

PROBLEM: A particle of mass m moves in one dimension according to the Hamiltonian

$$H_0 = \frac{p^2}{2m} + V(x) \quad (I)$$

$$H_0 \psi_n(x) = E_n \psi_n(x) \quad (II)$$

All eigenfunctions $\psi_n(x)$ and eigenvalues E_n are known. Suppose we add a term to the Hamiltonian, where λ and m are constants and p is the momentum operator:

$$H = H_0 + \frac{\lambda}{m} p \quad (III)$$

Derive an expression for the eigenvalues and eigenstates of the new Hamiltonian H .

SOLUTION: The first step is to rewrite the Hamiltonian by completing the square on the momentum operator:

$$H = \frac{p^2}{2m} + \frac{\lambda}{m} p + V(x) = \frac{(p+\lambda)^2}{2m} - \frac{\lambda^2}{2m} + V(x) \quad (1)$$

The constant just shifts the zero of the momentum operator. The rewritten Hamiltonian in (1) suggests the perturbed eigenstates

$$\psi_n^-(x) = e^{-ix\lambda/\hbar} \psi_n(x) \quad (2)$$

The action of the displaced momentum operator $p+\lambda$ on the new eigenstates is

$$(p + \lambda) \psi_n^-(x) = e^{-ix\lambda/\hbar} p \psi_n(x) \quad (3)$$

so the Hamiltonian gives

$$H \psi_n^-(x) = e^{-ix\lambda/\hbar} \left[H_0 - \frac{\lambda^2}{2m} \right] \psi_n(x) = \left[E_n - \frac{\lambda^2}{2m} \right] \psi_n^-(x) \quad (4)$$

and the eigenvalues are simply

$$E_n^- = E_n - \frac{\lambda^2}{2m} \quad (5)$$