PERIOD-ENERGY RELATION

Here is a solution of the problem from "Foundations of Mechanics" by Abraham and Marsden: problem 5.2G on p. 401 in the second edition.

Ley $H = H(z)$ be a Hamiltonian function of a Hamiltonian system on the symplectic manifold (M, ω) with the simplectic local coordinates $z = (p, x)$.

A submanifold $\Sigma \subset M, \quad \dim \Sigma = l$ consists of periodic trajectories. Let $\tau(z)$ be a period of the trajectory $\gamma(z) \subset \Sigma$ passing though the point $z \in \Sigma$ and

$$J(z) := \int_{\gamma(z)} p_i dx^i, \quad \omega = dp_i \wedge dx^i.$$

Theorem 1. *The following formula holds*

$$dJ(z) = -\tau(z) dH(z), \quad z \in \Sigma.$$

Proof. By v_H we denote the Hamiltonian vector field. Recall that $i_{v_H}\omega = dH$.

A vector field $u(z) = \tau(z) v_H(z)$ has the same periodic trajectories and the period of each trajectory equals 1.

From here and till the end of the text all the constructions are inside the manifold Σ.

Fix a trajectory $\gamma(\tilde{z})$. In its neighbourhood there are local coordinates

$$(y, \varphi) = (y^1, \ldots, y^{l-1}, \varphi), \quad \varphi \pmod 1$$

such that

$$u = (0, \ldots, 0, 1)^T.$$

In these coordinates define the following vector fields

$$v_1 = (1, 0, \ldots, 0)^T, \quad v_2 = (0, 1, 0, \ldots, 0), \ldots, v_l = u.$$

Let g_i^s be the flow of the vector field $v_i, \quad i = 1, \ldots, l$.

Each flow g_i^s takes a trajectory to a trajectory:

$$g_i^s(\gamma(z)) = \gamma(g_i^s(z)). \tag{1}$$

The function H is constant on each trajectory:

$$H(g_i^s(\xi)) = H(g_i^s(z)), \quad \forall \xi \in \gamma(z).$$

Differentiate this equality in s and put $s = 0$:

$$dH(\xi) v_i(\xi) = dH(z) v_i(z), \quad \forall \xi \in \gamma(z). \tag{2}$$

From formula (1) it follows that

$$J(g_i^s(z)) = \int_{g_i^s(\gamma(z))} p_i dx^i.$$

Differentiate this equality in s and put $s = 0$:

$$dJ(z)v_i = \int_{\gamma(z)} L_{v_i}(p_i dx^i) = \int_{\gamma(z)} i_{v_i}\omega + \int_{\gamma(z)} di_{v_i}(p_i dx^i).$$

(Here we use $L_v = di_v + i_v d$.)

The last integral equals zero and by using (2) we obtain

$$dJ(z)v_i = \int_{\gamma(z)} i_{v_i}\omega = -\int_0^{\tau(z)} dH v_i dt = -\tau(z) dH(z) v_i(z).$$

The theorem is proved.

Remark 1. *The functions τ and H are dependent in Σ. Indeed,*

$$ddJ = -d\tau \wedge dH = 0.$$