Area Bounded by a Function in Polar Coordinates:

Suppose we define an arbitrary polar function $f(\theta)$ that is bounded on the closed interval $[a, b]$. We then define a partition P of the closed interval $[a, b]$ such that,

$$P = \{t_0, t_1, \dots, t_{n-1}, t_n\} \text{ where } a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

Since our polar function maps how the radius $f(\theta_i)$ varies with the angle θ_i, we can approximate the region R bounded by $f(\theta)$ using sectors of circles, each with area,

$$A_i = \frac{f(\theta_i)^2 \theta_i}{2}$$

Given the area for a sector of a circle, we can define upper and lower sums of $f(\theta)$ for P in the following manner,

$$L(f, P) = \sum_{i=1}^{n} \frac{r_i^2}{2(t_i - t_{i-1})} \text{ where } r_i = \inf\{f(\theta) : \theta \in [t_{i-1}, t]\}$$

$$U(f, P) = \sum_{i=1}^{n} \frac{R_i^2}{2(t_i - t_{i-1})} \text{ where } R_i = \sup\{f(\theta) : \theta \in [t_{i-1}, t_i]\}$$

It should immediately be clear that, given any partition P of $[a, b]$,

$$L(f, P) \leq A \leq U(f, P)$$

However, given this last inequality, it may occur that,

$$\sup\{L(f, P)\} = \inf\{U(f, P)\}$$

This value must be the area bounded by the function $f(\theta)$ since there is only one convergent value between the lower and upper sums of $f(\theta$) for P. Moreover, if this is the case, we define the area of R as follows,

$$A = \inf\left\{\sum_{i=1}^{n} \left(\frac{r_i^2}{2(t_i - t_{i-1})}\right)\right\} = \sup\left\{\sum_{i=1}^{n} \frac{R_i^2}{2(t_i - t_{i-1})}\right\} = \int_a^b \frac{f(\theta)^2}{2} \, d\theta$$

While our previous definition for the area bounded by the function $f(\theta$) relies on the criterion that $\sup\{L(f,P)\} = \inf\{U(f,P)\}$, we can quantify this qualification in the following manner,

$$U(f,P) - L(f,P) < \varepsilon, \forall\varepsilon > 0$$

We can prove this requirement by noting that,

$$\inf\{U(f,P)\} - \sup\{L(f,P)\} < \varepsilon, \forall\varepsilon > 0, \because \inf\{U(f,P)\} \leq U(f,P) \text{ and } \sup\{L(f,P)\} \geq L(f,P)$$

$$\therefore \sup\{L(f,P)\} = \inf\{U(f,P)\}$$

Consequently, the area bounded by the region R is given by,

$$A = \int_a^b \frac{f(\theta)^2}{2} d\theta, \text{ if } \forall\varepsilon > 0, U(f,P) - L(f,P) < \varepsilon$$

In order to consider the Riemann Sum approximating the area bounded by the function $f(\theta)$, we make the choose an arbitrary angle $\theta_i \in [t_{i-1}, t_i]$, hence,

$$L(f,P) \leq \sum_{i=1}^{n} \frac{f(\theta_i)^2}{2(t_i - t_{i-1})} \leq U(f,P)$$

Given this inequality, the Riemann sum converges to the area bounded by the region R if,

$$\forall\varepsilon > 0, \exists\delta > t_i - t_{i-1} > 0 \text{ such that } \left|\sum_{i=1}^{n} \frac{f(\theta_i)^2}{2(t_i - t_{i-1})} - \int_a^b \frac{f(\theta)^2}{2} d\theta\right| < \varepsilon$$

We prove this proposition by noting that,

$$L(f,P) \leq \sum_{i=1}^{n} \frac{f(\theta_i)^2}{2(t_i - t_{i-1})} \leq U(f,P) \text{ and } L(f,P) \leq \int_a^b \frac{f(\theta)^2}{2} d\theta \leq U(f,P)$$

$$\therefore 0 < \left|\sum_{i=1}^{n} \frac{f(\theta_i)^2}{2(t_i - t_{i-1})} - \int_a^b \frac{f(\theta)^2}{2} d\theta\right| \leq U(f,P) - L(f,P) < \varepsilon$$

$$0 < \sum_{i=1}^{n} \frac{R_i^2}{2(t_i - t_{i-1})} - \sum_{i=1}^{n} \frac{r_i^2}{2(t_i - t_{i-1})} = \sum_{i=1}^{n} \frac{(R_i^2 - r_i^2)}{2(t_i - t_{i-1})} < \varepsilon$$

Since $f(\theta)$ is bounded on the closed interval $[a, b]$ and $R_i^2 \geq r_i^2 \geq 0$,

$$0 < \frac{M^2}{2}\sum_{i=1}^{n}(t_i - t_{i-1}) < \varepsilon, \textbf{ where } M = \sup\{|f(\theta)|: \theta \in [a, b]\}$$

$$\therefore 0 < \sum_{i=1}^{n}(t_i - t_{i-1}) < \frac{2\varepsilon}{M^2}$$

Because the sum in the last inequality above is greater than $t_i - t_{i-1}$, we let,

$$\delta = \frac{2\varepsilon}{M^2}$$

This completes the proof and consequently,

$$A = \int_a^b \frac{f(\theta)^2}{2} d\theta \textbf{ if } \forall\varepsilon > 0, \exists\delta > t_i - t_{i-1} > 0 \textbf{ such that } \left|\sum_{i=1}^{n}\frac{f(\theta_i)^2}{2(t_i - t_{i-1})} - \int_a^b \frac{f(\theta)^2}{2} d\theta\right| < \varepsilon \blacksquare$$