ON THE FINITENESS OF THE BROCARD–RAMANUJAN DIOPHANTINE EQUATION

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Abstract

Through the transformation of the Brocard-Ramanujan Diophantine Equation (referred to further as Brocard's Problem) with the use of the Pythagorean Theorem, I explain and prove the finiteness and solutions to Brocard's Problem.

Introduction

The Brocard's Problem was conjectured by Henri Brocard in 1876 and 1885, as well as by Ramanujan in 1913. The Brocard Problem asks to find the integer values for *n* which make the following equation result in an integer for *m*.

$$n! + 1 = m^2$$

The known values for **(n,m)** are known as the Brown Numbers and they are...

- (4,5)
- (5,11)
- (7,71)

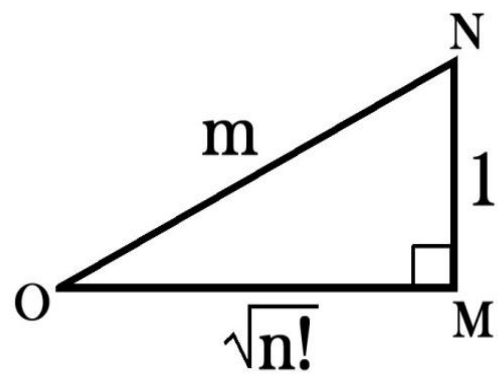
This challenge required a transformation of the equation into a form that would reveal a unique and critical property of the equation that would allow it to fall to logic.

Transformation

To begin the proof, I took the original equation and transformed it using the Pythagorean Theorem.

$$(\sqrt{n!})^2 + (1)^2 = (m)^2$$

This coupled with the following diagram allowed me to use the Law of Sines to transform the equation even further.



$$\frac{\sqrt{n!}}{\sin N} = \frac{1}{\sin O} = m$$

With this transformation and slight modifications with squares, the following equations result.

$$\frac{n!}{sin^2\, N} = n! + 1 = m^2$$

This can be equated to,

$$sin^2 N = \frac{n!}{n! + 1}$$

This results in the following transformation for *N*.

$$tan^{-1}\left(\sqrt{n!}\right) = N = sin^{-1}\sqrt{\frac{n!}{n! + 1}}$$

So the final form of the equation required for the proof is,

$$\frac{\sqrt{n!}}{\sin\left(tan^{-1}(\sqrt{n!})\right)} = m$$

From this point onward, we will label the sine-arctanget function as,

$$\sin(\tan^{-1}(\sqrt{n!})) = O(\sqrt{n!})$$

<u>Proof of Finality</u>

First we must make a point that the square root of a factorial is ALWAYS irrational (via quadratic irrationals); because we know that the square root of a factorial *n* can never be equal to *m*, which is always an integer. Or in equation form...

$$\sqrt{n!} \neq m$$

We must also realize an interesting of the sine-arctangent function. The property of this function is that as *n* approaches infinity, the sine-arctangent function will approach one. The equation form of the preceding argument is

$$\lim_{n\to\infty} O(\sqrt{n!}) = 1$$

So as *n* grows larger, the sine-arctangent function approaches one. But, if *n* could reach infinity (which is necessary for non-finite equations) then it would be a complete contradiction of the original equation because

$$\lim_{n \to \infty} \sqrt{n!} = m\, O(\sqrt{n!})$$

But as *n* grows larger and the sine-arctangent approaches one, you inevitably arrive at the equation

$$\sqrt{n!} = m$$

Which is a contradiction of the first equation, therefore *n cannot reach infinity*. Why does this matter though? Well, if we assume that there are an infinite number of solutions; then that implies that you could explore *n* all the way to infinity and keep finding solutions. If, however, you cannot reach infinity; this implies that eventually you will come to a point where no matter how much farther you explore; you will never find a solution. Therefore, there MUST be a *FINITE* number of solutions.

Proof of Solutions

To uncover all of the solutions, you must remember that the square root of *n!* is irrational and that by dividing by the sine-arctangent function you arrive at either another irrational number or a whole number (*m*). What more as *n* arrives closer to infinity (which was shown as not possible in the previous section) and the sine-arctangent function approaches closer to one, the impact that the sine-arctangent function has on the square root of *n!* diminishes. This results in the following realization.

$$\lim_{n\to\infty} \frac{\sqrt{n!}}{O(\sqrt{n!})} = \sqrt{n!} \neq m$$

So as *n* grows larger, the potential for it to reach a whole number approaches zero. And in fact when *n* is greater than 7, the sine-arctangent only affects the decimal points and the impact of the sine-arctangent function on the square root of the *n!* will eventually have no impact at all and will never approach a whole number beyond *n*=7. *Therefore the only values for n that result in a whole number are n=4, 5, and 7.*