**Schwarzschild Metric From Kepler's 3rd Law**

http://mathpages.com/rr/s5-05/5-05.htm

The Schwarzschild metric is a solution of Einstein's field equations for an uncharged, non-rotating, spherically symmetric body of mass [M]. While most authoritative texts on general relativity will highlight that there is no simple derivations of the Schwarzschild metric, it would seem the author of this paper provides a basic derivation of the Schwarzschild metric apparently anchored in the inverse square law of gravity, Kepler's third law for circular orbits and the spacetime interval of light. It starts by declaring [r] to be the radial spatial coordinate and [t] to be the time coordinate. As such, any surface of constant [r] and [t] must possess the two-dimensional geometry of a 2-sphere, i.e. a surface defined by [$4\pi r^2$].

[1] $$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

In [1], [ds] is the incremental distance on the surface of a sphere of radius [r] described in terms of the spherical coordinates [dθ] and [dϕ], where the coordinate [θ] represents latitude and the [ϕ] represents the longitude. The complete spacetime metric, inclusive of time [t] takes the following form rationalise using coefficients:

[2] $$ds^2 = c^2 d\tau^2 = c^2 d\tau^2 - \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right)$$

$$c^2 d\tau^2 = g_{tt} c^2 dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2$$

At this point, we might describe [$g_{tt}$] and [$g_{rr}$] as unknown functions of [r] and mass [M], although it is known that when [M=0], the functions [$g_{tt}$] and [$-g_{rr}$] must both equal unity in order to align with the flat spacetime metric and that as [r] increases to infinity these functions will also approach unity. In order to proceed from [2], a circular/Kepler orbits around [M] is assumed, where [r] might now be described in terms Newtonian physics as the radial distance from the centre of the mass. The circular orbit around [M] is maintained by an angular velocity [ω], which complies with Kepler's third law:

[3] $$T^2 = \frac{4\pi^2}{GM} r^3 \Rightarrow M = \frac{4\pi^2}{GT^2} r^3$$

In [3], [T] is the orbital period that equates to the circumference [2πr] divided by the orbital velocity [ω=v/r], such that we might reduced [3] to a simplified form, where the gravitational constant [G] is required to resolve the units to [kg].

[4] $$M = \frac{4\pi^2}{G\left(\frac{2\pi r}{\omega r}\right)^2} r^3 = \frac{\omega^2 r^3}{G}$$

If we consider a circular orbit of constant radius, i.e. [dr=0], on the equatorial plane (θ=π/2), the metric in [2] reduces to:

[5] $$c^2 d\tau^2 = \left(g_{tt} c^2 dt^2 + 0 + 0 + g_{\phi\phi} d\phi^2\right) = g_{tt} c^2 dt^2 - r^2 d\phi^2$$

If we now divide [5] by [$d\tau^2$] we get:

[6] $$c^2 \frac{d\tau^2}{d\tau^2} = g_{tt} c^2 \left(\frac{dt}{d\tau}\right)^2 - r^2 \left(\frac{d\phi}{d\tau}\right)^2 = 1 \Rightarrow g_{tt} c^2 \left(\frac{dt}{d\tau}\right)^2 = 1 + r^2 \left(\frac{d\phi}{d\tau}\right)^2$$

Partially differentiating [6] with respect to [r] to isolate the coefficient [$g_{tt}$] in terms of the radius [r]:

[7] $$\frac{\partial(g_{tt})}{\partial r} = 2r \left(\frac{d\phi}{d\tau}\right)^2$$

We might also recognise that the quantity [dϕ/dt] is equivalent to angular velocity [ω] as defined in [4]:

[8] $$\frac{\partial g_{tt}}{\partial r} = 2r \left(\frac{d\phi}{d\tau}\right)^2 = 2r\omega^2 = 2r\left(\frac{GM}{r^3}\right) = \frac{2GM}{r^2}$$

Integrating [8], we are left with a constant [C] of integration. However, we know that [$g_{tt}$] must equal unity when [M=0] and/or as [r] approaches infinity, such that we may set [C=1] as shown in [9]:

[9] $$g_{tt} = -\frac{2GM}{r} + C \Rightarrow 1 - \frac{2GM}{r} = \frac{m^3}{kg.s}\frac{kg}{m} = \frac{m^2}{s^2}?$$

The inference that the dimensionless coefficient [$g_{tt}$] appears to require the units [$m^2/s^2$] is addressed by dividing through by [$c^2$], which then allows the Schwarzschild radius [Rs] to be substituted:

[10] $$g_{tt} = 1 - \frac{2GM}{rc^2} = \left(1 - \frac{Rs}{r}\right); \quad where\ Rs = \frac{2GM}{c^2}$$

At this point, 3 out of the 4 coefficients have been resolved, leaving only the remaining [$g_{rr}$] coefficient. If we return to [2], but only consider the radial equatorial path, where [$d\theta$] and [$d\phi$] equal zero, we may proceed with the simplified form:

[11] $$c^2 d\tau^2 = g_{tt} c^2 dt^2 + g_{rr} dr^2$$

If we divide through by [$dt^2$] we get:

[12] $$c^2 \frac{d\tau^2}{dt^2} = g_{tt} c^2 \frac{dt^2}{dt^2} + g_{rr} \frac{dr^2}{dt^2}$$

***Note: While [12] would seem a reasonable starting point to resolve [$g_{rr}$] the approach is not obvious. However, the mathpages article proceeds on the basis of a previously defined 'radial geodesic equation'. This said, neither the physics or mathematical rationale supporting [13] is understood and the primary reason for raising this PF thread.***

[13] $$\frac{d^2r}{d\tau^2} = \frac{1}{2g_{rr}} \frac{\partial g_{tt}}{\partial r} \left(\frac{dt}{d\tau}\right)^2$$

While the form of [13] is not clear, the various components in [13] are rationalised as follows:

[14] $$\frac{d^2r}{d\tau^2} = g = -\frac{GM}{r^2}; \quad \frac{\partial g_{tt}}{\partial r} = \frac{2GM}{r^2}; \quad \left(\frac{dt}{d\tau}\right)^2 = \frac{1}{g_{tt}}$$

Inserting the expressions in [14] back into [13] we get the required result:

[15] $$-\frac{GM}{r^2} = \frac{1}{2g_{rr}} \left(\frac{2GM}{r^2}\right)\left(\frac{1}{g_{tt}}\right) \Rightarrow g_{tt} g_{rr} = -1$$

So based on the result for [$g_{tt}$] in [10], we can resolve the final coefficient [$g_{rr}$]:

[16] $$g_{rr} = -\left(1 - \frac{Rs}{r}\right)^{-1}$$

As such, we can complete the Schwarzschild metric:

[17] $$c^2 d\tau^2 = g_{tt} c^2 dt^2 + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2$$

$$c^2 d\tau^2 = \left(1 - \frac{Rs}{r}\right) c^2 d\tau^2 - \left(\left(1 - \frac{Rs}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right)$$