Quantum mechanics - Problem Set 2

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1. Two observables A_1 and A_2, which do not involve time explicitly, are known not to commute,
$$[A_1, A_2] \neq 0,$$
yet we also know that A_1 and A_2 both commute with the Hamiltonian:
$$[A_1, H] = 0,\ [A_2, H] = 0.$$
Prove that the energy eigenstates are, in general, degenerate. Are there exceptions? As an example, you may think of the central-force problem $H = \mathbf{p}^2/2m + V(r)$, with $A_1 \to L_z$, $A_2 \to L_x$.

Proof. My strategy is postulating an eigenvalue equation of H first, then I think I can derive another similar eigenvalue equation which shows the energy eigenstates are, in general, degenerate once we use all above given conditions. Let us assume that
$$H\left|\psi\right\rangle = E\left|\psi\right\rangle. \tag{1}$$
In order to use commutation relations, I let AH operator act on the ket $\left|\psi\right\rangle$,
$$AH\left|\psi\right\rangle = EA\left|\psi\right\rangle. \tag{2}$$
Note that $A\left|\psi\right\rangle$ is still a ket living in the Hilbert Space. Therefore, we define $\left|\phi\right\rangle \equiv A\left|\psi\right\rangle$. With the commutation relation and Eq.(2),
$$HA\left|\psi\right\rangle = H\left|\phi\right\rangle = AH\left|\psi\right\rangle = E\left|\phi\right\rangle. \tag{3}$$
$$\therefore \begin{cases} H\left|\psi\right\rangle = E\left|\psi\right\rangle \\ H\left|\phi\right\rangle = E\left|\phi\right\rangle \end{cases} \tag{4}$$
Now, we need to explain the possible interpretations of Eq.(4). We assume that the energy eigenstates are degenerate. It is what we want. That is to say
$$\left|\phi\right\rangle \neq C\left|\psi\right\rangle \implies A\left|\psi\right\rangle \neq C\left|\psi\right\rangle, \tag{5}$$
where C is a complex number. That means $\left|\psi\right\rangle$ is not the eigenket of operator A if the energy eigenstates are degenerate. On the other hand, $\left|\psi\right\rangle$ is the eigenket of operator A if and only if Eq.(4) is just one equation which

means we have not found any possible degeneracy of the Hamiltonian, H. Anyway, there are always lots of state kets which are not the eigenkets of operator A so that the energy eigenstates are, in general, degenerate. Note that operator A could be either A_1 or A_2. Therefore, we get a general degenerate principle:

> If $|\psi\rangle$ is not the eigenket of A_1, yet it is still the eigenket of H, then its degenerate ket is $A_1 |\psi\rangle$. And if $|\psi\rangle$ is not the eigenket of A_2, yet it is still the eigenket of H, then its degenerate ket is $A_2 |\psi\rangle$.

However, there is one more equation we've not used. That is

$$[A_1, A_2] \neq 0 \tag{6}$$

My concern is, if $|\phi\rangle = A_1 |\psi\rangle$, then does the above degenerate principle still apply to $|\phi\rangle$? I think this is into where the incompatible relation of A_1 and A_2 play. Suppose that $|\phi\rangle = A_1 |\psi\rangle$, then we need to know where it would be the eigenket of A_1 or A_2. Needless to say, $|\phi\rangle$ must be the eigenket of A_1. What about A_2?

$$A_2 |\phi\rangle = A_2 A_1 |\psi\rangle . \tag{7}$$

With the definition of commutator, we assume that $|\phi\rangle$ is the eigenket of A_2,

$$\begin{aligned} (A_1 A_2 - [A_1, A_2]) |\psi\rangle &= \lambda |\phi\rangle \\ \Longrightarrow \ A_2 |\psi\rangle = \lambda |\psi\rangle \ &\text{and} \ [A_1, A_2] = 0, \end{aligned} \tag{8}$$

where λ is a complex number. However, we know that $[A_1, A_2] \neq 0$ so that $|\psi\rangle$ is not the eigenket of A_2. Finally, we can say that if $|\alpha\rangle$ is the eigenket of Hamiltonian, rather than A_1, then its energy degeneracies are

$$\{A_1 |\alpha\rangle , A_2 A_1 |\alpha\rangle , A_1 A_2 A_1 |\alpha\rangle , \cdots\}.$$

Similarly, if $|\beta\rangle$ is the energy eigenstate, yet it is not the eigenket of A_2, then its energy degeneracies are

$$\{A_2 |\beta\rangle , A_1 A_2 |\beta\rangle , A_2 A_1 A_2 |\beta\rangle , \cdots\}.$$

Now we are in the position of finding the exceptions. Is it possible that premises are not valid? First, there must be energy eigenstates. But, maybe all of them are also the eigenkets of A_1(or A_2) just with different eigenvalues (except for $A_1 \propto H$ or $A_2 \propto H$, for it leads to $[A_1, A_2] = 0$)? That is to say, A_1(or A_2) and H have simultaneous but non-degenerate eigenkets. I think it is mathematically possible because I've not thought of any physical example.

As for the central-force problem example,

$$\begin{cases} H = \mathbf{p}^2/2m + V(r) \\ A_1 \to L_z \\ A_2 \to L_x \end{cases} \tag{9}$$

For hydrogen atom,

$$H |n, l, m, s\rangle = E_n |n, l, m, s\rangle \tag{10}$$

Since $|n, l, m, s\rangle$ is the simultaneous eigenket of H and L_z, yet it is not the eigenket of L_x, I think $L_x |n, l, m, s\rangle$ is still the eigenket of H. Besides, from Griffiths p.170-171,

$$H = \frac{1}{2mr^2}\left[-\hbar^2 \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + L^2\right] + V(r), \tag{11}$$

$$L_x = \frac{\hbar}{i}\left(-\sin\frac{\partial}{\partial \theta} - \cos\phi \cot\theta \frac{\partial}{\partial \phi}\right), \tag{12}$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi}. \tag{13}$$

Therefore, we know that

$$\begin{cases} [L_x, L_z] \neq 0, \\ [L_x, H] = 0,\ [L_z, H] = 0. \end{cases} \tag{14}$$

Hence, $|n, m, l, s\rangle$ is the eigenket of H, yet it is not the eigenket of L_x, so that

$$H(L_x |n, l, m, s\rangle) = L_x H |n, l, m, s\rangle = E_n(L_x |n, l, m, s\rangle). \tag{15}$$

From Eq.(10) and Eq.(15), we conclude that the degenerate ket of $|n, l, m, s\rangle$ is $L_x |n, l, m, s\rangle$. Moreover, the following eigenkets are also the degenerate kets of $|n, l, m, s\rangle$,

$$\{L_z L_x |n, l, m, s\rangle, L_x L_z L_x |n, l, m, s\rangle, \cdots\}.$$

□

2. Find the linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the uncertainty product

$$\langle(\Delta S_x)^2\rangle \langle(\Delta S_y)^2\rangle.$$

Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated.

Proof. We want to show that

$$\langle(\Delta S_x)^2\rangle \langle(\Delta S_y)^2\rangle \geq \frac{1}{4}|\langle[S_x, S_y]\rangle|^2. \tag{16}$$

We know the definition of uncertainty is

$$\Delta A \equiv A - \langle A\rangle \mathbb{I}. \tag{17}$$

Therefore,

$$\langle(\Delta S_x)^2\rangle \langle(\Delta S_y)^2\rangle = (\langle S_x^2\rangle - \langle S_x\rangle^2)(\langle S_y^2\rangle - \langle S_y\rangle^2) \tag{18}$$

Suppose that

$$|\psi\rangle = \alpha\,|+\rangle + \beta\,|-\rangle \tag{19}$$

In addition,

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}\,|x;+\rangle + \frac{1}{\sqrt{2}}\,|x;-\rangle \\ |-\rangle &= \frac{1}{\sqrt{2}}\,|x;+\rangle - \frac{1}{\sqrt{2}}\,|x;-\rangle \end{aligned} \tag{20}$$

$$\begin{aligned} |+\rangle &= \frac{1}{\sqrt{2}}\,|y;+\rangle + \frac{1}{\sqrt{2}}\,|y;-\rangle \\ |-\rangle &= -\frac{i}{\sqrt{2}}\,|y;+\rangle + \frac{i}{\sqrt{2}}\,|y;-\rangle \end{aligned} \tag{21}$$

By Eq.(18), Eq.(19), and Eq.(20), the $|\psi\rangle$ can be rewritten as

$$\begin{aligned} |\psi\rangle &= \frac{\alpha+\beta}{\sqrt{2}}\,|x;+\rangle + \frac{\alpha-\beta}{\sqrt{2}}\,|x;-\rangle \\ &= \frac{\alpha-i\beta}{\sqrt{2}}\,|y;+\rangle + \frac{\alpha+i\beta}{\sqrt{2}}\,|y;-\rangle \end{aligned} \tag{22}$$

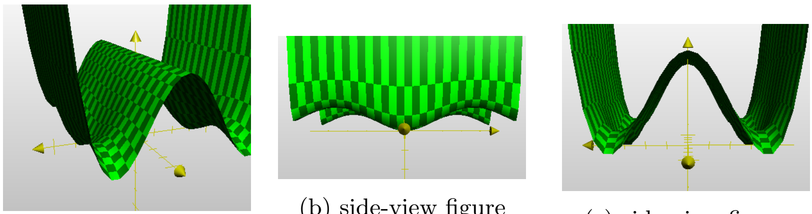
Therefore,

$$\begin{aligned} \langle S_x\rangle = &\left(\frac{\overline{\alpha}+\overline{\beta}}{\sqrt{2}}\,\langle x;+| + \frac{\overline{\alpha}-\overline{\beta}}{\sqrt{2}}\,\langle x;-|\right)\times \\ &\frac{\hbar}{2}\left(\frac{\alpha+\beta}{\sqrt{2}}\,|x;+\rangle - \frac{\alpha-\beta}{\sqrt{2}}\,|x;-\rangle\right) \\ &= \frac{\hbar}{2}\left(\overline{\alpha}\beta + \overline{\beta}\alpha\right) \end{aligned} \tag{23}$$

$$\begin{aligned} \langle S_y\rangle = &\left(\frac{\overline{\alpha}+i\overline{\beta}}{\sqrt{2}}\,\langle y;+| + \frac{\overline{\alpha}-i\overline{\beta}}{\sqrt{2}}\,\langle y;-|\right)\times \\ &\frac{\hbar}{2}\left(\frac{\alpha-i\beta}{\sqrt{2}}\,|y;+\rangle - \frac{\alpha+i\beta}{\sqrt{2}}\,|y;-\rangle\right) \\ &= \frac{i\hbar}{2}\left(-\overline{\alpha}\beta + \overline{\beta}\alpha\right) \end{aligned} \tag{24}$$

$$\begin{aligned} \langle S_x^2\rangle = &\left(\frac{\overline{\alpha}+\overline{\beta}}{\sqrt{2}}\,\langle x;+| + \frac{\overline{\alpha}-\overline{\beta}}{\sqrt{2}}\,\langle x;-|\right)\times \\ &\frac{\hbar^2}{4}\left(\frac{\alpha+\beta}{\sqrt{2}}\,|x;+\rangle + \frac{\alpha-\beta}{\sqrt{2}}\,|x;-\rangle\right) \\ &= \frac{\hbar^2}{4}\left(|\alpha|^2 + |\beta|^2\right) \end{aligned} \tag{25}$$

$$\begin{aligned} \langle S_y^2\rangle = &\left(\frac{\overline{\alpha}+i\overline{\beta}}{\sqrt{2}}\,\langle y;+| + \frac{\overline{\alpha}-i\overline{\beta}}{\sqrt{2}}\,\langle y;-|\right)\times \\ &\frac{\hbar^2}{4}\left(\frac{\alpha-i\beta}{\sqrt{2}}\,|y;+\rangle + \frac{\alpha+i\beta}{\sqrt{2}}\,|y;-\rangle\right) \\ &= \frac{\hbar^2}{4}\left(|\alpha|^2 + |\beta|^2\right) \end{aligned} \tag{26}$$



(a) Overview (b) side-view figure (c) side-view figure

Figure 1: Search for the maximum points

With Born's rule,

$$|\alpha|^2 + |\beta|^2 = 1, \tag{27}$$

we define β as

$$\beta \equiv \sqrt{1-\alpha^2} e^{i\delta},$$

where δ is a real number. Therefore,

$$\langle(\Delta S_x)^2\rangle = \frac{\hbar^2}{4}\left(1 - 4\alpha^2(1-\alpha^2)\cos^2\delta\right) \tag{28}$$

Similarly,

$$\langle(\Delta S_y)^2\rangle = \frac{\hbar^2}{4}\left(1 - 4\alpha^2(1-\alpha^2)\sin^2\delta\right) \tag{29}$$

Let us plug Eq.(28), Eq.(29) into Eq.(18), then we will get

$$\begin{aligned}\langle(\Delta S_x)^2\rangle \langle(\Delta S_y)^2\rangle &= (\langle S_x^2\rangle - \langle S_x\rangle^2)(\langle S_y^2\rangle - \langle S_y\rangle^2) \\ = \frac{\hbar^4}{16}\left(1 - 4\alpha^2(1-\alpha^2)\cos^2\delta\right)&\left(1 - 4\alpha^2(1-\alpha^2)\sin^2\delta\right) \\ &= \frac{\hbar^4}{16}\left[1 - 4\alpha^2(1-\alpha^2) + 4\alpha^4(1-\alpha^2)^2\sin^2 2\delta\right]\end{aligned} \tag{30}$$

Then I use "Grapher" to determine the maximum. I found that there is maximum once $\alpha = 0, \pm 1$ (see fig.(1-a) - fig.(1-c)). Then the maximum is

$$\langle(\Delta S_x)^2\rangle \langle(\Delta S_y)^2\rangle \Big|_{\alpha=0,\pm 1} = \frac{\hbar^4}{16} \tag{31}$$

On the other hand, we know

$$\begin{aligned}[S_x, S_y] &= i\hbar S_z \\ \Longrightarrow \frac{1}{4}|\langle[S_x,S_y]\rangle|^2 &= \frac{\hbar^2}{4}|\langle S_z\rangle|^2 \\ = \frac{\hbar^2}{4}\left(\frac{\hbar}{2}\right)^2|\alpha-\beta|^2 &= \frac{\hbar^4}{16}|\alpha-\beta|^2\end{aligned} \tag{32}$$

Therefore, when $\alpha = 0, \pm 1$,

$$\frac{1}{4}|\langle[S_x,S_y]\rangle|^2 = \frac{\hbar^4}{16} \tag{33}$$

I conclude that the uncertainty relation for S_x and S_y really holds. $\square$

3. (a) Suppose that $f(A)$ is a function of a Hermitian operator A with the property $A\,|a'\rangle = a'\,|a'\rangle$. Evaluate $\langle b''|f(A)|b'\rangle$ when the transformation matrix from the a' basis to the b' basis is known.

Proof.

$$\begin{aligned}\langle b''|f(A)|b'\rangle &= \sum_{i,j} \langle b''|a^{(i)}\rangle \langle a^{(i)}|f(A)|a^{(j)}\rangle \langle a^{(j)}|b'\rangle \\ &= \sum_{i,j} f(a^{(j)})\delta_{ij} \langle b''|a^{(i)}\rangle \langle a^{(j)}|b'\rangle \\ &= \sum_{i} f(a^{(i)}) \langle b''|a^{(i)}\rangle \langle a^{(i)}|b'\rangle \\ &= \sum_{i} f(a^{(i)}) \langle a''|U^{\dagger}|a^{(i)}\rangle \langle a^{(i)}|U|a'\rangle\end{aligned} \tag{34}$$

□

(b) Using the continuum analogue of the result obtained in (a), evaluate

$$\langle \mathbf{p}''|F(r)|\mathbf{p}'\rangle .$$

Simplify your expression as far as you can. Note that r is $\sqrt{x^2+y^2+z^2}$, where x, y, z are *operators*.

Proof. For the continuous spectra, the transformation function from the x-representation to the p-representation is

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ip'x'}{\hbar}\right) \tag{35}$$

By using the completeness relations

$$\int d^3x'\,|x'\rangle\langle x'| = 1, \tag{36}$$

$$\int d^3p'\,|p'\rangle\langle p'| = 1, \tag{37}$$

the $\langle \mathbf{p}''|F(r)|\mathbf{p}'\rangle$ could be rewritten as

$$\begin{aligned}\langle \mathbf{p}''|F(r)|\mathbf{p}'\rangle &= \iint d^3x'd^3x''\,\langle \mathbf{p}''|x''\rangle \langle x''|F(r)|x'\rangle \langle x'|\mathbf{p}'\rangle \\ &= \frac{1}{2\pi\hbar}\iint d^3x'd^3x''F(r')\delta(r'-r'')\exp\left[\frac{i(\mathbf{p}'\cdot\mathbf{r}'-\mathbf{p}''\cdot\mathbf{r}'')}{\hbar}\right] \\ &= \frac{1}{2\pi\hbar}\int d^3r'F(r')\exp\left[\frac{i(\mathbf{p}'-\mathbf{p}'')\cdot r'}{\hbar}\right]\end{aligned} \tag{38}$$

□

4. (a) Verify (7.39a) and (7.39b) for the expectation value of p and p^2 from the Gaussian wave packet (7.35).

Proof. By the following identity, we can verify the $\langle p\rangle$ and $\langle p^2\rangle$ without using the momentum-space wave function (7.42):

$$\langle\beta|p^n|\alpha\rangle = \int dx'\psi_\beta^*(x')(-i\hbar)^n\frac{\partial^n}{\partial x'^n}\psi_\alpha(x'). \tag{39}$$

Therefore,

$$\langle p\rangle = \int dx'\psi_\alpha^*(x')\left(-i\hbar\frac{\partial}{\partial x'}\right)\psi_\alpha(x'). \tag{40}$$

The Gaussian wave packet (7.35) is

$$\psi_\alpha(x') = \langle x'|\alpha\rangle = \left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\exp\left(ikx' - \frac{x'^2}{2d^2}\right). \tag{41}$$

Let us plug Eq.(41) into Eq.(40), we get

$$\begin{aligned}\langle p\rangle &= \frac{-i\hbar}{\sqrt{\pi}d}\int dx'\,\exp\left(-\frac{x'^2}{d^2}\right)\left(ik - \frac{x'}{d^2}\right)\\ &= \frac{\hbar k}{\sqrt{\pi}d}\int\exp\left(-\frac{x'^2}{d^2}\right)dx' + \frac{i\hbar}{\sqrt{\pi}d^3}\int x'\exp\left(-\frac{x'^2}{d^2}\right)dx'\end{aligned} \tag{42}$$

By Gaussian integral,

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}dx = e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}. \tag{43}$$

We know that

$$\int_{-\infty}^{\infty} dx'\exp\left(-\frac{x'^2}{d^2}\right) = \sqrt{\pi}d \tag{44}$$

Besides,

$$\int x'\exp\left(-\frac{x'^2}{d^2}\right)dx' = 0, \tag{45}$$

since the integrand is an odd function. Hence, after plugging Eq.(44) and Eq.(45) into Eq.(43), we get

$$\langle p\rangle = \frac{\hbar k}{\sqrt{\pi}d}\sqrt{\pi}d = \hbar k \tag{46}$$

Similarly, for $\langle p^2\rangle$,

$$\begin{aligned}\langle p^2\rangle &= \int dx'\psi_\alpha^*(x')(-i\hbar)^2\frac{\partial^2}{\partial x'^2}\psi_\alpha(x')\\ &= \frac{-\hbar^2}{\sqrt{\pi}d}\int dx'\exp\left(-ikx' - \frac{x'^2}{2d^2}\right)\frac{\partial^2}{\partial x'^2}\left[\exp\left(ikx' - \frac{x'^2}{2d^2}\right)\right]\\ &= \frac{-\hbar^2}{\sqrt{\pi}d}\int dx'\exp\left(-ikx' - \frac{x'^2}{2d^2}\right)\times\\ &\quad\frac{\partial}{\partial x'}\left[\left(ik - \frac{x'}{d^2}\right)\exp\left(ikx' - \frac{x'^2}{2d^2}\right)\right]\end{aligned} \tag{47}$$

The partial differentiation term is:

$$
\begin{aligned}
&\frac{\partial}{\partial x'}\left[\left(ik-\frac{x'}{d^2}\right)\exp\left(ikx'-\frac{x'^2}{2d^2}\right)\right] \\
&= ik\left(ik-\frac{x'}{d^2}\right)\exp\left(ikx'-\frac{x'^2}{2d^2}\right) \\
-\frac{1}{d^2}\exp\left(ikx'-\frac{x'^2}{2d^2}\right)&-\frac{x'}{d^2}\left(ik-\frac{x'}{d^2}\right)\exp\left(ikx'-\frac{x'^2}{2d^2}\right) \\
&= \left(-k^2-\frac{1}{d^2}-\frac{i2kx'}{d^2}+\frac{x'^2}{d^4}\right)\exp\left(ikx'-\frac{x'^2}{2d^2}\right).
\end{aligned} \tag{48}
$$

Therefore

$$
\begin{aligned}
\langle p^2\rangle &= \frac{-\hbar^2}{\sqrt{\pi}d}\int dx'\exp\left(-ikx'-\frac{x'^2}{2d^2}\right)\times \\
&\left(-k^2-\frac{1}{d^2}-\frac{i2kx'}{d^2}+\frac{x'^2}{d^4}\right)\exp\left(ikx'-\frac{x'^2}{2d^2}\right) \\
&= -\frac{\hbar^2}{\sqrt{\pi}d}\int dx'\left(-k^2-\frac{1}{d^2}+\frac{x'^2}{d^4}\right)\exp\left(-\frac{x'^2}{d^2}\right),
\end{aligned} \tag{49}
$$

where I've used Eq.(45). Then, after taking Gaussian integral Eq.(43) partial differentiation with respect to a, we get

$$
\int_{-\infty}^{\infty} x'^2\exp\left(-ax'^2\right)dx' = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}. \tag{50}
$$

Therefore,

$$
\int\frac{x'^2}{d^4}\exp\left(-\frac{x'^2}{d^2}\right)dx' = \frac{\sqrt{\pi}d^3}{2d^4} = \frac{\sqrt{\pi}}{2d}. \tag{51}
$$

Finally,

$$
\begin{aligned}
\langle p^2\rangle &= -\frac{\hbar^2}{\sqrt{\pi}d}\left[\left(-k^2-\frac{1}{d^2}\right)\sqrt{\pi}d+\frac{\sqrt{\pi}}{2d}\right] \\
&= -\hbar^2\left(-k^2-\frac{1}{2d^2}\right) \\
&= \frac{\hbar^2}{2d^2}+\hbar^2k^2
\end{aligned} \tag{52}
$$

□

(b) Evaluate the expectation value of p and p^2 using the momentum-space wave function (7.42).

Proof. From the Gaussian wave packet (7.35),

$$
\langle x'|\alpha\rangle = \left(\frac{1}{\pi^{1/4}\sqrt{d}}\right)\exp\left(ikx'-\frac{x'^2}{2d^2}\right), \tag{53}
$$

we can use Fourier transform to get the $\langle p'|\alpha\rangle$.

$$\begin{aligned}\langle p'|\alpha\rangle &= \int dx' \langle p'|x'\rangle \langle x'|\alpha\rangle \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int dx' \exp\left(\frac{-ip'x'}{\hbar}\right) \langle x'|\alpha\rangle \\ &= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{1}{\pi^{1/4}\sqrt{d}}\right) \int \exp\left(-\frac{x'^2}{2d^2} + i(k - \frac{p'}{\hbar})x'\right) dx'\end{aligned} \tag{54}$$

By Gaussian integral,

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}}. \tag{55}$$

We know that

$$\begin{aligned}\langle p'|\alpha\rangle &= \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{1}{\pi^{1/4}\sqrt{d}}\right) \exp\left(\frac{-(k-\frac{p'}{\hbar})^2}{2/d^2}\right) \sqrt{\frac{\pi}{1/2d^2}} \\ &= \frac{1}{\sqrt{2\pi\hbar}} \frac{\sqrt{2\pi}d}{\pi^{1/4}\sqrt{d}} \exp\left(\frac{-(k-\frac{p'}{\hbar})^2 d^2}{2}\right) \\ &= \sqrt{\frac{d}{\hbar\sqrt{\pi}}} \exp\left[\frac{-(p'-\hbar k)^2 d^2}{2\hbar^2}\right]\end{aligned} \tag{56}$$

Now we can verify $\langle p\rangle$ and $\langle p^2\rangle$ directly.

$$\begin{aligned}\langle p\rangle &= \int dp' \langle \alpha|p'\rangle p' \langle p'|\alpha\rangle \\ &= \frac{d}{\hbar\sqrt{\pi}} \int p' \exp\left[\frac{-(p'-\hbar k)^2 d^2}{\hbar^2}\right] dp'\end{aligned} \tag{57}$$

Let $u \equiv p' - \hbar k$,

$$\begin{aligned}\langle p\rangle &= \frac{d}{\hbar\sqrt{\pi}} \int_{-\infty}^{\infty} (u + \hbar k) \exp\left(-\frac{u^2 d^2}{\hbar^2}\right) du \\ &= \frac{kd}{\sqrt{\pi}} \int \exp\left(-\frac{u^2 d^2}{\hbar^2}\right) du \\ &= \frac{kd}{\sqrt{\pi}} \sqrt{\frac{\pi\hbar^2}{d^2}} = \hbar k.\end{aligned} \tag{58}$$

Similarly,

$$\langle p^2\rangle = \frac{d}{\hbar\sqrt{\pi}} \int p'^2 \exp\left[\frac{-(p'-\hbar k)^2 d^2}{\hbar^2}\right] dp' \tag{59}$$

Let $u \equiv p' - \hbar k$,

$$\begin{aligned}\langle p^2\rangle &= \frac{d}{\hbar\sqrt{\pi}} \int (u + \hbar k)^2 \exp\left(-\frac{u^2 d^2}{\hbar^2}\right) du \\ &= \frac{d}{\hbar\sqrt{\pi}} \int u^2 \exp\left(-\frac{u^2 d^2}{\hbar^2}\right) du + \frac{k^2 \hbar d}{\sqrt{\pi}} \int \exp\left(-\frac{u^2 d^2}{\hbar^2}\right) du \\ &= \frac{d}{\hbar\sqrt{\pi}} \frac{1}{2} \sqrt{\frac{\pi\hbar^6}{d^6}} + \frac{k^2 \hbar d}{\sqrt{\pi}} \sqrt{\frac{\pi\hbar^2}{d^2}} = \frac{\hbar^2}{2d^2} + \hbar^2 k^2\end{aligned} \tag{60}$$

□

5. (a) Prove the following:

   i. $\langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle$,

   ii. $\langle\beta|x|\alpha\rangle = \int dp'\phi^*_\beta(p')i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')$,

   where $\phi_\alpha(p') = \langle p'|\alpha\rangle$ and $\phi_\beta(p') = \langle p'|\beta\rangle$ are momentum-space wave functions.

Proof. We know the commutation relation of x and p,

$$[x, p] = i\hbar. \tag{61}$$

Therefore

$$\begin{aligned}\langle p'|[x,p]|\alpha\rangle &= \langle p'|xp - px|\alpha\rangle \\ &= \langle p'|xp|\alpha\rangle - \langle p'|px|\alpha\rangle\end{aligned} \tag{62}$$

Since momentum operator p is Hermitian,

$$\langle p'|[x,p]|\alpha\rangle = \langle p'|xp|\alpha\rangle - p'\langle p'|x|\alpha\rangle \tag{63}$$

With the commutation relation Eq.(64),

$$\Longrightarrow\; p'\langle p'|x|\alpha\rangle = \langle p'|xp|\alpha\rangle - i\hbar\langle p'|\alpha\rangle \tag{64}$$

Now I need to rewritten the $\langle p'|xp|\alpha\rangle$, by the identity relation,

$$\begin{aligned}\langle p'|xp|\alpha\rangle &= \int dx'\,\langle p'|x|x'\rangle\langle x'|p|\alpha\rangle \\ = \frac{-i\sqrt{\hbar}}{\sqrt{2\pi}}&\int x'\exp\left(-\frac{ip'x'}{\hbar}\right)\left(\frac{\partial}{\partial x'}\langle x'|\alpha\rangle\right)dx'\end{aligned} \tag{65}$$

Using integration by parts,

$$\begin{aligned}\langle p'|xp|\alpha\rangle = i\sqrt{\frac{\hbar}{2\pi}}\int\langle x'|\alpha\rangle\left(1-\frac{ip'x'}{\hbar}\right)\exp\left(-\frac{ip'x'}{\hbar}\right)dx' \\ = i\frac{\hbar}{\sqrt{2\pi\hbar}}\int\exp\left(-\frac{ip'x'}{\hbar}\right)\langle x'|\alpha\rangle\,dx' \\ -i\frac{\hbar}{\sqrt{2\pi\hbar}}\int\frac{ip'x'}{\hbar}\exp\left(-\frac{ip'x'}{\hbar}\right)\langle x'|\alpha\rangle\,dx'\end{aligned} \tag{66}$$

Since

$$\begin{aligned}\langle p'|\alpha\rangle &= \int dx'\,\langle p'|x'\rangle\langle x'|\alpha\rangle \\ = \frac{1}{\sqrt{2\pi\hbar}}&\int dx'\exp\left(-\frac{ip'x'}{\hbar}\right)\langle x'|\alpha\rangle,\end{aligned} \tag{67}$$

we can derive that

$$\begin{aligned}\langle p'|xp|\alpha\rangle &= i\hbar\langle p'|\alpha\rangle \\ +ip'\hbar\frac{\partial}{\partial p'}&\left[\frac{1}{\sqrt{2\pi\hbar}}\int\exp\left(-\frac{ip'x'}{\hbar}\right)\langle x'|\alpha\rangle\,dx'\right] \\ &= i\hbar\langle p'|\alpha\rangle + i\hbar p'\frac{\partial}{\partial p'}\langle p'|\alpha\rangle\,.\end{aligned} \tag{68}$$

Now, let us plug Eq.(68) into Eq.(64),

$$p'\langle p'|x|\alpha\rangle = i\hbar p'\frac{\partial}{\partial p'}\langle p'|\alpha\rangle \tag{69}$$

$$\Longrightarrow\ \langle p'|x|\alpha\rangle = i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle \tag{70}$$

For the second formula,

$$\langle\beta|x|\alpha\rangle = \iint dp'dp''\,\langle\beta|p'\rangle\langle p'|x|p''\rangle\langle p''|\alpha\rangle \tag{71}$$

Besides,

$$\begin{aligned}\langle p'|x|p''\rangle &= \iint dx'dx''\,\langle p'|x'\rangle\langle x'|x|x''\rangle\langle x''|p''\rangle \\ &= \iint dx'dx''\,\langle p'|x'\rangle\, x''\delta(x'-x'')\langle x''|p''\rangle \\ &= \int x'\langle p'|x'\rangle\langle x'|p''\rangle\,dx' \\ &= \int x'\exp\left(i\frac{(p''-p')x'}{\hbar}\right)dx' \\ = -i\hbar\frac{\partial}{\partial p''}&\int\exp\left(-i\frac{(p'-p'')x'}{\hbar}\right)dx' = -i\hbar\frac{\partial}{\partial p''}\delta(p'-p'')\end{aligned} \tag{72}$$

Plugging Eq.(72) into Eq.(71),

$$\langle\beta|x|\alpha\rangle = -i\hbar\iint dp'dp''\,\langle\beta|p'\rangle\langle p''|\alpha\rangle\frac{\partial}{\partial p''}\delta(p'-p'') \tag{73}$$

Using integration by parts,

$$\begin{aligned}\langle\beta|x|\alpha\rangle = -i\hbar\Bigg[&\int dp'\,\langle\beta|p'\rangle\langle p''|\alpha\rangle\,\delta(p'-p'')\Big|_{-\infty}^{\infty} \\ &-\iint dp'dp''\,\langle\beta|p'\rangle\,\delta(p'-p'')\frac{\partial}{\partial p''}\langle p''|\alpha\rangle\Bigg]\end{aligned} \tag{74}$$

$$\begin{aligned}\Longrightarrow\ \langle\beta|x|\alpha\rangle &= \int dp'\,\langle\beta|p'\rangle\, i\hbar\frac{\partial}{\partial p'}\langle p'|\alpha\rangle \\ &= \int dp'\,\phi_\beta^*(p')i\hbar\frac{\partial}{\partial p'}\phi_\alpha(p')\end{aligned} \tag{75}$$

□

(b) What is the physical significance of

$$\exp\left(\frac{ix\Xi}{\hbar}\right),$$

where x is the position operator and Ξ is some number with the dimension of momentum? Justify your answer.

Proof. Since the translation operator in position space is

$$T(\Delta x) = \exp\left(\frac{ip\Delta x}{\hbar}\right), \tag{76}$$

I guess its physical significance should be the translation operator in momentum space. After all, they have some symmetries of x and p. If it is the translation operator in momentum space, then

$$\exp\left(\frac{ix\Xi}{\hbar}\right)|p'\rangle = |p' + \Xi\rangle, \tag{77}$$

just like

$$\exp\left(\frac{ip\Delta x}{\hbar}\right)|x'\rangle = |x' + \Delta x\rangle. \tag{78}$$

That means, if we can prove the following equation, then everything is done!

$$p\left[\exp\left(\frac{ix\Xi}{\hbar}\right)|p'\rangle\right] = (p' + \Xi)\left[\exp\left(\frac{ix\Xi}{\hbar}\right)|p'\rangle\right] \tag{79}$$

There is one way to prove it. Consider the commutator

$$\begin{aligned}\left[p, \exp\left(\frac{ix\Xi}{\hbar}\right)\right] &= \left[p, \sum_k \frac{1}{k!}\left(\frac{ix\Xi}{\hbar}\right)^k\right] \\ &= \sum_k \frac{1}{k!}\frac{(i\Xi)^k}{\hbar^k}\left[p, x^k\right]\end{aligned} \tag{80}$$

By this theorem,

$$[A, B^n] = \sum_{i=0}^{n-1} B^i [A, B] B^{n-i-1}, \tag{81}$$

we can derive that

$$\begin{aligned}\left[p, \exp\left(\frac{ix\Xi}{\hbar}\right)\right] &= \sum_k \frac{1}{k!}\frac{(i\Xi)^k}{\hbar^k}\sum_{j=0}^{k-1} x^j [p, x] x^{k-j-1} \\ = \sum_k \frac{(i\Xi)^k}{k!\hbar^k}\sum_{j=0}^{k-1}(-i\hbar)x^{k-1} &= \sum_k \frac{(i\Xi)^k}{k!\hbar^k} k(-i\hbar)x^{k-1} \\ &= \sum_{k=1} \frac{1}{(k-1)!}\left(\frac{ix\Xi}{\hbar}\right)^{k-1}(-i\hbar)\left(\frac{i\Xi}{\hbar}\right) \\ &= \Xi\sum_{k=0}\frac{1}{k!}\left(\frac{ix\Xi}{\hbar}\right)^k = \Xi\exp\left(\frac{ix\Xi}{\hbar}\right)\end{aligned} \tag{82}$$

Let this commutator act on the $|p'\rangle$,

$$\begin{aligned}
\left[p, \exp\left(\frac{ix\Xi}{\hbar}\right)\right]|p'\rangle &= p\exp\left(\frac{ix\Xi}{\hbar}\right)|p'\rangle - \exp\left(\frac{ix\Xi}{\hbar}\right)p\,|p'\rangle \\
\therefore \Xi\exp\left(\frac{ix\Xi}{\hbar}\right)|p'\rangle + p'\exp\left(\frac{ix\Xi}{\hbar}\right)|p'\rangle &= p\exp\left(\frac{ix\Xi}{\hbar}\right)|p'\rangle \\
\implies p\left[\exp\left(\frac{ix\Xi}{\hbar}\right)|p'\rangle\right] &= (p' + \Xi)\left[\exp\left(\frac{ix\Xi}{\hbar}\right)|p'\rangle\right] \\
\implies \exp\left(\frac{ix\Xi}{\hbar}\right)|p'\rangle &= |p + \Xi\rangle
\end{aligned} \tag{83}$$

□