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Optimum Stirling engine geometry

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SUMMARY

This paper combines the author's work on mechanical efficiency of reciprocating engines with the classic Schmidt thermodynamic model for Stirling engines and revisits the problem of identifying optimal engine geometry. All previous optimizations using the Schmidt theory focused on obtaining a maximal specific *indicated* cyclic work. This does not necessarily produce the highest shaft output. Indeed, some optima based upon indicated work would yield engines that cannot run at all due to excessive intrinsic mechanical losses. The analysis presented in this paper shows how to optimize for *shaft* or *brake* work output. Specifically, it presents solutions to the problem of finding the piston-to-displacer swept volume ratio and phase angle which will give the maximum brake output for a given total swept volume, given temperature extremes, a given mean operating pressure, and a given engine mechanism effectiveness. The paper covers the split-cylinder or gamma-type Stirling in detail, serving as a model for similar analysis of the other Stirling engine configurations. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: Stirling engines; mechanical efficiency; Schmidt analysis; swept volume ratio; phase angle

INTRODUCTION

This paper treats the central problem which occurs to everyone who sets out to design a gamma Stirling engine for the first time. What should the ratio of the swept volume of the piston be to that of the displacer, and what should the phase angle be between them? In approaching these questions, a basic mathematical model is essential. It not only guides the formulation and refinement of the inquiry to a meaningful form, but also guides the interpretation of results whether they are experimental, computational, or analytic.

In an outstanding piece of mathematical analysis, Gustav Schmidt, over 130 years ago, developed a basic model of the Stirling engine and obtained a closed-form expression for its indicated cyclic work (Schmidt, 1871). It is an idealized model, but one which captures the essential features of the engine and their basic interplay. It provides a best case analysis, which at the very least can identify general Stirling engine characteristics and behaviour. Schmidt used it to quantify and interpret the performance potential of contemporary engines.

The form of the Schmidt indicated work formula is far too complex to reveal directly what the exact optimum geometry of a Stirling engine should be. In addition, the calculations required to obtain numerical approximations of the optima were too many and too long to do before the advent of the electronic computer. It was about 90 years after Schmidt's pioneering

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mathematical work that the optimization problem was first taken up (Finkelstein, 1960; Kirkley, 1962; Walker, 1962). The work focused on finding the maximum indicated cyclic work relative to total mass of the working fluid or to the total swept volume, and relative to a characteristic cycle pressure. Higher-order computer simulations were subsequently developed to model heat exchanger limitiations and various internal thermal and fluid flow losses (Martini, 1978). These second- and third-order programs could be used to analyse and optimize specific engine designs with respect to thermal efficiency or indicated power.

However, obtaining the maximum indicated output does not ensure getting the maximum shaft output. As has recently become clear, shaft work is not a simple multiple of indicated work (Senft, 1987). Not only do the characteristics of the mechanism determine output at the shaft, but so also do the shape of the cycle and the external or buffer space pressure. This paper initiates the combination of the author's general results on mechanical efficiency with the classical Schmidt analysis to help identify engine geometries which give optimum shaft performance. The scope of this paper is limited to the elemental case of the split-cylinder or gamma-type Stirling where the problem is most familiar and clear. The implications of the results are identified and compared with observation and experience. Treatment of other Stirling engine configurations will follow along similar lines.

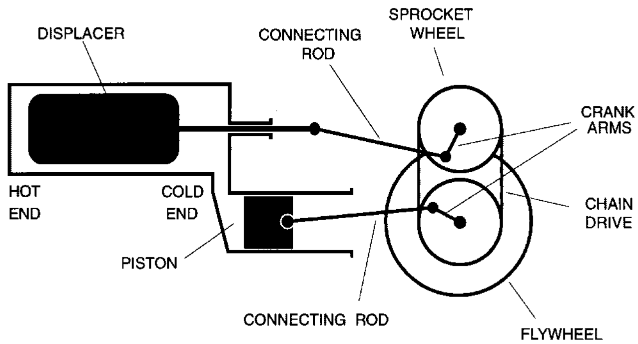
THE SCHMIDT MODEL FOR GAMMA ENGINES

Figure 1 is a schematic representation of the split-cylinder or gamma-type Stirling engine.

The gamma type engine can be described by the following parameters and variables:

V_1 = displacer swept volume
V_2 = piston swept volume
V_D = dead volume
T_H = hot space temperature
T_C = cold space temperature
T_D = dead space temperature = $(T_H + T_C)/2$
$\bar{p}$ = average cycle pressure = $\sqrt{p_{max}\, p_{min}}$
p_b = external buffer pressure = $\bar{p}$
ω = angular velocity of crankshaft
α = angle by which displacer crank leads piston crank
τ = T_C/T_H = cold-to-hot space temperature ratio
κ = V_2/V_1 = piston to displacer swept volume ratio
χ = V_D/V_1 = dead volume ratio
ωt = instantaneous angle of piston crank
V = instantaneous total engine volume
p = instantaneous pressure throughout engine spaces

In spite of the finite length connecting rods shown in the figure, a standard assumption of the Schmidt model is pure sinusoidal motion of the piston and displacer. This is taken as a reasonable and tractable approximation of the motions in most real engines. The other Schmidt assumptions include an ideal gas working fluid, isothermal hot, cold and dead spaces, uniform instantaneous pressure throughout the engine spaces, and no leakage of working gas into or out of the engine.



SPLIT STIRLING ENGINE

Figure 1. Diagram of the split-cylinder or gamma-type Stirling engine.

Convenient expressions for instantaneous total engine volume V and pressure p are the following:

$$V = \frac{V_{\mathrm{T}}}{\kappa+1}\left(1+\frac{\kappa}{2}(1+\cos(\omega t))+\chi\right)$$

$$p = \bar{p}\frac{\sqrt{Y^2-X^2}}{Y+X\cos(\omega t-\theta)} \quad \text{where} \quad \theta = \cos^{-1}\left(\frac{\kappa-(1-\tau)\cos\alpha}{X}\right) \tag{1}$$

$$X = \sqrt{\kappa^2-2\kappa(1-\tau)\cos\alpha+(1-\tau)^2} \quad \text{and} \quad Y = 1+\tau+\kappa+\frac{4\tau\chi}{1+\tau}$$

The appendix below outlines the derivation of the Schmidt formulas used in this paper. Figure 2 shows a p–V diagram of a gamma engine with some practical parameter values.

INDICATED CYCLIC WORK

A closed-form expression for the *indicated work* per cycle $W = \oint p\,\mathrm{d}V$ of a Schmidt gamma Stirling can be written in the following way:

$$W = \frac{V_{\mathrm{T}}\bar{p}}{\kappa+1}\frac{\pi(1-\tau)\kappa\sin\alpha}{\left(\sqrt{Y^2-X^2}+Y\right)} \tag{2}$$

where X and Y are complex functions of the parameters defined above (1). Note that the only factors having dimension in the expression for W are the combined swept volume of the piston and displacer, $V_{\mathrm{T}} = V_1 + V_2 = (\kappa+1)V_1$, and $\bar{p}$, the integral average cycle pressure (also equal to the root mean of the maximum and minimum cycle pressures). For simplicity in this treatment, it is assumed that all the dead space within the engine is at the arithmetic average of the extreme cycle temperatures; this is reflected in the last term of the parameter Y which involves the dead space ratio χ.

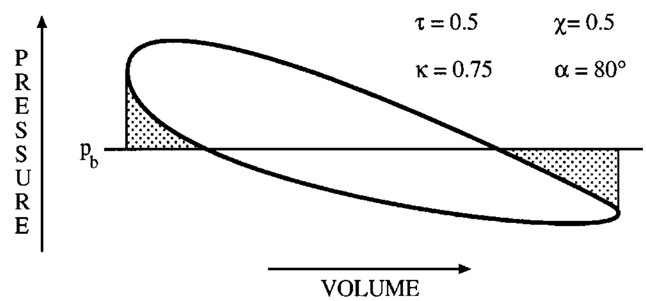


Figure 2. A pressure–volume diagram of a gamma-type Stirling engine.

SHAFT WORK FORMULA

How much useful work appears at the shaft of a reciprocating engine depends upon the forces applied to the mechanism by the piston on one side and by the flywheel on the other, and upon how well the mechanical section of the engine transfers the work done by these forces. Each transmission of mechanical energy through the mechanism is subject to loss due to friction. In a reciprocating engine, work is typically transmitted through the mechanism in both directions, from the piston to the shaft during some parts of the cycle, and from the flywheel to the piston in other parts of the cycle. The work transfers in both directions affect the mechanical efficiency of the engine over a complete cycle. A general invesitgation of mechanical efficiency can be found in the references (Senft, 1987, 1993, 1997). From this work, the following basic theorem emerged which is very useful in making broad performance comparisons:

Theorem
If the instantaneous effectiveness ε of an engine mechanism is bounded above by the constant E, that is if $\varepsilon \leqslant E$ throughout the cycle, then the cyclic shaft work of the engine is bounded as follows:

$$W_S \leqslant E\,W - \left(1/E - E\right)\,W_-$$

Equality holds for an engine having a mechanism with constant effectiveness E.

The quantity W_- is termed the *forced work* of the cycle. Its integral definition is the following:

$$W_- = \oint [(p - p_b)\,\mathrm{d}V]^- \tag{3}$$

It is work that must be done on the piston to carry out compressions when the workspace pressure is above the external buffer pressure, plus the work required by the piston to perform expansions when workspace pressure is below the buffer pressure. The forced work associated with the cycle and buffer pressure of Figure 2 is shown as the shaded area in the figure.

In keeping with the best case analysis afforded by the Schmidt thermodynamic model, it will be assumed in the following that the mechanism has a *constant effectiveness* throughout its range of operation. A constant effectiveness E means that whatever work is put into one side of the engine's mechanism, the fixed fraction E of that work comes out the other side (Senft, 1987, 1993, 1997). The upper bound of the theorem above then becomes the formula

$$W_S = E\,W - \left(1/E - E\right)\,W_- \tag{4}$$

where $0 < E \leqslant 1$ is the constant effectiveness of the engine mechanism, $W = \oint p\,dV$ is the indicated cyclic work and W_- is the forced work per cycle as defined by (3).

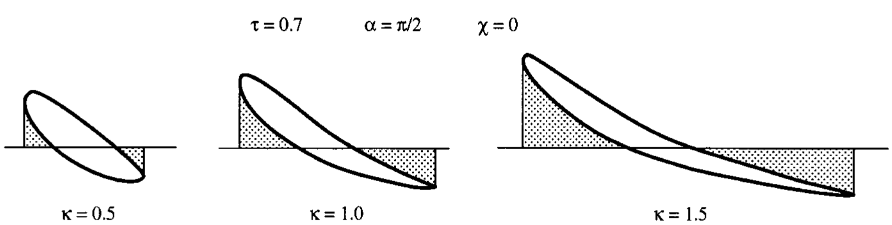
Most Stirling engines operate with the external or buffer pressure essentially equal to the mean workspace pressure. This is usually automatically established in engines having close-clearance-type piston and rod seals because of the very small but unavoidable leakage past these seals. After a relatively short period of operation, the net effect of the leakage is to more or less equalize the mean pressures of the work and buffer spaces. This situation is modelled in our analysis here by taking $p_b = \bar{p}$ throughout.

Actually, this is a fortunate circumstance, because it yields a high mechanical efficiency. It has been shown that the optimum constant buffer pressure with respect to mechanical efficiency for ideal Stirling engines is very nearly equal to the mean cycle pressure (Senft, 1991, 1997). For Schmidt cycles, the situation is similar, as the figures presented here indicate. In engines with cup seals, bellows, or diaphragms, a small intentional leak path between the spaces produces the same beneficial effect.

PARAMETER EFFECTS ON BRAKE WORK

The values of the parameters influence the indicated work W of the Schmidt cycle according to formula (2) given above. Basically, they determine the shape and size of the cycle and hence the area within. They also affect the shaft work output in a second way. The shape of the cycle and its position relative to the buffer pressure determines the forced work W_- of the cycle and formula (4) shows how both influence the shaft work output.

Shown in Figure 3 is a sequence of three Schmidt gamma engine cycles where the piston to displacer swept volume ratio $\kappa = V_2/V_1$ differs. In each of the cycles, the displacer swept volume V_1 and the mean cycle pressure $\bar{p}$ are the same which may be taken as unity. Each engine cycle



Indicated work, forced work, shaft work, and mechanical efficiency for three gamma engines with $\tau = 0.7$, $\alpha = \pi/2$, $\chi = 0$, and $E = 0.8$			
κ	0.5	1.0	1.5
W	.109	.182	.235
W_-	.022	.119	.273
W_S	.077	.092	.066
η_m	.71	.50	.28

Figure 3. The effect of changing the swept volume ratio on the pressure–volume cycle of a Schmidt gamma engine with a fixed displacer swept volume.

has the same the temperature ratio $\tau = 0.7$ and phase angle $\alpha = \pi/2$, and no dead space. The values of the swept volume ratio κ for the cycles vary through 0.5, 1.0 and 1.5.

The cycle diagrams are all drawn to the same scale. It is obvious that the indicated work W increases with κ. The table gives the exact values from which it is seen that the indicated work of the largest is over twice that of the smallest. Also shown is how quickly forced work, the shaded area, increases. The largest cycle has over ten times the forced work of the smallest, and it exceeds the indicated work that the cycle produces. The last two lines of the table show the shaft work output W_S and the mechanical efficiency $\eta_m = W_S/W$ of the cycles when coupled to a mechanism of constant effectiveness $E = 0.8$. The shaft work output is largest for the middle cycle, and lowest for the largest cycle. Mechanical efficiency is highest for the smallest cycle, and decreases as κ increases. This initial example makes clear how the simple mechanical model used here can guide the choice of a good overall engine geometry.

OPTIMUM SWEPT VOLUME AND PHASE ANGLE

When making comparisons of engine output, it is only fair to consider engines of the same size. A natural nominal measure of size for gamma engines is the total combined swept volumes of the piston and the displacer. This is the measure advocated in the earlier work on optimizing with respect to indicated output (Kirkley, 1962; Walker, 1962). The optimization problem then becomes that of partitioning the given total swept volume between the piston and displacer sections of the engine to maximize work output. The other parameter subject to optimization is the phase angle α by which the motion of the displacer leads that of the piston.

In this work, the mean cycle pressure $\bar{p}$ will be taken as a common basis when optimizing performance. This is the most natural and intuitive reference pressure against which to compare engines, especially the gamma type, and it is the easiest characteristic pressure to measure and control in practice, since it automatically tends to equalize to the buffer or surrounding atmospheric pressure as has been already noted.

Presented in Table I are the results of numerical calculations using Equations (1)–(4) for a range of values of temperature ratio τ, mechanism effectiveness E, and dead space ratio χ. The table shows the values κ^* and α^* that yield the maximum $\hat{W}_S^*$ specific shaft work $\hat{W}_S = W_S/(\bar{p}\,V_T)$ for the given values of τ, E, and χ. Also shown in the table are the corresponding specific values of $\hat{W} = W/(\bar{p}\,V_T)$ and $\hat{W}_- = W_-/(\bar{p}\,V_T)$, and the mechanical efficiency $\eta_m = W_S/W$ of the engine at the maximum specific shaft work $\hat{W}_S^*$ output point.

SWEPT VOLUME RATIOS

In applying this table, it is very important to keep in mind the interchange between mechanical efficiency and maximum shaft output. As can be seen in the table, at the maximum shaft output point, the mechanical efficiency η_m is most often significantly below the mechanism effectiveness E. A better mechanical efficiency can be obtained by taking a lower swept volume ratio than κ^* as the example of Figure 3 above indicated. In most cases, only a slightly smaller κ will considerably improve mechanical efficiency while resulting in only a relatively small reduction of shaft output. This is illustrated in Figure 4 which gives a graphical picture of how the swept

Table I. Swept volume ratio κ^* and phase angle α^* that yield the maximum specific shaft work $\hat{W}_S^*$ for given values of temperature ratio τ, mechanism effectiveness E and dead volume ratio χ are shown. Also shown are corresponding values of specific indicated $\hat{W}$ and forced work $\hat{W}_-$, and cyclic mechanical efficiency η_m at the maximum shaft work output point.

τ	χ	E	κ^*	α^*	$\hat{W}_S^*$	$\hat{W}$	$\hat{W}_-$	η_m
0.3	0	0.7	0.78	84°	0.157	0.245	0.0199	0.64
0.3	1	0.7	0.89	82	0.107	0.170	0.0166	0.63
0.3	0	0.8	0.90	88	0.189	0.254	0.0318	0.74
0.3	1	0.8	1.05	85	0.130	0.178	0.0271	0.73
0.3	0	0.9	1.06	91	0.224	0.261	0.0482	0.86
0.3	1	0.9	1.28	89	0.157	0.184	0.0450	0.85
0.4	0	0.7	0.74	83	0.123	0.195	0.0188	0.63
0.4	1	0.7	0.83	81	0.0788	0.128	0.0146	0.62
0.4	0	0.8	0.87	86	0.149	0.204	0.0302	0.73
0.4	1	0.8	1.00	84	0.0973	0.136	0.0250	0.72
0.4	0	0.9	1.06	90	0.179	0.211	0.0499	0.85
0.4	1	0.9	1.28	88	0.119	0.143	0.0450	0.83
0.5	0	0.7	0.68	81	0.0921	0.149	0.0167	0.62
0.5	1	0.7	0.75	80	0.0565	0.0938	0.0126	0.60
0.5	0	0.8	0.82	85	0.114	0.158	0.0292	0.72
0.5	1	0.8	0.93	84	0.0710	0.102	0.0233	0.70
0.5	0	0.9	1.03	89	0.139	0.166	0.0503	0.84
0.5	1	0.9	1.25	88	0.0893	0.110	0.0451	0.81
0.6	0	0.7	0.60	80	0.0649	0.108	0.0146	0.60
0.6	1	0.7	0.64	81	0.0382	0.0653	0.0103	0.59
0.6	0	0.8	0.73	84	0.0817	0.117	0.0259	0.70
0.6	1	0.8	0.83	83	0.0492	0.0729	0.0202	0.68
0.6	0	0.9	0.98	88	0.103	0.126	0.0503	0.82
0.6	1	0.9	1.16	87	0.0639	0.0808	0.0418	0.79
0.7	0	0.7	0.49	79	0.0411	0.0705	0.0113	0.58
0.7	1	0.7	0.53	79	0.0234	0.0415	0.00785	0.56
0.7	0	0.8	0.64	84	0.0532	0.0799	0.0237	0.67
0.7	1	0.8	0.69	83	0.0309	0.0477	0.0160	0.65
0.7	0	0.9	0.89	87	0.0696	0.0885	0.0476	0.79
0.7	1	0.9	1.03	87	0.0420	0.0554	0.0371	0.76
0.8	0	0.7	0.37	80	0.0212	0.0388	0.00824	0.55
0.8	1	0.7	0.37	78	0.0116	0.0211	0.00433	0.55
0.8	0	0.8	0.49	83	0.0286	0.0453	0.0171	0.63
0.8	1	0.8	0.54	82	0.0160	0.0266	0.0117	0.60
0.8	0	0.9	0.74	87	0.0399	0.0537	0.0403	0.74
0.8	1	0.9	0.83	86	0.0232	0.0325	0.0286	0.71
0.9	0	0.7	0.21	78	0.00639	0.0127	0.00340	0.50
0.9	1	0.7	0.21	78	0.00336	0.00666	0.00179	0.50
0.9	0	0.8	0.29	82	0.00923	0.0160	0.00800	0.58
0.9	1	0.8	0.30	82	0.00494	0.00878	0.00462	0.56
0.9	0	0.9	0.50	86	0.0145	0.0220	0.0250	0.66
0.9	1	0.9	0.53	86	0.00806	0.0126	0.0155	0.64

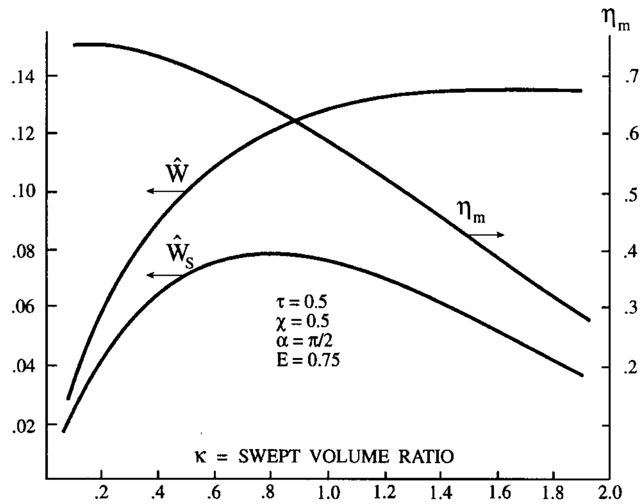


Figure 4. An example of the variation of specific indicated work $\hat{W}$, specific shaft work $\hat{W}_S$, and cyclic mechanical efficiency η_m with swept volume ratio κ for a gamma-type Schmidt Stirling engine.

volume ratio κ affects indicated work, shaft work, and mechanical efficiency in a specific case, this being an engine having the parameter values $\tau = 0.5$, $\chi = 0.5$, $E = 0.75$ and $\alpha = \pi/2$.

Although the peak specific shaft output occurs at a swept volume ratio κ between 0.7 and 0.8, there is not much reduction in shaft output with $\kappa = 0.6$ or even 0.5 and mechanical efficiency is much improved. At $\kappa = 0.8$, shaft work $\hat{W}_S = 0.077$ and mechanical efficiency $\eta_m = 0.64$, whereas at $\kappa = 0.5$, $\hat{W}_S = 0.071$ and $\eta_m = 0.71$. In this case, the small reduction in cyclic shaft work output would be well compensated for by the improved mechanical efficiency which translates to less wear and longer life.† On the other hand, it is seen that a value of κ larger than 0.8 would lower both output and mechanical efficiency. The general lesson for designers is that *it is always advisable to choose a swept volume ratio smaller than the shaft work optimum* shown in the table.

How much smaller is a matter of judicious compromise between W_S and η_m with due consideration of the range of temperature ratios at which the engine is expected to perform well. Note in Figure 4 that the work output curves drop off more and more rapidly as κ is taken farther below κ^*. A very small a choice for κ will give good mechanical efficiency, but output could be undesirably low. Thus the choice for κ should not be too much lower than κ^*. In a specific case, a plot of W_S should be made as in Figure 4, calculating from the equations given here, and from this a wise choice for κ can be made.

The analysis further shows that the point of maximum indicated work is quite far from the maximum shaft work point and is in fact on the unfavorable side of the brake output curve. In the case shown in Figure 4, the maximum indicated work $\hat{W} = 0.135$ occurs at about $\kappa = 1.7$ and $\alpha = 90°$. At this point, shaft work is a low $\hat{W}_S = 0.048$ and mechanical efficiency is a dismal $\eta_m = 0.35$.

† The parameters in this example are taken from the author's 10″ fan 'Moriya' (Senft, 1974) which was built with a swept volume ratio of 0.56.

As an extreme but real additional example of this, consider the design of a low temperature differential engine to operate on a temperature differential of $\tau = 0.9$ with, for simplicity, virtually no dead space, $\chi = 0$. The maximum indicated cyclic work per unit total swept volume and mean cycle pressure is $W = 0.0292$ and occurs at $\kappa = 1.6$ and $\alpha = 89^\circ$. At this point $W_- = 0.135$. If the mechanism effectiveness E is a fairly good 0.8, formula (4) shows that shaft work is negative:

$$W_S = E\,W - \left(1/E - E\right) W_- = (0.8)\,(0.0292) - (1/0.8 - 0.8)\,(0.135) = -0.0374$$

In other words, the engine could not run itself. Even if the mechanism effectiveness could be raised to 0.9, this engine still could not run itself. From Table I above, maximum shaft work for a Schmidt gamma engine with $\tau = 0.9$, $\chi = 0$ and $E = 0.8$ occurs for quite a different geometry, namely, $\kappa^* = 0.29$. This is a drastically lower value than $\kappa = 1.6$ for maximum indicated work, and by what was shown above, a slightly lower value still should be chosen to improve mechanical efficiency. This geometry is consistent with actual low temperature differential Stirling engine practice.

As already noted, the most natural and practical characteristic pressure to use when comparing gamma-type Stirling engines is the mean cycle pressure. However, the maximum cycle pressure appeared as another possible choice among some early researchers as a basis for comparison of indicated cyclic work. When this characteristic pressure is used in our brake work analysis, sample calculations show the optimum swept volume ratios tend to be lower than when mean cycle pressure is used. But even optimizing indicated cyclic work per unit maximum cycle pressure yields engine geometries which are not practical. In the above example engine with $\tau = 0.9$ and $\chi = 0$, the maximum indicated work per unit maximum cycle pressure occurs at $\kappa = 0.9$. This is still very much larger than that for the brake work optimum, $\kappa^* = 0.29$, and it still yields an engine which will not run coupled with a mechanism of effectiveness 0.8.

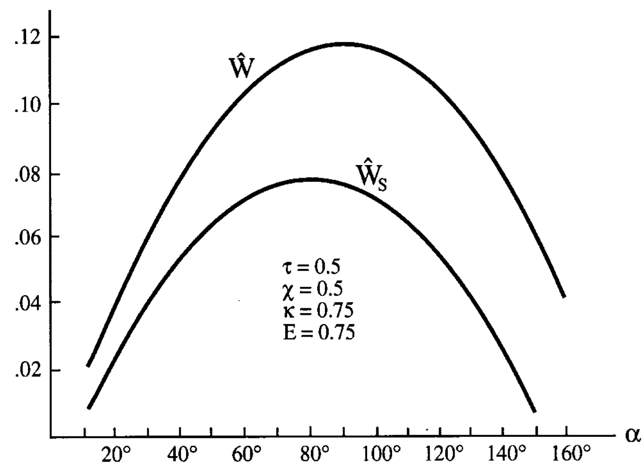
PHASE ANGLE

It should be noted from Table I that the optimum phase angle α^* tends to be lower for poorer mechanisms and for higher temperature ratios. While 90° is usually considered near enough to the optimum for practical purposes, the table shows values lower by more than 10° in some cases. Of course, in most gamma-type engines, the phase angle can be easily adjusted or modified to best suit operating conditions. On the other hand, engine performance is not extremely sensitive to phase angle. Figure 5 shows how the engine of Figure 4 responds to variation of its phase angle.

The swept volume ratio for this example was taken to be $\kappa = 0.75$ for which the optimum phase angle is near 80°. The table in the figure gives the exact values in a neighbourhood of this point and shows shaft work output is virtually the same 10° either side of the optimum. Note that the maximum indicated work in this example occurs at 90°.

DEAD SPACE

The negative effect of dead volume on the indicated work of the Stirling cycle is well known. The analysis presented in this paper clearly shows how shaft output is affected. For example, from



Dependence of gamma engine performance on phase angle α							
α	50°	60°	70°	80°	90°	100°	110°
$\hat{W}$	0.089	.101	.110	.116	.118	.117	.112
$\hat{W}_-$	0.008	.010	.012	.016	.020	.025	.031
$\hat{W}_s$	0.062	.070	.076	.078	.077	.073	.066
η_m	0.70	.69	.68	.67	.65	.63	.59

Figure 5. Variation of specific indicated work $\hat{W}$ and shaft work $\hat{W}_S$ with phase angle α for the gamma engine of Figure 4 with a swept volume ratio $\kappa = 0.75$.

Table I, a high-performance engine with $\tau = 0.3$ and $E = 0.8$ has a maximum specific shaft output of 0.189 with no dead space, and 0.130 with $\chi = 1$. This amount of dead volume reduces output potential by about 31%. It is important to keep this in mind when designing internal heat exchangers to improve Stirling engine performance. The increase in exchanger surface area must be large enough to more than offset in increased speed the decrease in cyclic work caused by the dead volume of the heat exchanger. The dead volume effect is greater for engines operating on smaller temperature differences as is seen for example from the table entries for $\tau = 0.8$ and $E = 0.8$. Here, maximum shaft work with no dead space is 0.0286, while $\chi = 1$ lowers it to 0.0160, a reduction of 44%.

Figure 6 shows how maximum specific shaft work varies with increasing dead space over the range from $\chi = 0$ to 10 for a specific engine with $\tau = 0.5$ and $E = 0.7$. Note that performance drops off at a high rate over the entire usual range of relative dead volume, only becoming what one might call gradual for dead volume ratios much higher than necessary in practice.

Another point to observe from Table I is that optimum swept volume ratio increases as relative dead volume is increased. However, the increase is quite small, and most importantly, dead volume effects on work output are not anywhere near being neutralized by increasing compression ratio as one might have surmised beforehand.

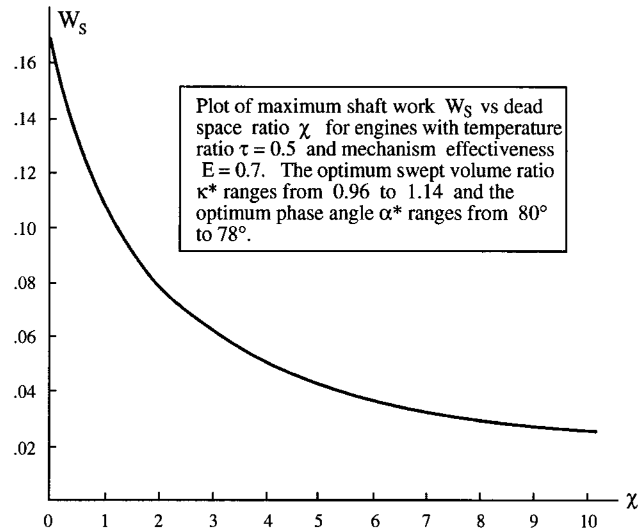


Figure 6. Variation of maximum specific shaft work with dead space ratio for a gamma Schmidt engine with $\tau = 0.5$ and $E = 0.7$.

INTERNAL TEMPERATURES

It is important to keep in mind when applying these results that they are derived from the assumption that the ratio τ describes the temperature extremes of the working gas. These temperatures will always be closer together than those of the metal surfaces of the heater and cooler sections of the engine because of heat transfer limitations. This makes the actual gas temperature ratio larger than the ratio of the cold to hot end surface temperatures. The temperature ratio discrepancy becomes more pronounced as engine speeds increase. This needs to be taken into account in the design of an engine. The table above shows that as the effective gas temperature ratio increases, the optimal swept volume ratios are lower. Once again the general advice that emerges from this analysis is to favor a lower swept volume ratio.

ALTERNATE ENGINE CONFIGURATIONS

The analysis presented here can be applied to the beta or single-cylinder Stirling engine configuration in much the same way as in the gamma case. The Schmidt equations need be slightly changed to reflect overlapping of the piston and displacer strokes. From there the indicated cyclic and forced works of the cycle can be calculated using Equation (3), after which the computation of shaft work and mechanical efficiency is exactly the same using Equation (4).

In the case of the alpha or two-piston configuration, the shaft work formula (4) must be applied to each piston, and then combined. The theoretical and general mathematical aspects involved in this, and in other more complex engine arrangements, have been worked out and reported in the literature (Senft, 2000).

FINAL POINTS

This paper shows for the first time how swept volume ratio and phase angle affect the shaft power output of a Schmidt gamma Stirling engine. The analysis is idealized of course, but idealized in such a way that the general behavior of engines shown by it will be reflected beneath all the additional losses and imperfections of actual machines.

Above all, this paper shows that optimum geometries are not only thermodynamically determined, but are also a heavily dependent function of the underlying effectiveness of the engine mechanism. Indicated work optimums do not occur at the same point as brake work optimums, and indeed, some indicated work optima yield engines that cannot run at all. Some of the more detailed findings that this paper reveals about Schmidt gamma Stirling engines can be listed as follows:

- Maximum shaft output occurs at smaller swept volume ratios than does maximum indicated work.
- Less effective mechanisms favour smaller swept volume ratios.
- Smaller swept volume ratios yield better mechanical efficiency.
- Low-temperature differential engines require small swept volume ratios.
- A high penalty in shaft output is incurred with the introduction of dead volume.
- Higher engine speeds favour lower swept volume ratios.
- Dead volume effects cannot be offset by increasing compression ratio.

In summary, this paper provides a mathematical model, methods, and examples for choosing an engine geometry with an optimum balance between shaft output and mechanical efficiency.

APPENDIX: DERIVATION OF FORMULAS

Volume and pressure

The instantaneous volume of gas in the hot space and cold spaces of the engine of the engine are

$$V_{\mathrm{H}} = \frac{V_1}{2}(1 + \cos(\omega t + \alpha)) \quad \text{and}$$

$$V_{\mathrm{C}} = V_1 - V_{\mathrm{H}} + \frac{V_2}{2}(1 + \cos\omega t) = \frac{V_1}{2}[1 + \kappa(1 + \cos\omega t) - \cos(\omega t + \alpha)]$$

The instantaneous total engine volume is then

$$\begin{aligned} V &= V_{\mathrm{H}} + V_{\mathrm{C}} + V_{\mathrm{D}} = V_1 + \frac{V_2}{2}(1 + \cos\omega t) + V_{\mathrm{D}} \\ &= V_1\left[1 + \frac{\kappa}{2}(1 + \cos\omega t) + \chi\right] \end{aligned}$$

In terms of the combined swept volume V_{T} of the piston and displacer, i.e. using the relation $V_{\mathrm{T}} = V_1 + V_2 = V_1(1 + \kappa)$, we find

$$\mathrm{d}V = -\frac{\omega\kappa V_T}{2(\kappa + 1)}(\sin\omega t)\,\mathrm{d}t$$

By the assumptions made, instantaneous pressure within the engine space is

$$p = \frac{mR}{V_H/T_H + V_C/T_C + V_D/T_D} = \frac{mRT_C}{\tau V_H + V_C + V_1\chi\, 2\tau/(1+\tau)}$$

where m is the total mass of gas captive within the engine and R is its ideal gas constant. Putting in the expressions obtained above for V_H and V_C in the denominator gives

$$p = \frac{mRT_C}{(V_1/2)\left[\tau(1+\cos(\omega t+\alpha)) + 1 + \kappa(1+\cos\omega t) - \cos(\omega t+\alpha) + 4\chi\tau/(1+\tau)\right]}$$

The factor in brackets in the denominator can be written in the following form:

$$\left(1+\tau+\kappa+\frac{4\chi\tau}{1+\tau}\right) + (\tau-1)\cos(\omega t+\alpha) + \kappa\cos\omega t = Y + A\cos\omega t + B\sin\omega t$$

where

$$Y = 1+\tau+\kappa+\frac{4\chi\tau}{1+\tau}, \qquad A = \kappa-(1-\tau)\cos\alpha \quad \text{and} \quad B = (1-\tau)\sin\alpha$$

Let

$$\theta = \cos^{-1}\frac{A}{\sqrt{A^2+B^2}} \quad \text{with} \quad 0\leqslant\theta\leqslant\pi$$

The analysis is limited to α in the range $0<\alpha<\pi$. Then $B>0$ and A is positive or negative according to whether $\kappa/(1-\tau) > \cos\alpha$ or not. As A is positive or negative, θ is in quadrant I or II. With θ defined as above,

$$A\cos\omega t + B\sin\omega t = X\cos(\omega t-\theta)$$

where

$$X = \sqrt{A^2+B^2} = \sqrt{\kappa^2 - 2\kappa(1-\tau)\cos\alpha + (1-\tau)^2}$$

Therefore,

$$p = \frac{mRT_C}{(V_1/2)[Y + X\cos(\omega t-\theta)]}$$

It follows then that

$$p_{\max} = \frac{mRT_C}{(V_1/2)\,[Y-X]} \quad \text{and} \quad p_{\min} = \frac{mRT_C}{(V_1/2)\,[Y+X]}$$

$$\text{so that} \quad \bar{p} = \sqrt{p_{\max}\,p_{\min}} = \frac{mRT_C}{(V_1/2)\sqrt{Y^2-X^2}}$$

whence p may be written more simply as

$$p = \frac{\bar{p}\,\sqrt{Y^2 - X^2}}{Y + X\cos(\omega t - \theta)}$$

Indicated work

From here the cyclic indicated work can be calculated as

$$W = \int_0^{2\pi/\omega} p\,\mathrm{d}V = -\frac{V_{\mathrm{T}}\bar{p}\omega\kappa\sqrt{Y^2 - X^2}}{2(\kappa + 1)} \int_0^{2\pi/\omega} \frac{\sin \omega t}{Y + X\cos(\omega t - \theta)}\,\mathrm{d}t$$

The last integral can be evaluated by well-known methods to yield the closed-form expression

$$W = \frac{\pi(1 - \tau)\,V_{\mathrm{T}}\,\bar{p}\,\kappa\sin\alpha}{(\kappa + 1)\left(\sqrt{Y^2 - X^2} + Y\right)}$$

Note that this expresses the cyclic indicated work in terms of the combined total swept volume V_{T}, mean cycle pressure $\bar{p}$, and the dimensionless parameters τ, χ, κ and α.

Forced work

The definition of the forced work of a cycle is $W_{-} = \oint[(p - p_{\mathrm{b}})\,\mathrm{d}V]^{-}$. This requires integrating the product of $p - p_{\mathrm{b}}$ and $\mathrm{d}V$ over those portions of the cycle, and only those portions, where they differ in sign. The forced work integral for a Schmidt engine cannot be obtained as a closed-form expression. It is therefore necessary to employ numerical integration techniques. The negative part function used in the integrand follows the standard definition:

$$Z^{-} = \begin{cases} 0 & \text{if } Z \geqslant 0 \\ -Z & \text{if } Z < 0 \end{cases}$$

An identity for this which is convenient for numerical integration is the following:

$$Z^{-} = \frac{|Z| - Z}{2}$$

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