \quad x_{1m} = 3.832,\ 7.016, \ldots$

↳ we can expand a function $f(\rho)$, $0 \leq \rho \leq a$ in a Bessel series

$$f(\rho) = \sum_{m=1}^{\infty} A_{\nu m} J_\nu\left(x_{\nu m} \frac{\rho}{a}\right)$$

where

$$A_{\nu m} = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu m})} \int_0^a \rho f(\rho) J_\nu\left(x_{\nu m}\frac{\rho}{a}\right) d\rho$$

– this expansion is useful for functions that vanish at $\rho = a$

↳ other expansions are possible

Neumann series : $\sum_{n=0}^{\infty} a_n J_{\nu+n}(z)$

Kapteyn series : $\sum_{n=0}^{\infty} a_n J_{\nu+n}((\nu+n)z)$

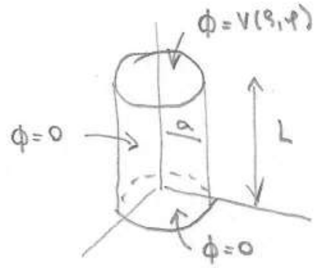
Schlömilch series : $\sum_{n=1}^{\infty} a_n J_\nu(nx)$

↳ finally, if $k^2 < 0$ ⇒ then k is imaginary and solutions to Bessel eq. are modified Bessel functions

$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$
$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix)$$

– note: $I_\nu(x)$, $K_\nu(x)$ are real functions if x is real

example: boundary value problem for a cylinder



boundary conditions:

$$\phi\big|_{\rho=a} = 0$$
$$\phi\big|_{z=0} = 0 \qquad \phi\big|_{z=L} = V(\rho,\varphi)$$

↳ the separation of variables

$$Q(\varphi) = A \sin\nu\varphi + B\cos\nu\varphi$$
$$Z(z) = a\, e^{-kz} + b\, e^{kz}$$

b.c.

$$Q\big|_{\varphi=0} = Q\big|_{\varphi=2\pi} \Rightarrow \quad Q(\varphi) = A\sin n\varphi + B\cos n\varphi \qquad n \text{ is integer}$$

$$Z(0) = 0 \Rightarrow \quad a + b = 0 \Rightarrow \quad Z(z) = \sinh(kz)$$
$$a = -b$$

↳ the radial factor is

$$R(\rho) = C\, J_m(k\rho) + D\, N_m(k\rho)$$

– for $\rho \to 0$ $N_m(k\rho)$ diverges as either $\ln(x)$ ($m=0$) or x^{-m} for $m \neq 0$

$$\Rightarrow D = 0$$

– since b.c. is $\phi\big|_{\rho=a} = 0 \Rightarrow J_m(ka) = 0$

so ka needs to be a zero of $J_m(x)$

$$\Rightarrow k_{mn} = \frac{x_{mn}}{a} \qquad n = 1,2,3\ldots$$

x_{mn} is n-th root of $J_m(x)$, i.e. $J_m(x_{mn}) = 0$

↳ the solution to boundary value problem is thus

$$\phi(\rho,\varphi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\sinh(k_{mn}z)\left(A_{mn}\sin m\varphi + B_{mn}\cos m\varphi\right)$$

– the coefficients A_{mn}, B_{mn} are obtained from b.c. at $z=L$

$$\phi\Big|_{z=L} = V(\rho,\varphi) = \sum_{m,n} J_m(k_{mn}\rho)\sinh(k_{mn}L)\left(A_{mn}\sin m\varphi + B_{mn}\cos m\varphi\right)$$

$$\Rightarrow \quad A_{mn} = \frac{2}{\pi}\frac{1}{a^2 J_{m+1}^2(k_{mn}a)}\frac{1}{\sinh(k_{mn}L)}\int_0^{2\pi} d\varphi \int_0^a d\rho\, \rho\, V(\rho,\varphi)\, J_m(k_{mn}\rho)\sin m\varphi$$

$$B_{mn} = \quad -\|- \qquad\qquad -\|- \qquad \cos m\varphi$$

(note: for $m=0$ one need to use $\frac{1}{2}B_{0n}$ in the series)

↳ note: this series is appropriate for finite interval in ρ, $0 \le \rho \le a$

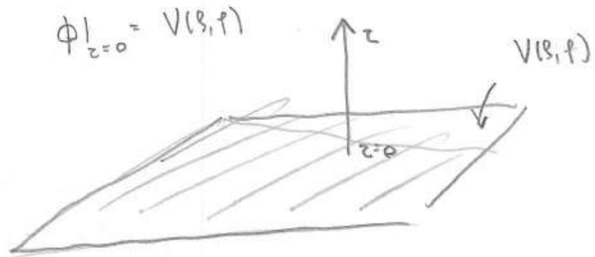
↳ if ρ can go to ∞ we have

(for $z \geq 0$, no free charges and $\phi \to 0$ for $z \to \infty$)

$$\phi(\rho,\varphi,z) = \sum_{m=0}^{\infty}\int_0^{\infty} dk\, e^{-kz} J_m(k\rho)\left[A_m(k)\sin(m\varphi) + B_m(k)\cos(m\varphi)\right]$$

– the coefficients obtained from B.C.

$$\phi\Big|_{z=0} = V(\rho,\varphi)$$



$$\phi\Big|_{z=0} = V(\rho,\varphi) = \sum_{m=0}^{\infty}\int_0^{\infty} dk\, J_m(k\rho)\left[A_m(k)\sin(m\varphi) + B_m(k)\cos(m\varphi)\right]$$

– can use orthogonality of $\sin(m\varphi)$, $\cos(n\varphi)$ and the relation

$$\int_0^{\infty}\rho\, J_m(k\rho)\, J_m(k'\rho)\, d\rho = \frac{1}{k}\,\delta(k-k')$$

$$\Rightarrow \quad A_m(k) = \frac{k}{\pi}\int_0^{\infty} d\rho\, \rho \int_0^{2\pi} d\varphi\, V(\rho,\varphi)\, J_m(k\rho)\sin(m\varphi)$$

$$B_m(k) = \qquad -\|- \qquad \cos(m\varphi)$$

Problem 3.9 : A hollow right circular cylinder of radius b has its axis coincident with the z axis and its ends at z=0 and z=L. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\varphi, z)$. Using the appropriate separation of variables in cylindrical coordinates, find a series solution for the potential anywhere inside the cylinder.

Solution: no charge ⇒ we have Laplace eq. in cylindrical coordinates

$$\nabla^2 \phi = 0 \quad \Rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

- separation of variables

$$\phi = R(\rho)\, Q(\phi)\, Z(z)$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = k^2 \quad \Rightarrow \quad \begin{cases} Z(z) = e^{\pm kz} & \text{for } k \neq 0 \\ Z(z) = A + Bz & \text{for } k = 0 \end{cases}$$

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \varphi^2} = -\nu^2 \quad \Rightarrow \quad \begin{cases} Q(\varphi) = e^{\pm i\nu\varphi} & \text{for } \nu \neq 0 \\ Q(\varphi) = A + B\varphi & \text{for } \nu = 0 \end{cases}$$

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)R = 0 \qquad k \neq 0,\ x = \rho k \quad \Rightarrow \begin{cases} J_\nu(x), J_{-\nu}(x) & x = k\rho,\ \nu \neq \text{integer} \\ J_\nu(x), N_\nu(x) & \nu \text{ is integer} \end{cases}$$

$$\text{if } k = 0 \quad \Rightarrow \quad \frac{1}{R}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{dR}{d\rho}\right) + \frac{1}{\rho^2}(-\nu^2) = 0$$

$$\Downarrow$$

$$\begin{cases} \nu = 0 & R = A + B\ln\rho \\ \nu \neq 0 & R = A\rho^\nu + B\rho^{-\nu} \end{cases}$$

↳ the general solution to Laplace eq. in cylindrical coord. thus

$$\begin{aligned}
\phi(\rho,\varphi,z) = {} & (A_{00} + B_{00}\ln\rho)(C_{00} + D_{00}\varphi)(F_{00} + G_{00}z) && \leftarrow k=0, \nu=0 \\
& + \sum_{\nu\neq 0}^{\infty} (A_{\nu 0}\rho^\nu + B_{\nu 0}\rho^{-\nu})(C_{\nu 0}e^{i\nu\varphi} + D_{\nu 0}e^{-i\nu\varphi})(F_{\nu 0} + G_{\nu 0}z) && \leftarrow k=0, \nu\neq 0 \\
& + \sum_{k\neq 0} (A_{0k}J_0(k\rho) + B_{0k}N_0(k\rho))(C_{0k} + D_{0k}\varphi)(F_{0k}e^{kz} + G_{0k}e^{-kz}) && \leftarrow k\neq 0, \nu=0 \\
& + \sum_{\nu\neq, k\neq 0} (A_{\nu k}J_\nu(k\rho) + B_{\nu k}N_\nu(k\rho))(C_{\nu k}e^{i\nu\varphi} + D_{\nu k}e^{-i\nu\varphi})(F_{\nu k}e^{kz} + G_{\nu k}e^{-kz}) && \leftarrow k\neq 0, \nu\neq 0
\end{aligned}$$

↳ we need periodic function

$$\phi\big|_{\varphi=0} = \phi\big|_{\varphi=2\pi} \quad \Rightarrow \quad \begin{aligned} D_{00} &= 0 \\ D_{0k} &= 0 \end{aligned}$$

ν are integers

↳ the solution is thus

$$\phi(\rho,\varphi,z) = \left(A_{00} + B_{00}\ln\rho\right)\left(F_{00} + G_{00} z\right)$$
$$+ \sum_{n\neq 0}\left(A_{n0}\rho^{n} + B_{n0}\rho^{-n}\right)\left(C_{n0}e^{in\varphi} + D_{n0}e^{-in\varphi}\right)\left(F_{n0} + G_{n0} z\right)$$
$$+ \sum_{k\neq 0}\left(A_{0k} J_0(k\rho) + B_{0k} N_0(k\rho)\right)\left(F_{0k}e^{kz} + G_{0k}e^{-kz}\right)$$
$$+ \sum_{n\neq 0,\, k\neq 0}\left(A_{nk} J_n(k\rho) + B_{nk} N_n(k\rho)\right)\left(C_{nk}e^{in\varphi} + D_{nk}e^{-in\varphi}\right)\left(F_{nk}e^{kz} + G_{nk}e^{-kz}\right)$$

– no charges at $\rho = 0$ $\Rightarrow$ $B_{00} = 0$, $B_{n0} = 0$, $B_{0k} = B_{nk} = 0$

↑ since $N_n(k\rho) \to \infty$ $k\rho \to 0$

$$\Rightarrow \phi(\rho,\varphi,z) = \left(A_{00} + B_{00} z\right) + \sum_{n\neq 0}\rho^{n}\left(A_{n0}e^{in\varphi} + B_{n0}e^{-in\varphi}\right)\left(F_{n0} + G_{n0} z\right)$$
$$+ \sum_{k\neq 0} J_0(k\rho)\left(A_{0k}e^{kz} + B_{0k}e^{-kz}\right) + \sum_{n\neq 0,\, k\neq 0} J_n(k\rho)\left(A_{nk}e^{in\varphi} + B_{nk}e^{-in\varphi}\right)\left(C_{nk}e^{kz} + D_{nk}e^{-kz}\right)$$

↳ boundary conditions:

$$\phi\big|_{z=0} = 0 \qquad \phi\big|_{z=L} = 0$$

↳ from $\phi\big|_{z=0} = 0$ $\Rightarrow$ $A_{00} = 0$, $F_{n0} = 0$, $A_{0k} = -B_{0k}$, $C_{nk} = -D_{nk}$

$$\Rightarrow \phi(\rho,\varphi,z) = B_{00} z + \sum_{n=0}\rho^{n}\left(A_{n0}e^{in\varphi} + B_{n0}e^{-in\varphi}\right)G_{n0} z + \sum_{k\neq 0} J_0(k\rho)\, A_{0k}\sinh(kz)$$
$$+ \sum_{n\neq 0,\, k\neq 0} J_n(k\rho)\left(A_{nk}e^{in\varphi} + B_{nk}e^{-in\varphi}\right)\sinh(kz)$$

↳ from $\phi\big|_{z=L} = 0$ $\Rightarrow$ $G_{n0} = 0$, $B_{00} = 0$

$\sinh(kL) = 0$ $\Rightarrow$ $k = i\frac{\pi m}{L}$; $m = 1, 2, \dots$

$$\Rightarrow \phi(\rho,\varphi,z) = \sum_{m=1}^{\infty} J_0\!\left(i\frac{\pi m}{L}\rho\right) A_{0m}\sin\!\left(\frac{\pi m}{L}z\right) +$$
$$+ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} J_n\!\left(i\frac{\pi m}{L}\rho\right)\left(A_{nm}e^{in\varphi} + B_{nm}e^{-in\varphi}\right)\sin\!\left(\frac{\pi m}{L}z\right)$$

we have

$$J_\nu(ix) = i^\nu I_\nu(x)$$

↑ modified Bessel functions

$$\Rightarrow \phi(\rho,\varphi,z) = \sum_{m=1}^{\infty} I_0\left(\frac{\pi m}{L}\rho\right) A_{0m} \sin\left(\frac{\pi m}{L} z\right) + \sum_{m,n=1}^{\infty} i^n I_n\left(\frac{\pi m}{L}\rho\right)\left(A_{nm} e^{in\varphi} + B_{nm} e^{-in\varphi}\right)\sin\left(\frac{\pi m}{L} z\right)$$

↳ boundary condition on the sides of cylinder

$$\phi\Big|_{\rho=b} = V(\varphi,z)$$

$$\phi(\rho=b) = \sum_{m=1}^{\infty} I_0\left(\frac{\pi m}{L} b\right) A_{0m} \sin\left(\frac{\pi m}{L} z\right) + \sum_{m,n=1}^{\infty} i^n I_n\left(\frac{\pi m}{L} b\right)\left(A_{nm} e^{in\varphi} + B_{nm} e^{-in\varphi}\right)\sin\left(\frac{\pi m}{L} z\right)$$

can also double Fourier expand $V(\varphi,z)$ $\Rightarrow$ coeff. A:

$$\int_0^{2\pi}\int_0^L V(\varphi,z)\, \underset{e^{+in\varphi}}{e^{-in\varphi}}\, d\varphi \sin\left(\frac{\pi m}{L} z\right) dz = 2\pi\, I_n\left(\frac{\pi m}{L} b\right) \underset{B_{nm}}{A_{nm}} \cdot \frac{L}{2}$$

$\Rightarrow A_{nm}, B_{nm}$

↳ the solution is thus

$$\phi(\rho,\varphi,z) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{I_n\left(\frac{\pi m}{L}\rho\right)}{I_n\left(\frac{\pi m}{L} b\right)} \sin\left(\frac{\pi m}{L} z\right)\Big(\overbrace{\tfrac{1}{\pi L}\int V(\varphi,z)\, e^{-in\varphi} \sin\left(\tfrac{\pi m}{L} z\right) d\varphi\, dz}^{a_{nm}}\; e^{in\varphi} + a_{nm}^* e^{-in\varphi}\Big)$$

$$\phi(\rho,\varphi,z) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{I_n\left(\frac{\pi m}{L}\rho\right)}{I_n\left(\frac{\pi m}{L} b\right)} \sin\left(\frac{\pi m}{L} z\right)\left[a_{nm} e^{in\varphi} + a_{nm}^* e^{-in\varphi}\right]$$

$$a_{nm} = \frac{1}{\pi L}\int_0^{2\pi} d\varphi \int_0^L dz\; V(\varphi,z)\, e^{-in\varphi} \sin\left(\frac{\pi m}{L} z\right)$$

Problem 3.12: An infinite, thin, plane sheet of conducting material has a circular hole of radius a cut in it. A thin, flat disc of the same material and slightly smaller radius lies in the plane, filling the hole, but separated from the sheet by a very narrow insulating ring. The disc is maintained at a fixed potential V, while the infinite sheet is kept at zero potential.

a) Using appropriate cylindrical coordinates, find an integral expression involving Bessel functions for the potential at any point above the plane.

b) Show that the potential a perpendicular distance z above the center of the disc is

$$\phi_0(z) = V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

c) Show that the potential a perpendicular distance z above the edge of the disc is

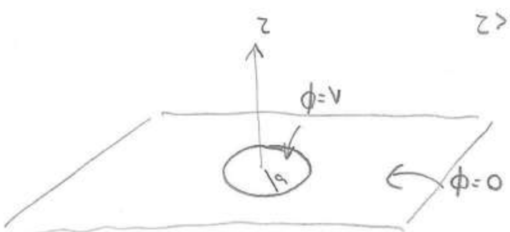
$$\phi_a(z) = \frac{V}{2}\left(1 - \frac{kz}{\pi a}\, K(k)\right)$$

where $k = \frac{2a}{(z^2+4a^2)^{1/2}}$, and $K(k)$ is the complete elliptic integral of the first kind

Solution

$z>0$ is our region

a)



↳ cylindrical coordinates since cylindrical symmetry

no free charges, laplace eq.

$$\nabla^2\phi = 0$$

$z>0$ separation of variables gives

$$\begin{aligned}\phi(\rho,\varphi,z) = & \left(A_{00} + B_{00}\ln\rho\right)\left(C_{00} + D_{00}\varphi\right)\left(F_{00} + G_{00}z\right) \\ & + \sum_{\nu\neq 0}\left(A_{\nu 0}\rho^{\nu} + B_{\nu 0}\rho^{-\nu}\right)\left(C_{\nu 0}e^{i\nu\varphi} + D_{\nu 0}e^{-i\nu\varphi}\right)\left(F_{\nu 0} + G_{\nu 0}z\right) \\ & + \sum_{k\neq 0}\left(A_{0k}J_0(k\rho) + B_{0k}N_0(k\rho)\right)\left(C_{0k} + D_{0k}\varphi\right)\left(F_{0k}e^{kz} + G_{0k}e^{-kz}\right) \\ & + \sum_{\nu\neq 0}\sum_{k\neq 0}\left(A_{\nu k}J_\nu(k\rho) + B_{\nu k}N_\nu(k\rho)\right)\left(C_{\nu k}e^{i\nu\varphi} + D_{\nu k}e^{-i\nu\varphi}\right)\left(F_{\nu k}e^{kz} + G_{\nu k}e^{-kz}\right)\end{aligned}$$

↳ it has axial symmetry ⇒ ϕ does not depend on φ

only φ indep. terms survive

$$\phi(\rho,\varphi,z) = \left(A_{00} + B_{00}\ln\rho\right)\left(F_{00} + G_{00} z\right) + \sum_{k\neq 0}\left(A_{0k} J_0(k\rho) + B_{0k} N_0(k\rho)\right)\left(F_{0k} e^{kz} + G_{0k} e^{-kz}\right)$$

when $z \to \pm\infty$ ϕ needs to be finite ⇒ $G_{00} = 0$, $F_{0k} = 0$

$\rho \to 0$ —‖— ⇒ $B_{00} = 0$, $B_{0k} = 0$

↳ the solution is

$$\phi(\rho,\varphi,z) = A_{00} + \sum_{k\neq 0} A_{0k} J_{0}(k\rho) e^{-kz} = \sum_{k=0} A_{0k} J_0(k\rho) e^{-kz}$$

note: we have infinite boundary, thus k is continuous variable

$$\phi = \int_0^\infty dk\ A(k) J_0(k\rho) e^{-kz}$$

↳ boundary condition is

$$\phi(z=0) = V(\rho)$$

$$\Rightarrow \quad V(\rho) = \int_0^\infty dk\, A(k) J_0(k\rho)$$

we can use

$$\int_0^\infty dx\ x\, J_0(kx) J_0(k'x) = \frac{1}{k}\delta(k-k')$$

$$\Rightarrow \int_0^\infty V(\rho) J_0(k'\rho)\rho\, d\rho = \int_0^\infty\int_0^\infty dk\, A(k) J_0(k\rho) J_0(k'\rho)\rho\, d\rho = \int \frac{1}{k}\delta(k-k')\, dk\, A(k) = \frac{1}{k'} A(k')$$

in our case $V(\rho) = \begin{cases} V\ ;\ \rho < a \\ 0\ ;\ \rho \geq a \end{cases}$

$$V\int_0^a J_0(k'\rho)\rho\, d\rho = \frac{1}{k'} A(k')$$

$$\text{or} \quad A(k) = Vk\int_0^a J_0(k\rho)\rho\, d\rho = Va\, J_1(ka)$$

thus we have

$$\left[\ \phi(\rho,\varphi,z) = Va\int_0^\infty J_1(ka) J_0(k\rho) e^{-kz}\, dk\ \right]$$

b) above the center of the disc:

$$\phi(s=0) = Va \int_0^\infty J_1(ka)\underbrace{J_0(k\cdot 0)}_{1} e^{-kz} dk = Va \int_0^\infty J_1(\underbrace{ka}_{x}) e^{-kz} dk =$$

$ka = x$
$k = \frac{x}{a}$

we can use the representation

$$J_1(x) = \frac{1}{2\pi i}\int_0^{2\pi} e^{i(x\cos\theta+\theta)} d\theta$$

$$\phi\Big|_{s=0} = V\int_0^\infty J_1(x) e^{-xz/a} dx = \frac{V}{2\pi i}\int_0^\infty dx \int_0^{2\pi} d\theta\, e^{-x\frac{z}{a}} e^{i(x\cos\theta+\theta)} = \frac{V}{2\pi i}\int_0^{2\pi} d\theta\, e^{i\theta}\int_0^\infty dx\, e^{x(i\cos\theta - \frac{z}{a})} =$$

$$= \frac{V}{2\pi}\int_0^{2\pi} d\theta\, e^{i\theta}\frac{1}{\left(\cos\theta + i\frac{z}{a}\right)} = \frac{V}{2\pi}\int_0^{2\pi} d\theta\,\frac{(\cos\theta + i\sin\theta)(\cos\theta - i\frac{z}{a})}{\left(\cos^2\theta + (\frac{z}{a})^2\right)} = \frac{V}{2\pi}\int_0^{2\pi} d\theta\,\frac{\cos^2\theta}{\left(\cos^2\theta + (\frac{z}{a})^2\right)}$$

$$= V\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)$$

c) above the edge of the disc

$$\phi\Big|_{s=a} = Va\int_0^\infty J_1(ka)\, J_0(ka)\, e^{-kz} dk$$

use representation
$$J_1(x) = \frac{1}{2\pi i}\int_0^{2\pi} e^{i(x\cos\theta+\theta)} d\theta$$
$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\theta'} d\theta'$$

$$\phi\Big|_{s=a} = V\int_0^\infty J_1(x) J_0(x)\, e^{-\frac{xz}{a}} dx = V\frac{1}{(2\pi)^2}\frac{1}{i}\int_0^\infty dx\int_0^{2\pi} d\theta\int_0^{2\pi} d\theta'\, e^{i(x\cos\theta+\theta)} e^{ix\cos\theta'} e^{-\frac{xz}{a}} =$$

$$= -i\frac{V}{(2\pi)^2}\int_0^{2\pi} d\theta\int_0^{2\pi} d\theta'\int_0^\infty dx\, e^{i\theta} e^{x(i\cos\theta + i\cos\theta' - \frac{z}{a})} =$$

$$= \frac{V}{(2\pi)^2}\int_0^{2\pi} d\theta\int_0^{2\pi} d\theta'\, e^{i\theta}\frac{1}{\left(\cos\theta' + \cos\theta + i\frac{z}{a}\right)} = \frac{V}{2\pi^2}\int_0^{2\pi} d\theta\int_0^{\pi} d\theta'\,\frac{\cos\theta\cos\theta' + \cos^2\theta}{(\cos\theta' + \cos\theta)^2 + \frac{z^2}{a^2}}\, d\theta' d\theta$$

$$= \frac{V}{2}\left(1 - \frac{1}{\pi}\int_0^{\pi}\frac{d\theta}{\sqrt{1 + 4(\frac{a}{z})^2\cos^2\theta}}\right)$$

use $k = \dfrac{2}{\sqrt{(\frac{z}{a})^2 + 4}}$ $\Rightarrow \phi = \frac{V}{2}\left(1 - \frac{k}{\pi}\frac{z}{a}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}\right)$

or

$$\phi = \frac{V}{2}\left(1 - \frac{kz}{\pi a}K(k)\right)$$

since $K(k) = \int_0^{\pi/2}\frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}$

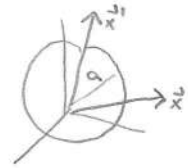
EXPANDING GREEN FUNCTIONS IN SPHERICAL COORDINATES

↳ often useful to have $G(\vec{x},\vec{x}')$ expanded in spherical or cylindrical coord.

↳ start with Green function for the case of no boundary:

$$\frac{1}{|\vec{x}-\vec{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi)$$

↳ Green function for the sphere



$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} - \frac{a}{x'} \frac{1}{|\vec{x}-\frac{a^2}{x'^2}\vec{x}'|}$$

$$\Rightarrow G(\vec{x},\vec{x}') = 4\pi \sum_{l,m} \frac{1}{2l+1} \left(\frac{r_<^l}{r_>^{l+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1} \right) Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi)$$

for the <u>exterior</u> of the sphere

↳ one can construct such expansion more generally

- consider a Green function for a Dirichlet problem:

$$\nabla^2_x G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$$

and $\quad G(\vec{x},\vec{x}') = 0 \quad$ for $\vec{x}$ or $\vec{x}'$ on the boundary surface S

- in spherical coordinates

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r')\, \underbrace{\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')}_{\displaystyle\downarrow}$$

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi) \qquad \text{completeness relation}$$

- we can thus write

$$\delta(\vec{x}-\vec{x}') = \frac{1}{r^2}\delta(r-r') \sum_{l,m} Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi)$$

↳ if we expand the Green function in terms of spherical harmonics

$$G(\vec{x},\vec{x}') = \sum_{l,m} A_{lm}(r\,|\,r',\theta',\varphi')\, Y_{lm}(\theta,\varphi)$$

where

$$A_{lm}(r|r',\theta',\varphi') = g_l(r,r')\, Y^*_{lm}(\theta',\varphi')$$

the radial Green function satisfies

$$\frac{1}{r}\frac{d^2}{dr^2}\left(r\, g_\ell(r,r')\right) - \frac{\ell(\ell+1)}{r^2}\, g_\ell(r,r') = -\frac{4\pi}{r^2}\,\delta(r-r') \qquad (*)$$

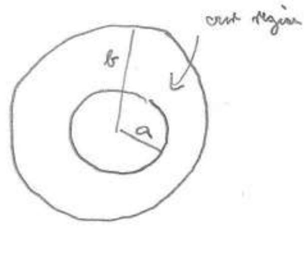
$\Rightarrow$ the solution for $r \neq r'$ is

$$g_\ell(r,r') = \begin{cases} A r^\ell + B r^{-(\ell+1)} & ; \ r<r' \\ A' r^\ell + B' r^{-(\ell+1)} & ; \ r>r' \end{cases}$$

$$A = A(r') \qquad B = B(r') \qquad A' = A'(r') \qquad B' = B'(r')$$

↳ the coeffs. are determined from b.c., and from symmetry $g(r,r') = g(r',r)$

↳ example: boundaries are two spheres with radii a and b



b.c. $g_\ell(r,r') = 0$ for $r=a$ or $r'=a$ or $r=b$ or $r'=b$

$$g_\ell\Big|_{r=a} = 0 \ \Rightarrow \ A a^\ell + B a^{-(\ell+1)} = 0 \qquad (a<r' \text{ always})$$
$$B = -A\, a^{2\ell+1}$$

$$g_\ell\Big|_{r=b} = 0 \ \Rightarrow \ A' b^\ell + B' b^{-(\ell+1)} = 0 \qquad (r'<b \text{ always})$$
$$B' = -A'\, b^{(2\ell+1)}$$

$$\Rightarrow \quad g_\ell(r,r') = \begin{cases} A\left(r^\ell - \dfrac{a^{2\ell+1}}{r^{\ell+1}}\right) & , \ r<r' \\ B'\left(\dfrac{1}{r^{\ell+1}} - \dfrac{r^\ell}{b^{2\ell+1}}\right) & , \ r>r' \end{cases}$$

- the symmetry in r and r' requires that

$$g_\ell(r,r') = C\left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)$$

here $r_<$ ($r_>$) is the smaller (larger) of r, r'

↳ the constant C is determined from δ function in $(*)$

- integrate $(*)$ over $r \in [r'-\varepsilon, r'+\varepsilon]$

$$\int_{r'-\varepsilon}^{r'+\varepsilon} dr\ \frac{1}{r}\frac{d^2}{dr^2}(r g_\ell) - \int_{r'-\varepsilon}^{r'+\varepsilon} \frac{\ell(\ell+1)}{r^2}\, g_\ell\, dr = -\int_{r'-\varepsilon}^{r'+\varepsilon} \frac{4\pi}{r^2}\,\delta(r-r')\, dr$$

$$\frac{1}{r'}\frac{d}{dr}(r g_\ell)\Big|_{r'-\varepsilon}^{r'+\varepsilon} = -\frac{4\pi}{r'^2} \qquad \Rightarrow \qquad \frac{d}{dr} g_\ell \Big|_{r'-\varepsilon}^{r'+\varepsilon} = -\frac{4\pi}{r'^2}$$

$$\frac{d}{dr}(r g_\ell)\Big|_{r=r'+\varepsilon} = \frac{d}{dr}\left[C\left(r'^\ell - \frac{a^{2\ell+1}}{r'^{\ell+1}}\right)\left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{b^{2\ell+1}}\right)\right]\Bigg|_{r=r'+\varepsilon} = C\left(r'^\ell - \frac{a^{2\ell+1}}{(r')^{\ell+1}}\right)\left(-\frac{\ell+1}{r^{\ell+2}} - \frac{\ell r^{\ell-1}}{b^{2\ell+1}}\right)\Bigg|_{r=r'}$$

$$\frac{d}{dr}(g_\ell)\Big|_{r=r'-\varepsilon} = \frac{d}{dr}\left[c\left(r^\ell - \frac{a^{2\ell+1}}{r^{\ell+1}}\right)\left(\frac{1}{r'^{\ell+1}} - \frac{r'^\ell}{b^{2\ell+1}}\right)\right]\Big|_{r=r'-\varepsilon} = C\left(\ell r^{\ell-1} + \frac{a^{2\ell+1}(\ell+1)}{r^{\ell+2}}\right)\left(\frac{1}{(r')^{\ell+1}} - \frac{r'^\ell}{b^{2\ell+1}}\right)\Big|_{r=r'-\varepsilon}$$

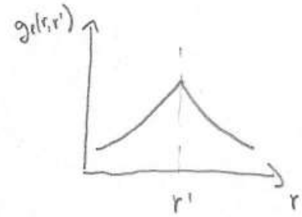
$$\Rightarrow \quad \frac{d}{dr} g_\ell \Big|_{r'-\varepsilon}^{r'+\varepsilon} = -(2\ell+1)\left(1 - \left(\frac{a}{b}\right)^{2\ell+1}\right)\frac{C}{r'^2} = -\frac{4\pi}{r'^2}$$

so that $$C = \frac{4\pi}{(2\ell+1)\left(1-\left(\frac{a}{b}\right)^{2\ell+1}\right)}$$

↳ finally this gives a Green function for a spherical shell bounded by $r=a$ and $r=b$

$$G(\vec{x},\vec{x}') = 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)}{(2\ell+1)\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]} \left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}}\right)$$

note: – the radial Green function ~~thus~~ is continuous in r, but it's derivative is not



the derivative changes at $r=r'$

3.10 POTENTIAL PROBLEMS AND SPHERICAL GREEN FUNCTION EXPANSION

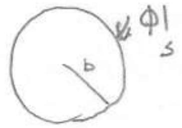
↳ including charge distributions, the solution to Poisson eq.

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}$$

– for Dirichlet b.c. the solution is

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x' \; - \frac{1}{4\pi}\oint_S \phi(\vec{x}')\,\frac{\partial G}{\partial n'}\, da' \qquad (*)$$

↳ an example: potential <u>inside</u> a sphere of radius b



we can use that:

– any function of θ,φ can be expanded in spherical harmonics

$$g(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_{lm} Y_{lm}(\theta,\varphi)$$

– a solution for boundary value problem (no free charges)

$$\phi(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[A_{lm} r^{l} + B_{lm} r^{-(l+1)}\right] Y_{lm}(\theta,\varphi)$$

↳ we want to show that above solution for $\phi(r,\theta,\varphi)$ is the same as $(*)$ when $G(\vec{x},\vec{x}')$ is expanded in spherical harmonics

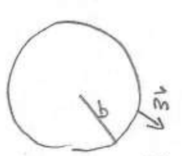
– using the form of Green's function for spherical shell $r=a$, $r=b$

$$G(\vec{x},\vec{x}') = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{Y^{*}_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)}{(2l+1)\left[1-\left(\frac{a}{b}\right)^{2l+1}\right]}\left(r_<^{l} - \frac{a^{2l+1}}{r_<^{l+1}}\right)\left(\frac{1}{r_>^{l+1}} - \frac{r_>^{l}}{b^{2l+1}}\right)$$

can set $a=0$

– normal derivative

$$\frac{\partial G}{\partial n'} = \left.\frac{\partial G}{\partial r'}\right|_{r'=b} = -\frac{4\pi}{b^2}\sum_{l,m}\left(\frac{r}{b}\right)^{l} Y_{lm}(\theta',\varphi')\,Y_{lm}(\theta,\varphi)$$



$\frac{\partial G}{\partial n}$ is derivative along $\vec{n}$ (oriented outward from the volume)

$\Rightarrow$ the solution of the Laplace eq. $\nabla^2\phi = 0$ inside the sphere

$$\phi(\vec{x}) = \sum_{lm} \left[\int d\Omega' \, V(\theta',\varphi') Y^*_{lm}(\theta',\varphi') \right] \left(\frac{r}{b}\right)^l Y_{lm}(\theta,\varphi)$$

this is exactly A_{lm} on the previous side

example: ring of charge radius a and total charge Q inside a grounded, conducting sphere of radius b

ring in xy plane

$$\rho(\vec{x}') = \frac{Q}{2\pi a^2} \delta(r'-a)\,\delta(\cos\theta')$$

- can not separated Green function $(a=0)$

$$G(\vec{x},\vec{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi)}{(2l+1)} r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right)$$

in Dirichlet b.c. solution

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') G(\vec{x},\vec{x}') d^3\vec{x}' - \frac{1}{4\pi} \oint_S \underset{0}{\phi(\vec{x}')} \frac{\partial G}{\partial n'} da' =$$

$$= \frac{1}{\varepsilon_0} \int \frac{Q}{2\pi a^2} \delta(r'-a)\,\delta(\cos\theta')\, r'^2 dr' \, d\cos\theta' \, \underset{\text{only } m=0 \text{ survives}}{d\varphi'} \sum_{ml} \frac{Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi)}{(2l+1)} r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right) =$$

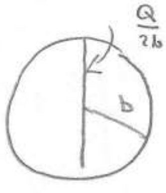
$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

$$\left[\phi(\vec{x}) = \frac{Q}{4\pi\varepsilon_0} \sum_{l=0}^{\infty} P_l(0) P_l(\cos\theta) \, r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right) \right] \Leftarrow \quad \begin{array}{l} r_< = \min\{a, r\} \\ r_> = \max\{a, r\} \end{array}$$

- note: $P_{2m+1}(0) = 0 \qquad P_{2m}(0) = \frac{(-1)^m (2m-1)!!}{2^m \, m!}$

example: uniform line charge of length $2b$ and total charge Q inside a grounded, conducting sphere of radius b

$\frac{Q}{2b}$ charge density (linear)



$$\rho(\vec{x}') = \frac{Q}{2b} \frac{1}{2\pi r'^2} \left[\delta(\cos\theta' - 1) + \delta(\cos\theta' + 1) \right]$$

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') G(\vec{x},\vec{x}') d^3\vec{x}' = \frac{Q}{8\pi\varepsilon_0 b} \sum_{l=0}^{\infty} \left[P_l(1) + P_l(-1) \right] P_l(\cos\theta) \underbrace{\int_0^b r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right) dr'}_{\mathrm{I}}$$

$$I = \int_0^r r'^{l}\left(\frac{1}{r^{2l+1}} - \frac{r^l}{b^{2l+1}}\right)dr' + \int_r^b r^l\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right)dr' = \frac{2l+1}{l(l+1)}\left[1-\left(\frac{r}{b}\right)^l\right]$$

for $l=0$ we need to take

$$\lim_{l\to 0} I = \lim_{l\to 0}\frac{1-\left(\frac{r}{b}\right)^l}{l} = \lim_{l\to 0}\frac{\frac{d}{dl}\left(1-\left(\frac{r}{b}\right)^l\right)}{\frac{d}{dl}\,l} = \ln\left(\frac{b}{r}\right)$$

also use $P_l(-1) = (-1)^l$

$$\Rightarrow \quad \phi(\vec{x}) = \frac{Q}{4\pi\varepsilon_0 b}\left\{\ln\left(\frac{b}{r}\right) + \sum_{j=1}^{\infty}\frac{4j+1}{2j(2j+1)}\left(1-\left(\frac{r}{b}\right)^{2j}\right)P_{2j}(\cos\theta)\right\}$$

note: the series diverges for $\cos\theta = \pm 1$ (i.e. along z-axis)

↳ the surface charge density on the grounded sphere

$$\sigma(\theta) = \varepsilon_0 \frac{\partial\phi}{\partial r}\bigg|_{r=b} = -\frac{Q}{4\pi b^2}\left[1 + \sum_{j=1}^{\infty}\frac{(4j+1)}{(2j+1)}P_{2j}(\cos\theta)\right]$$

3.11 EXPANSION OF GREEN FUNCTIONS IN CYLINDRICAL COORDINATES

Green function satisfies

$$\nabla_x^2\, G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$$

– in cylindrical coordinates:

$$\nabla_x^2\, G(\vec{x},\vec{x}') = -\frac{4\pi}{\rho}\,\delta(\rho-\rho')\,\delta(\varphi-\varphi')\,\delta(z-z') \qquad (*)$$

↳ the φ and z delta functions we expand as

$$\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')} = \frac{1}{\pi}\int_0^{\infty} dk\,\cos(k(z-z'))$$

← here we assume z is on infinite interval

$$\delta(\varphi-\varphi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')}$$

↳ we similarly expand the Green function

$$G(\vec{x},\vec{x}') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, e^{im(\varphi-\varphi')}\cos\left[k(z-z')\right] g_m(k,\rho,\rho')$$

from $(*)$ one then obtains the equation for radial Green function

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg_m}{d\rho}\right) - \left(k^2+\frac{m^2}{\rho^2}\right)g_m = -\frac{4\pi}{\rho}\,\delta(\rho-\rho')$$

↳ if $\varrho \neq \varrho'$ the above is equation for modified Bessel functions

$$\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)R = 0$$

⇒ solutions $I_\nu(x)$, $K_\nu(x)$

– so in our case the solutions are $I_m(k\varrho)$, $K_m(k\varrho)$

↳ we need to split the solutions for $\varrho > \varrho'$ and $\varrho < \varrho'$

$$\psi_1(k\varrho) = a_1 I_m(k\varrho) + b_1 K_m(k\varrho)$$

solution that satisfies b.c. for $\varrho < \varrho'$

$$\psi_2(k\varrho) = a_2 I_m(k\varrho) + b_2 K_m(k\varrho)$$

–II– $\varrho > \varrho'$

⇒ the symmetry of Green funct. under $\varrho \leftrightarrow \varrho'$ requires

$$g_m(k,\varrho,\varrho') = \psi_1(k\varrho_<)\,\psi_2(k\varrho_>)$$

↳ the normalization of g_m follows from discontinuity of the g_m derivative due to δ function

$$\underbrace{\left.\frac{dg_m}{d\varrho}\right|_{\varrho=\varrho'+\varepsilon} - \left.\frac{dg_m}{d\varrho}\right|_{\varrho=\varrho'-\varepsilon}}_{} = -\frac{4\pi}{\varrho'}$$

$$= k(\psi_1\psi_2' - \psi_2\psi_1') = k W[\psi_1,\psi_2] \leftarrow$$ Wronskian of ψ_1 and ψ_2

$$W[\psi_1,\psi_2] = \det\begin{pmatrix} \psi_1 & \psi_2 \\ \psi_1' & \psi_2' \end{pmatrix}$$

↳ note that the equation for radial Green function

$$\frac{1}{\varrho}\frac{d}{d\varrho}\left(\varrho\frac{dg_m}{d\varrho}\right) - \left(k^2 + \frac{m^2}{\varrho^2}\right)g_m = -\frac{4\pi}{\varrho}\delta(\varrho-\varrho')$$

this is of Sturm – Liouville type (apart from δ function)

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = 0$$

these types of equations satisfy the following property: Taking two lin. indep. solutions ψ_1, ψ_2

⇒ the Wronskian is $W[\psi_1,\psi_2] \propto \frac{1}{p(x)}$

↳ in our case we need to demand normalization

$$W[\psi_1(x),\psi_2(x)] = -\frac{4\pi}{x} \quad ; \quad x = k\varrho$$

↳ Example: consider a case of no boundary surfaces

$g_m(k,\varrho,\varrho')$ need to be finite for $\varrho \to 0$

vanish for $\varrho \to \infty$

the modified Bessel functions satisfy

$$x \ll 1 \qquad I_\nu(x) \to \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu$$

$$K_\nu(x) \to \begin{cases} -\left(\ln\left(\frac{x}{2}\right) + \gamma_E\right) & \nu = 0 \\ \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu & \nu \neq 0 \end{cases}$$

$$x \gg 1, \nu \qquad I_\nu(x) \to \frac{1}{\sqrt{2\pi x}}\, e^{x}\left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right]$$

$$K_\nu(x) \to \frac{1}{\sqrt{2\pi x}}\, e^{-x}\left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right]$$

$$\Rightarrow \quad \psi_1(k\rho) = A I_m(k\rho)$$
$$\psi_2(k\rho) = K_m(k\rho)$$

↳ we can use the expanded version of $I_m(x)$, $K_m(x)$ for $x \ll 1$ to find

$$W[I_m(x), K_m(x)] = -\frac{1}{x}$$

$\Rightarrow$ $A = 4\pi$ is the proper normalization, so $g_m = 4\pi\, I_m(k\rho_<)\, K_m(k\rho_>)$

↳ the Green function for this case is simply $G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$ and thus

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\, e^{im(\varphi-\varphi')}\cos[k(z-z')]\; I_m(k\rho_<)\, K_m(k\rho_>)$$

or in terms of real functions

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{4}{\pi}\int_0^\infty dk\, \cos[k(z-z')]\left\{\frac{1}{2} I_0(k\rho_<)K_0(k\rho_>) + \sum_{m=1}^{\infty}\cos[m(\varphi-\varphi')]\, I_m(k\rho_<)\, K_m(k\rho_>)\right\}$$

Taking $\vec{x}' \to 0$ gives

$$\frac{1}{\sqrt{\rho^2+z^2}} = \frac{2}{\pi}\int_0^\infty \cos kz\; K_0(k\rho)\, dk$$

↳ one can also obtain (see Jackson for details, p. 126) the expansion of Green function for 2D polar coordinates

$$\ln\left(\frac{1}{\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')}\right) = 2\ln\left(\frac{1}{\rho_>}\right) + 2\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\rho_<}{\rho_>}\right)^m\cos[m(\varphi-\varphi')]$$

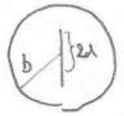
Problem 3.14 : A line charge of length 2d with a total charge Q has a linear charge density varying as (d^2-z^2), where z is the distance from the midpoint. A grounded, conducting, spherical shell of inner radius $b>d$ is centered at the midpoint of the line charge.

a) Find the potential everywhere inside the spherical shell as an expansion in Legendre polynomials

b) Calculate the surface charge density induced on the shell

c) Discuss your answers to part a) and b) in the limit that $d \ll b$.

Solution



using Green function

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}')\, G_D(\vec{x},\vec{x}')\, d^3\vec{x}' - \frac{1}{4\pi}\oint_S \underbrace{\Phi}_{0 \text{ on } S} \frac{\partial G}{\partial n'}\, da'$$

charge density

$$\rho(\vec{x}) = A\,\underbrace{(d^2-z^2)}_{\text{same as saying } (d^2-r^2)}\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]\cdot \underbrace{\frac{1}{r^2}}_{\text{since we are writing } \delta \text{ function in spherical coordinates}}\ \overbrace{H(d-r)}^{\text{step function}}$$

$$\int \rho(\vec{x})\, d^3\vec{x} = Q = A\int r^2 dr\, d\cos\theta\, d\varphi\ A\,\frac{(d^2-r^2)}{r^2}\, H(d-r)\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right] =$$

$$= A\,4\pi\int_0^d (d^2-r^2)\,dr = A\,4\pi\left(d^3-\frac{d^3}{3}\right) = A\,\frac{8\pi d^3}{3}$$

$$\Rightarrow \rho(\vec{x}) = \frac{3Q}{8\pi d^3}\,\frac{(d^2-r^2)}{r^2}\, H(d-r)\left[\delta(\cos\theta-1)+\delta(\cos\theta+1)\right]$$

↳ the Green function expanded in spherical harmonics (p.71 $a \to 0$)

$$G_D = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{b^{2l+1}}\right) Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)$$

– can use it in the solution for $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}')\, G_D(\vec{x},\vec{x}')\, d^3\vec{x}'$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \underbrace{\rho(\vec{x}')}_{\varphi \text{ indep.}}\, 4\pi\sum_{l,m}\frac{1}{(2l+1)}\left(\frac{r_<^l}{r_>^{l+1}} - \frac{(rr')^l}{b^{2l+1}}\right) Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)\, r'^2 dr'\, d\varphi'\, d\cos\theta'$$

φ indep. $\Rightarrow$ only $m=0$ is left

$$Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$$

77)

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \underset{\uparrow}{\rho(\vec{x}')}\; 2\pi \sum_\ell \left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{(rr')^\ell}{b^{2\ell+1}} \right) P_\ell(\cos\theta') P_\ell(\cos\theta)\, r'^2 dr'\, d\cos\theta' =$$

$$\frac{3Q}{8\pi d^3} \frac{d^2-r'^2}{r'^2} H(d-r) \left[\delta(\cos\theta-1) + \delta(\cos\theta+1)\right]$$

$$= \frac{1}{2\varepsilon_0} \frac{3Q}{8\pi d^3} \int_0^d dr' \,(d^2-r'^2) \sum_\ell \left(\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{(rr')^\ell}{b^{2\ell+1}} \right) \left[\underset{\substack{\uparrow \\ 1}}{P_\ell(1)} + \underset{\substack{\uparrow \\ (-1)^\ell}}{P_\ell(-1)} \right]$$

$$\int_0^d dr' \,(d^2-r'^2)\, r'^\ell = d^2 \frac{d^{\ell+1}}{\ell+1} - \frac{d^{\ell+3}}{\ell+3} = \frac{2}{(\ell+1)(\ell+3)} d^{\ell+3}$$

$$\int_0^d dr' \,(d^2-r'^2) \frac{r_<^\ell}{r_>^{\ell+1}} \begin{cases} \overset{r>d}{=} \dfrac{1}{r^{\ell+1}} \displaystyle\int_0^d dr' \,(d^2-r'^2)\, r'^\ell = \frac{1}{r^{\ell+1}} \frac{2}{(\ell+1)(\ell+3)} d^{\ell+3} \\[2ex] \overset{r<d}{=} \displaystyle\int_0^r dr' \,(d^2-r'^2) \frac{r'^\ell}{r^{\ell+1}} + \int_r^d dr' \,(d^2-r'^2) \frac{r^\ell}{r'^{\ell+1}} = \end{cases}$$

$$= \frac{d^2}{\ell+1} - \frac{r^2}{3+\ell} + \frac{d^2}{\ell-2} - \frac{2d^2}{\ell(\ell-2)} + \frac{r^2}{\ell-2} - \frac{2d^{2-\ell}r^\ell}{\ell(\ell-2)} =$$

$$= r^2 \left(-\frac{2\ell+1}{(\ell+3)(\ell-2)} + \frac{2}{\ell(\ell-2)} \left(\frac{d}{r}\right)^{2-\ell} + \frac{2\ell+1}{\ell(\ell+1)} \left(\frac{d}{r}\right)^2 \right)$$

so for $r>d$ we get

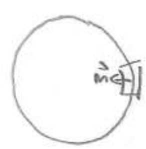
$$\phi(\vec{x}) = \frac{3Q}{4\pi\varepsilon_0} \sum_{\ell=\text{even}} P_\ell(\cos\theta) \frac{d^\ell}{(\ell+1)(\ell+3)\, b^{\ell+1}} \left[\left(\frac{b}{r}\right)^{\ell+1} - \left(\frac{r}{b}\right)^\ell \right]$$

for $r<d$ we get

$$\phi(\vec{x}) = \frac{3Q}{8\pi\varepsilon_0 d^3} \sum_{\ell=\text{even}} P_\ell(\cos\theta) \left[d^2 \frac{(2\ell+1)}{\ell(\ell+1)} + r^2 \frac{(2\ell+1)}{(2-\ell)(3+\ell)} + r^\ell \left(\frac{2}{\ell(\ell-2)} \frac{1}{d^{\ell-2}} - \frac{2}{(\ell+3)(\ell-2)} \frac{d^{\ell+3}}{b^{2\ell+1}} \right) \right]$$

b) the surface charge density on the shell

$$\sigma = -\varepsilon_0 \frac{\partial\phi}{\partial n}\Big|_S$$



$\vec{n}$ in this case points inward

$$\Rightarrow \sigma = \varepsilon_0 \frac{\partial\phi}{\partial r}\Big|_{r=b} \overset{\substack{\text{since } b>d \\ \downarrow \\ r>d}}{=} \varepsilon_0 \frac{3Q}{4\pi\varepsilon_0} \frac{\partial}{\partial r} \sum_{\ell=\text{even}} P_\ell(\cos\theta) \frac{d^\ell}{(\ell+1)(\ell+3)\, b^{\ell+1}} \left[\left(\frac{b}{r}\right)^{\ell+1} - \left(\frac{r}{b}\right)^\ell \right] \Bigg|_{r=b} =$$

$$= -\frac{3Q}{4\pi b^2} \sum_{\ell=\text{even}} P_\ell(\cos\theta) \frac{2\ell+1}{(\ell+1)(\ell+3)} \left(\frac{d}{b}\right)^\ell$$

c) take $d \ll b$

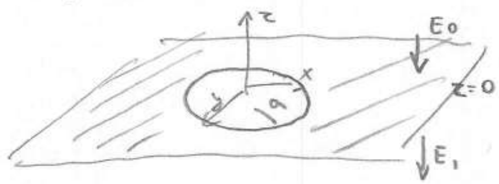
$$\phi(x) \to \underset{\substack{\uparrow \\ \text{comes from } \ell=0}}{\frac{Q}{4\pi\varepsilon_0} \left(\frac{1}{r} - \frac{1}{b}\right)} \quad \Leftarrow \text{equals } \phi \text{ from point charge in the center of the sphere}$$

$$\Rightarrow \sigma = -\frac{Q}{4\pi b^2}$$ (charge $-Q$ spread over the surface of the sphere uniformly)

3.13 MIXED BOUNDARY CONDITIONS

mixed boundary conditions more difficult to deal with

example: infinitely thin, grounded, conducting plane, with a circular hole of radius a cut in it



electric field far from the hole is normal to the plane, different values above and below the plane:

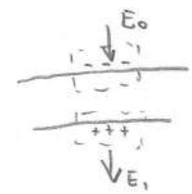
$$E_z = -E_0 \quad z>0 \qquad \text{for } r \gg a$$
$$E_z = -E_1 \quad z<0$$

⇒ the potential we can write as

$$\Phi = \begin{cases} E_0 z + \phi^{(1)} & ; \; z>0 \\ E_1 z + \phi^{(1)} & ; \; z<0 \end{cases}$$

↳ if the hole were not there:

- $\phi^{(1)} = 0$
- the top surface of the sheet: $\sigma = -\varepsilon_0 E_0$
- the bottom -11- $\sigma = \varepsilon_0 E_1$



⇒ $\phi^{(1)}$ describes the potential due to rearrangement of the surface charge because of the hole

- we can write

$$\phi^{(1)}(x,y,z) = \frac{1}{4\pi\varepsilon_0} \int \frac{\sigma^{(1)}(x',y')\, dx'dy'}{\sqrt{(x-x')^2+(y-y')^2+z^2}}$$

where $\sigma^{(1)}(x,y)$ is the rearranged charge surface density

⇒ note: $\phi^{(1)}(x,y,z)$ is even in z

$$\Rightarrow \quad E_x^{(1)} = -\frac{\partial\phi^{(1)}}{\partial x} \quad \text{even in } z$$

$$E_y^{(1)} = -\frac{\partial\phi^{(1)}}{\partial y} \quad \text{-11-}$$

$$E_z^{(1)} = -\frac{\partial\phi^{(1)}}{\partial z} \quad \text{odd in } z$$

, vanishes at z=0 (but the total E_z does not, since

$$E_z = E_0 + E_z^{(1)} \quad z>0$$
$$E_z = E_1 + E_z^{(1)} \quad z<0$$

note: $E_z^{(1)}$ is odd in z but does not vanish for z=0 (it is discontinuous)

- total E_z needs to be continuous in the hole at z=0

– so for $z=0$ and $\rho<a$

$$-E_0 + E_z^{(1)}\Big|_{z=0^+} = -E_1 + E_z^{(1)}\Big|_{z=0^-}$$

$$E_z^{(1)} \text{ is odd in } z \Rightarrow E_z^{(1)}\Big|_{z=0^+} = -E_z^{(1)}\Big|_{0^-}$$

↳ we thus have $$E_z^{(1)}\Big|_{z=0^+} = -E_z^{(1)}\Big|_{z=0^-} = \frac{1}{2}(E_0 - E_1)$$ for $0\leq\rho<a$

↳ the other b.c. : for $a\leq\rho<\infty$:

$$\phi\Big|_{z=0} = 0 \quad \Rightarrow \quad \phi = \begin{cases} \overset{0}{\overbrace{E_0 z}} + \phi^{(1)} \\ \underset{0}{\underbrace{E_1 z}} + \phi^{(1)} \end{cases} \quad \Rightarrow \quad \phi^{(1)}\Big|_{z=0} = 0$$

↳ mixed boundary conditions

$$\frac{\partial\phi^{(1)}}{\partial z}\Big|_{z=0^+} = -\frac{1}{2}(E_0 - E_1) \qquad 0\leq\rho<a$$

$$\phi^{(1)}\Big|_{z=0} = 0 \qquad a\leq\rho$$

↳ there is azimuthal symmetry, in the expansion

$$\phi(\rho,\varphi,z) = \sum_{m=0}^{\infty}\int_0^{\infty} dk\, e^{-kz} J_m(k\rho)\left[A_m(k)\sin m\varphi + B_m(k)\cos m\varphi\right]$$

only $m=0$ survives

$$\phi^{(1)}(\rho,z) = \int_0^{\infty} dk\, J_0(k\rho)\, e^{-k|z|} A(k) \qquad (\phi^{(1)} \text{ has } z\to -z \text{ symmetry})$$

↳ asymptotic behaviour of ~~the~~ $A(k)$:

– for large ρ $\quad J_0(k\rho)$ oscillates rapidly
– for large z $\quad e^{-kz}$ small
$\Big\}$ ⇒ in these two limits only $k=0$ important

↳ assume that $A(k)$ can be Taylor expanded around $k=0$

$$A(k) = \sum_{l=0}^{\infty}\frac{k^l}{l!}\frac{d^l A(k)}{dk^l}\Big|_{k=0}$$

⇒ we can thus write

$$\phi^{(1)}(\rho,z) = \sum_{l=0}^{\infty}\frac{d^l A(k)}{dk^l}\Big|_{k=0} B_l(\rho,z) \quad ; \text{ where } B_l(\rho,z) = \frac{1}{l!}\int_0^{\infty} dk\, k^l e^{-k|z|} J_0(k\rho)$$

from here:

$$B_\ell = \frac{1}{\ell!}\left(-\frac{d}{d|z|}\right)^\ell \int_0^\infty dk\, e^{-k|z|} J_0(k\rho) = \frac{1}{\ell!}\left(-\frac{d}{d|z|}\right)^\ell \frac{1}{\sqrt{\rho^2+z^2}} = \frac{P_\ell(|\cos\theta|)}{r^{\ell+1}}$$

where $\cos\theta = \frac{z}{r}$, $r = \sqrt{\rho^2+z^2}$

$$\Rightarrow \left[\ \Phi^{(1)} = \sum_{\ell=0}^\infty \left.\frac{d^\ell A}{dk^\ell}\right|_{k=0} \frac{P_\ell(|\cos\theta|)}{r^{\ell+1}}\ \right]$$

↳ this is a multipole expansion

- $A(0)$ is the total charge
- $\frac{d^1 A}{dk}(0)$ is the dipole moment in the z-direction

↳ obtaining $A(k)$

- the two b.c. are

$$\int_0^\infty dk\, A(k)\, k\, J_0(k\rho) = \frac{1}{2}(E_0 - E_1) \qquad \text{for } 0 \le \rho < a$$

$$\int_0^\infty dk\, A(k)\, J_0(k\rho) = 0 \qquad \text{for } a \le \rho$$

↳ dual integral equations ← general theory of solutions not well developed

one can find a solution

$$A(k) = \frac{E_0 - E_1}{\pi}\left(\frac{\sin ka}{k^2} - \frac{a\cos ka}{k}\right)$$

for small k

$$A(k) \simeq \frac{(E_0 - E_1)}{3\pi} a^2 \left(ka - \frac{(ka)^3}{10} + \dots\right)$$

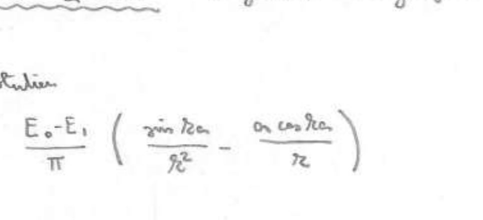
so for large r the limiting behaviour

$$\Phi^{(1)} \Rightarrow \frac{(E_0 - E_1)a^3}{3\pi}\frac{|z|}{r^3}$$

is a scalar potential for an effective dipole moment $\vec{p} = \pm \frac{4\varepsilon_0}{3}(\vec{E}_0 - \vec{E}_1)a^3$ (+ for $z>0$, − for $z<0$)

The effective dipole changes sign when observing from above, below (ϕ is z even, while dipole is z odd)

e.g. $E_1 = 0$:



dipole has origin from field penetrating through the hole