Solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \tag{1}$$

in a circular region subject to the boundary condition

$$\frac{\partial u}{\partial r}(a, \theta, t) = 0$$

with initial conditions

$$u(r, \theta, 0) = 0, \qquad \frac{\partial u}{\partial t}(r, \theta, 0) = \beta(r, \theta)$$

First we define the domain in polar coordinates and use separation of variables,

$$u(r, \theta, y) = \phi(r, \theta) h(t) \qquad \text{in } \Omega = [0, a] \times [-\pi, \pi]$$

$$\phi(r, \theta) h''(t) = c^2 \left(\nabla^2 \phi(r, theta) \right) h(t)$$
$$\frac{1}{c^2} \frac{h''(t)}{h(t)} = \frac{\nabla^2 \phi(r, theta)}{\phi(r, \theta)} = -\lambda$$

so, we have an equation for time and space,

$$\begin{cases} h''(t) + c^2 \lambda h(t) = 0 & (2) \\ \nabla^2 \phi(r, \theta) + \lambda \phi(r, \theta) & (3) \\ \dfrac{\partial \phi}{\partial r}(a, \theta) = 0 & (4) \\ & (5) \end{cases}$$

Next, we separate $\phi(r, \theta)$ so we get the form

$$u(r, \theta, t) = f(r) g(\theta) h(t)$$

We expand (7) in polar coordinates,

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}$$

So (7) becomes

$$\frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r f'(r) \right) g(\theta) + f(r) \frac{1}{r} g''(\theta) + \lambda f(r) g(\theta) = 0$$

or,

$$-\frac{g''(\theta)}{g(\theta)} = \frac{r \frac{\mathrm{d}}{\mathrm{d}r} \left(r f'(r) \right)}{f(r)} + \lambda r^2 = \mu$$

This gives us a differential equation for $f(r)$ and $g(\theta)$,

$$\begin{cases} g''(\theta) + \mu g(\theta) = 0 & (6) \\ r \dfrac{\mathrm{d}}{\mathrm{d}r} \left(r f'(r) \right) + \left(\lambda r^2 - \mu \right) f(r) = 0 & (7) \\ f'(a) = 0 & (8) \\ & (9) \end{cases}$$

and because $f(0)$ is finite, we can say

$$|f(0)| < \infty \tag{10}$$

Since $g(\theta)$ is a second order equation, we will need two boundary conditions. Since the domain is circular, we can assume,

$$\begin{cases} g(-\pi) = g(\pi) & (11) \\ \dfrac{\mathrm{d}g}{\mathrm{d}\theta}(-\pi) = \dfrac{\mathrm{d}g}{\mathrm{d}\theta}(\pi) & (12) \end{cases}$$

Solving for $g(\theta)$,

$$g(\theta) = C_1 \cos(\sqrt{\mu}\theta) + C_2 \sin(\sqrt{\mu}\theta)$$

From the boundary conditions, we find that if $\sin(\sqrt{\mu}\pi) \neq 0$, we will have a trivial solution, so we determine

$$\mu_m = m^2, \qquad m = 0, 1, 2, \ldots$$

so,

$$\boxed{g_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta)}$$

Then, we solve for $f(r)$,

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right) + \left(\lambda r^2 - m^2\right) f(r) = 0, \tag{13}$$

Rearranging the equation, we find

$$r^2\frac{\mathrm{d}^2 f(r)}{\mathrm{d}r^2} + r\frac{\mathrm{d}f(r)}{\mathrm{d}r} + \left(\lambda r^2 - m^2\right) f(r)$$

Defining $s^2 = \lambda r^2$, we change (13) to the form

$$s^2\frac{\mathrm{d}^2 f(s)}{\mathrm{d}s^2} + s\frac{\mathrm{d}f(s)}{\mathrm{d}s} + \left(s^2 - m^2\right) f(s) = 0 \tag{14}$$

with boundary conditions

$$f(\sqrt{\lambda}a) = 0 \qquad\qquad |f(0)| < \infty$$

To solve this, we introduce the Bessel function, which gives the solution

$$f(s) = C_1 J_m(s) + C_2 Y_m(s)$$

Because $Y_m(0)$ is unbounded, we are left with

$$f(s) = C_m J_m(s)$$

and converting back,

$$f(r) = C_m J_m(\sqrt{\lambda}r)$$

Applying the boundary condition in (10), we can state

$$f_{mn}(r) = c_m J_m(\sqrt{\lambda}r) \qquad m = 1, 2, \ldots$$

Hence, we can now define $\phi(r,\theta)$,

$$\boxed{\phi_{mn}(r,\theta) = a_m c_m J_m(\sqrt{\lambda_{mn}}r)\cos(m\theta) + b_m c_m J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta)} \tag{15}$$

We then solve for $h(t)$,

$$h''_{mn}(t) + c^2\lambda_{mn}h(t) = 0$$

$$h_{mn}(t) = D_1\sin(c\sqrt{\lambda_{mn}}t) + D_2\cos(c\sqrt{\lambda_{mn}}t)$$

Evaluating the first initial condition,

$$u(r,\theta,0) = f(r)g(\theta)h(0) = 0 \qquad \Rightarrow \qquad h(0) = 0$$

so,

$$\boxed{h_{mn}(t) = D_1\sin(c\sqrt{\lambda_{mn}}t)} \tag{16}$$

Finally, we solve for the solution,

$$u_{mn}(r,\theta,t) = A_{mn}J_m(\sqrt{\lambda_{mn}}r)\cos(m\theta)\sin(c\sqrt{\lambda_{mn}}t) + B_{mn}J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta)\sin(c\sqrt{\lambda_{mn}}t)$$

or,

$$\boxed{\begin{aligned} u(r,\theta,t) = &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}J_m(\sqrt{\lambda_{mn}}r)\cos(m\theta)\sin(c\sqrt{\lambda_{mn}}t) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta)\sin(c\sqrt{\lambda_{mn}}t) \end{aligned}} \tag{17}$$

Evaluating the initial conditions,

$$\begin{aligned} \frac{\partial u}{\partial t}(r,\theta,t) = &-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}J_m(\sqrt{\lambda_{mn}}r)\cos(m\theta)\cos(c\sqrt{\lambda_{mn}}t) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta)\cos(c\sqrt{\lambda_{mn}}t) \end{aligned}$$

$$\frac{\partial u}{\partial t}(r,\theta,0) = \beta(r,\theta) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}J_m(\sqrt{\lambda_{mn}}r)\cos(m\theta) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}J_m(\sqrt{\lambda_{mn}}r)\sin(m\theta)$$