References: some of this material can be found in chapter 1 of Green, Schwarz and Witten, although if you really like particle path integrals the best source is probably A. Polyakov, *Gauge fields and strings*. A general discussion of partial-wave decompositions in QFT can be found in Jacob and Wick, *Ann. Phys.* **7**, 404 (1959).

The Polyakov action

Our goal is to extend the path integral approach to relativistic particles, so the first thing we need is an action. That's easy enough. With the standard expressions for energy and momentum

$$E = \gamma mc^2 \qquad\qquad \mathbf{p} = \gamma m \mathbf{v}$$

we have the Lagrangian

$$L = \mathbf{p}\cdot\mathbf{v} - E = -mc\sqrt{1 - \frac{v^2}{c^2}}\,.$$

Then adopting the path integral prescription we have (at least naively) an expression for the relativistic propagator

$$\langle \mathbf{x}_f, t_f | \mathbf{x}_i, t_i \rangle = \int_{\substack{\mathbf{x}(t_i)=\mathbf{x}_i \\ \mathbf{x}(t_f)=\mathbf{x}_f}} \mathcal{D}\mathbf{x}(\cdot)\, e^{iS}$$

$$S = -mc \int_{t_i}^{t_f} dt\, \sqrt{1 - |\dot{\mathbf{x}}|^2/c^2}$$

I'm not sure to what extent this path integral makes sense.[1] In any case we won't pursue it, for two reasons.

- We'd like manifest Lorentz invariance, something which is lacking in our description so far.
- As a practical matter, we'd like a Gaussian path integral that we know how to evaluate.

[1] Or what it calculates… since $E > 0$ maybe it gives a positive-frequency Wightman function?

Here we'll focus on the first problem, of obtaining a manifestly Lorentz invariant description. The fact that it leads to a Gaussian path integral will become clear in the next section.

Lorentz invariance is actually pretty simple. Setting $c = 1$ the action

$$\begin{aligned} S &= -m \int dt \sqrt{1 - |\dot{\mathbf{x}}|^2} \\ &= -m \int \sqrt{dt^2 - |d\mathbf{x}|^2} \\ &= -m \int ds \end{aligned}$$

is really just $(-m) \times$ the proper time measured along the particle worldline. Introducing a parameter τ and characterizing the worldline by functions $x^\mu(\tau)$, the action can be written in the manifestly Lorentz invariant form

$$S = -m \int d\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$$

where an overdot is $\partial/\partial\tau$ and $\eta_{\mu\nu} = \mathrm{diag}(- + ++)$ is the Minkowski metric. This action is invariant under reparameterizations of τ. One could fix this invariance in various ways, for example

$x^0(\tau) = \tau$ static gauge (takes us back to our starting point)

$x^+(\tau) = \tau$ light-front gauge

But for now we'll stick with the reparameterization-invariant form.

The next step, which at first sight is counterproductive but turns out to simplify things later, is to introduce an extra degree of freedom on the particle worldline, the "einbein" $e(\tau)$. For an action, we postulate the Polyakov action

$$S = \int d\tau \, \frac{1}{2e}\dot{x}^2 - \frac{1}{2}em^2 \,. \tag{1}$$

The equation of motion from varying e is

$$\frac{1}{e^2}\dot{x}^2 + m^2 = 0$$

which fixes

$$e = \frac{1}{m}\sqrt{-\dot{x}^2}\,. \tag{2}$$

This means e isn't an independent dynamical degree of freedom: given a trajectory $x^\mu(\tau)$, the einbein is completely determined. If we plug (2) back into the Polyakov action (1) we get

$$S = -m \int d\tau \, \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} \, .$$

That's exactly the particle action we constructed before. So the Polyakov action is equivalent to the usual action, in the sense that it ultimately gives rise to the same equations of motion.

* * *

The Polyakov action has a lovely geometric interpretation. Introduce a metric on the (one-dimensional) worldline of the particle, with line element

$$ds^2 = g_{\tau\tau} d\tau^2 \qquad g_{\tau\tau} \equiv e^2 \, .$$

Then the Polyakov action is

$$S = \int d\tau \sqrt{g_{\tau\tau}} \left(\frac{1}{2} g^{\tau\tau} \partial_\tau x^\mu \partial_\tau x^\nu \eta_{\mu\nu} - \frac{1}{2} m^2 \right) \, .$$

From the worldline point of view this is a theory of gravity (dynamical geometry) in a universe with 0+1 dimensions. Besides the metric $g_{\tau\tau}$ the dynamical variables are a collection of four scalar fields $x^\mu(\tau)$. (From the spacetime perspective they transform as a Lorentz 4-vector, but on the worldline they're scalar fields.)

Path integral for a relativistic particle

Let's quantize the Polyakov action using a path integral. This first expression one is tempted to write down is

$$\langle x^\mu_f, \tau_f | x^\mu_i, \tau_i \rangle = \int_{\substack{x^\mu(\tau_i) = x^\mu_i \\ x^\mu(\tau_f) = x^\mu_f}} \mathcal{D}\mathbf{x}(\cdot) \mathcal{D}e(\cdot) \, e^{iS}$$

$$S = \int_{\tau_i}^{\tau_f} d\tau \, \frac{1}{2e} \dot{x}^2 - \frac{1}{2} e m^2$$

However this expression ignores the reparameterization invariance: any two field configurations related by a diffeomorphism are physically equivalent and should only be counted once in the path integral. So the correct expression is

$$\langle x_f^\mu | x_i^\mu \rangle = \int_{\substack{x^\mu(\tau_i)=x_i^\mu \\ x^\mu(\tau_f)=x_f^\mu}} \frac{\mathcal{D}\mathbf{x}(\cdot)\mathcal{D}e(\cdot)}{\text{diffeos}}\, e^{iS}$$

$$S = \int_{\tau_i}^{\tau_f} d\tau\, \frac{1}{2e}\dot{x}^2 - \frac{1}{2}em^2$$

The only diffeomorphism-invariant quantity available to characterize the one-dimensional geometry of the particle worldline is its total length

$$s = \int_{\tau_i}^{\tau_f} d\tau\, e\,.$$

This means we can completely fix the diffeomorphism symmetry by setting

$$\tau_i = 0 \qquad \tau_f = s \qquad e = 1\,.$$

Having set $e = 1$ we no longer need to integrate over possible einbeins. However we still need to integrate over the possible total path lengths, so

$$\langle x_f^\mu | x_i^\mu \rangle = \int_0^\infty ds \int_{\substack{x^\mu(0)=x_i^\mu \\ x^\mu(s)=x_f^\mu}} \mathcal{D}\mathbf{x}(\cdot)\, e^{iS}$$

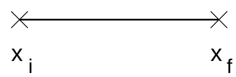
$$S = \int_0^s d\tau\, \frac{1}{2}\dot{x}^2 - \frac{1}{2}m^2$$

(One subtle point - how do we know the measure for integrating over {metrics}/diffeos is just ds?)

An unexpected bonus of this approach is that the path integral over x is Gaussian. We could directly calculate it, but it's a bit simpler to use operator methods. The action looks like a non-relativistic particle of unit mass in a constant potential (let's not worry about the Lorentzian signature). So we have

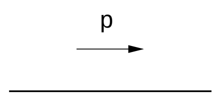
$$\begin{aligned}
\langle x_f^\mu | x_i^\mu \rangle &= \int_0^\infty ds \int_{\substack{x^\mu(0)=x_i^\mu \\ x^\mu(s)=x_f^\mu}} \mathcal{D}\mathbf{x}(\cdot)\, e^{iS} \\
&= \int_0^\infty ds\, \langle x_f | e^{-is\left(\frac{1}{2}p^2 + \frac{1}{2}m^2\right)} | x_i \rangle \\
&= \int_0^\infty ds\, \langle x_f | e^{-is\left(-\frac{1}{2}\Box + \frac{1}{2}m^2\right)} | x_i \rangle
\end{aligned}$$

Performing the integral over the "Schwinger parameter" s and introducing some obvious diagrammatic notation, we have



$$= \langle x_f^\mu | x_i^\mu \rangle = \langle x_f^\mu | \frac{2i}{\Box - m^2} | x_i^\mu \rangle \,.$$

At this point it's easiest to work in momentum space. We'll drop the pesky factor of two to agree with field theory conventions (maybe our measure should have been $ds/2$?). Then we're left with the scalar propagator



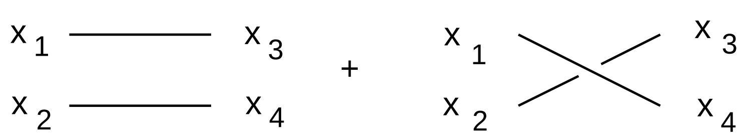
$$\frac{-i}{p^2 + m^2}$$

A few comments:

1. In position space, these expressions give the amplitude for a particle to propagate from x_i to x_f. In momentum space they give the amplitude for a particle to carry momentum p.
2. Note that, perhaps contrary to expectation, $p^2 = -m^2$ is not required. It's favored, since the propagator diverges on-shell, but off-shell propagation is also allowed.

Interactions and Feynman diagrams

So far we've understood how a single relativistic particle propagates from x_i to x_f. Of course we could work out propagation for a pair of particles, from initial positions (x_1, x_2) to final positions (x_3, x_4). In terms of diagrams we have (assuming identical particles obeying Bose statistics)

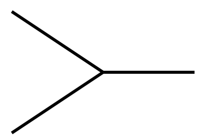


In terms of equations, the amplitude is given by

$$\langle x_1^\mu | x_3^\mu \rangle \langle x_2^\mu | x_4^\mu \rangle + \langle x_1^\mu | x_4^\mu \rangle \langle x_2^\mu | x_3^\mu \rangle \,.$$

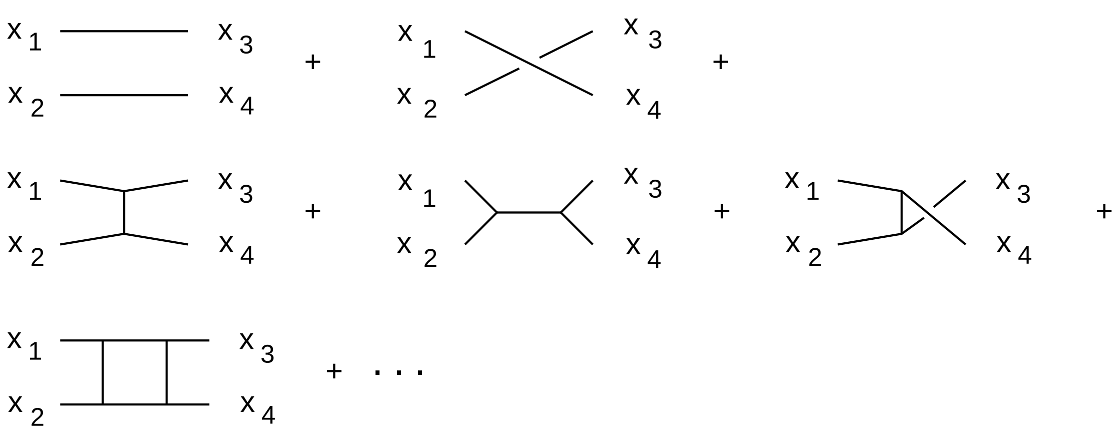
Here something like $\langle x_1^\mu | x_3^\mu \rangle$ denotes the free propagator given above. Note that, in the spirit of path integrals, we need to sum over all ways to get from the initial configuration to the final configuration. If the particles were distinct the second diagram wouldn't contribute.

So far this isn't so exciting. It gets more interesting if we let the particles interact, i.e. if we let the worldlines join or split. We'll just postulate that the amplitude for a worldline to join or split is determined by a fixed "coupling constant" g. In diagrams



$-ig$

Then the amplitude for propagation from (x_1, x_2) to (x_3, x_4) is given by an infinite sum over possible worldline topologies (a sum over possible graphs).



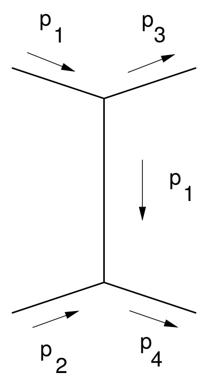
You can organize the sum as an expansion in powers of g. Diagrams in the first line are $\mathcal{O}(g^0)$, the second line are $\mathcal{O}(g^2)$, the third line starts at $\mathcal{O}(g^4)$. This corresponds to the perturbation expansion of QFT. For example the first diagram on the second line is

$$(-ig)^2 \int d^4y_1 d^4y_2 \, \langle x_1|y_1\rangle \langle y_1|x_3\rangle \langle y_1|y_2\rangle \langle x_2|y_2\rangle \langle y_2|x_4\rangle \, .$$

Note that (path integrals again) we integrate over the locations y_1, y_2 where the worldlines join or split.

Scattering and unitarity

For the most part we're interested in scattering, i.e. the evolution of a collection of widely separated particles in the far past to an (in general different) collection of widely separated particles in the far future. This is easiest to describe in momentum space. For example, to the diagram



we associate a scattering amplitude $\mathcal{M}$ given by

$$-i\mathcal{M} = (-ig)^2 \frac{-i}{(p_1 - p_3)^2 + m^2} \,.$$

By convention the sum of all diagrams gives $(-i) \times$ the scattering amplitude. Note that we don't include the "free" diagrams, since they don't contribute to scattering. Nor do we include the propagators for the incomming and outgoing particles, since they just describe the evolution of the widely-separated initial and final configurations (this is just as well since the propagators diverge on-shell). Finally note that we have momentum conservation at each vertex; if you want you can derive this by Fourier transforming our previous position-space diagrams.

So far we've understood how to calculate a scattering amplitude between initial and final momentum eigenstates. But it's sometimes useful to do a "partial wave" decomposition of the amplitude in order to describe scattering between angular momentum eigenstates. This is pretty easy to state. Let's work in the center of mass frame, with

$$\begin{aligned}
p_1 &= (E, 0, 0, p) \\
p_2 &= (E, 0, 0, -p) \\
p_3 &= (E, p\cos\theta, 0, p\sin\theta) \\
p_4 &= (E, -p\cos\theta, 0, -p\sin\theta)
\end{aligned}$$

Here I'm assuming the particles all have the same mass, so their energies are all the same. Then the partial-wave decomposition of the scattering amplitude is

$$\mathcal{M} = \frac{8\pi E}{ip} \sum_J (2J+1) \mathcal{M}_J P_J(\cos\theta)$$

where P_J is a Legendre polynomial. The cross-section for scattering with angular momentum J is simply given by

$$\sigma_J = \frac{\pi}{p^2}(2J+1) \left|\mathcal{M}_J\right|^2 .$$

This leads to a bound from unitarity (probability conservation). Scattering with impact parameter b corresponds to an initial angular momentum $J = bp$. So the cross-sectional area associated with having initial angular momentum between J and $J+1$ is the difference between the area of two disks,

$$A = \frac{\pi(J+1)^2}{p^2} - \frac{\pi J^2}{p^2} = \frac{\pi(2J+1)}{p^2}$$

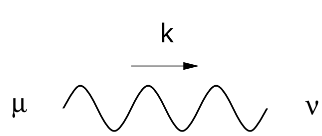
This leads to the bound

$$\sigma_J \leq \frac{\pi(2J+1)}{p^2} \quad \Rightarrow \quad \left|\mathcal{M}_J\right|^2 \leq 1 .$$

(This simple semiclassical argument somehow gives you the exact answer. A more careful analysis would trace this back to unitarity of the S-matrix.)

Particles with spin

So far our we've only discussed spinless particles, but it's not so hard to guess how we should treat particles with spin. For example, take a massless spin-1 particle like the photon. Besides its 4-momentum k^μ, a photon is characterized by specifying its polarization 4-vector ϵ_μ. These quantities correspond to a vector potential $A_\mu(x) = \epsilon_\mu e^{ik\cdot x}$. For a free (non-interacting) photon obeying the vacuum Maxwell equations, the polarization vector doesn't change with time. So a natural guess for the photon propagator is

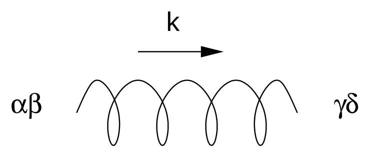


$$\frac{-ig_{\mu\nu}}{k^2}$$

What about the graviton, a massless spin-2 particle described by a symmetric polarization tensor $\epsilon_{\alpha\beta}$, corresponding to a spacetime metric

$$g_{\alpha\beta}(x) = \eta_{\alpha\beta} + \sqrt{32\pi G_N}\,\epsilon_{\alpha\beta} e^{ik\cdot x}\,?$$

A natural(?) guess for the propagator is



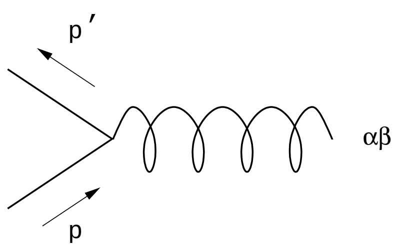
$$\frac{-i}{k^2}\cdot\frac{1}{2}\left(g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma} - g^{\alpha\beta}g^{\gamma\delta}\right)$$

(For a physical graviton, with a traceless polarization tensor, the last term drops out and the polarization tensor is conserved.)

What about interactions, say between a graviton and a spinless particle? As you saw on the homework, a graviton (thought of as a small variation in the spacetime metric) couples to a particle by making a small change in its action. The change is measured by the stress tensor

$$T^{\mu\nu}(x) = \int d\tau\,\frac{1}{e}\dot{X}^\mu\dot{X}^\nu\delta^4(x - X(\tau)) = \int d\tau\, e\, P^\mu P^\nu \delta^4(x - X(\tau))$$

It seems reasonable to guess that this determines the interaction amplitude. The precise rule is



$$i\sqrt{8\pi G_N}\Big(p_\alpha p'_\beta + p'_\alpha p_\beta - g_{\alpha\beta}\,\left(p\cdot p' + m^2\right)\Big)$$

If $p = p'$ is on-shell this reduces to $i\sqrt{32\pi G_N}$ $\times$ the stress tensor for the particle. The factor of i is easy enough to understand; it's because you do the path integral of e^{iS}. Newton's constant G_N, or better $\sqrt{32\pi G_N}$, acts as the coupling constant for gravity. Blame Newton for the funny normalization.