3. "3+1" split of the electromagnetic field:

An observer with 4-velocity $\vec{U}$ interacting with an electromagnetic field $\mathbf{F}$ measures electric and magnetic fields $\vec{E}_{\vec{U}}$ and $\vec{B}_{\vec{U}}$ in their instantaneous local inertial reference frame (that is, in an orthonormal basis with $\vec{e}_{\hat{0}} = \vec{U}$). These fields are 4-vectors with components

$$E^{\alpha}_{\vec{U}} = F^{\alpha\beta} U_{\beta} \ , \qquad B^{\alpha}_{\vec{U}} = -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} U_{\beta} F_{\gamma\delta} \ .$$

(a) Show that $\vec{E}_{\vec{U}}$ and $\vec{B}_{\vec{U}}$ lie orthogonal to observer's worldline. Thus, they are spatial vectors according to the observer, living entirely in that observer's hypersurface of simultaneity. (Hint: recall the projection tensor defined in Pset 1.)

(b) Show that the field tensor can be reconstructed from the observer's 4-velocity and the electric and magnetic fields they measure via the following tensor equation (valid for any basis):

$$F^{\alpha\beta} = U^{\alpha} E^{\beta}_{\vec{U}} - E^{\alpha}_{\vec{U}} U^{\beta} + \epsilon^{\alpha\beta}{}_{\gamma\delta} U^{\gamma} B^{\delta}_{\vec{U}} \ .$$

The identity $\epsilon^{\alpha\beta\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = 2\left(\delta^{\alpha}{}_{\nu}\delta^{\beta}{}_{\mu} - \delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu}\right)$ may prove useful.

(c) The wedge product between two vectors is defined as

$$\vec{A} \wedge \vec{B} = \vec{A} \otimes \vec{B} - \vec{B} \otimes \vec{A} \ .$$

The Hodge dual of a $(0,2)$ tensor is defined as

$${}^{*}C_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta}{}_{\mu\nu} C_{\alpha\beta} \ .$$

Show that the field tensor may be written

$$\mathbf{F} = a\,\vec{U} \wedge \vec{E}_{\vec{U}} + b\,{}^{*}\!\left(\vec{U} \wedge \vec{B}_{\vec{U}}\right) \ .$$

What are the values of the real constants a and b?

4. Transformation of Christoffel symbols:

(a) Show that, under a coordinate transformation, the components of the Christoffel symbol transform as follows:

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \Gamma^{\alpha}{}_{\beta\gamma} - \frac{\partial^2 x^{\alpha'}}{\partial x^{\beta}\,\partial x^{\gamma}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\gamma}}{\partial x^{\gamma'}}$$

Do this by considering the form of the Christoffel symbol in terms of derivatives of the metric.

(b) Show that, using this rule, the components of the covariant derivative of a vector transform as tensors should:

$$\nabla_{\alpha'} A^{\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\alpha'}} \frac{\partial x^{\beta'}}{\partial x^{\beta}} \nabla_{\alpha} A^{\beta} \ .$$