

16. Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Let

$$L^2(D) = \{f: D \rightarrow \mathbb{C} \mid \iint |f(x, y)|^2 dx dy < \infty\}$$

Define an inner product on $L^2(D)$ by

$$\langle f, g \rangle = \iint f(x, y) \overline{g(x, y)} dx dy$$

Let $\phi_n(x, y) = (x + iy)^n$, $n = 0, 1, 2, \dots$. Show that this collection of functions is orthogonal in $L^2(D)$ and compute $\|\phi_n\|$.

Let $\phi_j, \phi_k \in L^2(D)$, then

$$\langle \phi_j, \phi_k \rangle = \iint \phi_j(x, y) \overline{\phi_k(x, y)} dx dy = \iint (x + iy)^j (x - iy)^k dx dy = \iint (x^{j+k} + y^{j+k}) dx dy$$