Mathematical Challenges

May 2018 - December 2021

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1. Let G be a group with 3129 elements. Prove it is solvable.

   Solution: $3129 = 3 \cdot 7 \cdot 149$ is the product of three distinct primes, hence solvable. See

   `https://www.physicsforums.com/threads/math-challenge-february-2021.999180/page-2#post-6462158`

2. $$I(a) := \int_0^1 \left(\frac{\log x}{a+1-x} - \frac{\log x}{a+x} \right) dx \ ; \ a \in \mathbb{C} \backslash [-1,0]$$

   Solution: Define $F\ :]0,1] \longrightarrow \mathbb{C}$ by

   $$F(x) := \frac{x \log x}{a+x} - \log(a+x)$$

   then

   $$\begin{aligned} F'(x) &= \frac{(a+x)(1+\log x) - x\log x}{(a+x)^2} - \frac{1}{a+x} \\ &= \frac{a + a\log x + x + x\log x - x\log x - a - x}{(a+x)^2} \\ &= \frac{a \log x}{(a+x)^2} \\ &\Longrightarrow \end{aligned}$$

   $$\begin{aligned} a\int_0^1 \frac{\log x}{(a+x)^2}\,dx &= F(1) - F(0^+) = -\log(a+1) - (-\log a) \\ &= \log a - \log(a+1) \end{aligned}$$

   If we define $G\ :]0,1] \longrightarrow \mathbb{C}$ by

   $$G(x) := \frac{x\log x}{a+1-x} + \log(a+1-x)$$

   then

   $$\begin{aligned} G'(x) &= \frac{(a+1-x)(1+\log x) + x\log x}{(a+1-x)^2} - \frac{1}{a+1-x} \\ &= \frac{(a+1)\log x}{(a+1-x)^2} \\ &\Longrightarrow \end{aligned}$$

   $$(a+1)\int_0^1 \frac{\log x}{(a+1-x)^2} = G(1) - G(0^+) = \log a - \log(a+1)$$

Putting those integrals together and integrating by a gives

$$\begin{aligned}
&\int \left(\int_0^1 \left(\frac{\log x}{(a+x)^2} - \frac{\log x}{(a+1-x)^2} \right) dx \right) da \\
&\int_0^1 \left(\int \left(\frac{\log x}{(a+x)^2} - \frac{\log x}{(a+1-x)^2} \right) da \right) dx \\
&\quad = \int_0^1 \left(-\frac{\log x}{a+x} + \frac{\log x}{a+1-x} + C \right) dx = I(a) + C' \\
&\quad = \int \left(\left(\frac{1}{a} - \frac{1}{a+1} \right) \cdot (\log a - \log(a+1)) \right) da \\
&\quad = \frac{1}{2} (\log a - \log(a+1))^2 + C''
\end{aligned}$$

and we get $I(a) = \frac{1}{2} (\log a - \log(a+1))^2 + C$. Note that the limit $\lim_{a\to\infty} I(a) = 0$, i.e. $C = 0$, hence

$$I(a) = \frac{1}{2} (\log a - \log(a+1))^2 = \frac{1}{2} \log^2 \frac{a}{a+1}$$

3. Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic not 2. Prove that

$$\mathfrak{A}(\mathfrak{g}) = \{\alpha \in \mathfrak{gl}(\mathfrak{g}) \mid [\alpha(X), Y] + [X, \alpha(Y)] = 0 \text{ for all } X, Y \in \mathfrak{g}\}$$

is a Lie algebra. Determine $\mathfrak{A}(\mathfrak{B})$ for the two-dimensional non-abelian Lie algebra $\mathfrak{B}$.

Solution: $\mathfrak{B} = \langle X, Y \mid [X, Y] = Y \rangle$. Let $\alpha(X) = aX + bY$ and $\alpha(Y) = cX + dY$. Then the only defining equation is

$$Y = [aX + bY, cX + dY] = (ad - bc)Y \Rightarrow ad - bc = 1 \Rightarrow \mathfrak{A}(\mathfrak{B}) \cong \mathfrak{sl}(2)$$

which is a Lie algebra. To prove the general case, we only have to verify that $[\alpha, \beta] := \alpha\beta - \beta\alpha \in \mathfrak{A}(\mathfrak{g})$ since $\mathfrak{A}(\mathfrak{g})$ is obviously a vector space, and the Jacobi identity holds for the commutator multiplication.

$$\begin{aligned}
[[\alpha, \beta](X), Y] + [X, [\alpha, \beta](Y)] &= [\alpha(\beta(X)), Y] - [\beta(\alpha(X)), Y] \\
&\quad + [X, \alpha(\beta(Y))] - [X, \beta(\alpha(Y))] \\
&= -[\beta(X), \alpha(Y)] + [\alpha(X), \beta(Y)] \\
&\quad - [\alpha(X), \beta(Y)] + [\beta(X), \alpha(Y)] \\
&= 0
\end{aligned}$$

4. Show that a path connected set is connected but not vice versa and not necessarily simply connected.

   Solution: A set A of a topological space X is path connected, if for any two points $a, b \in A$ there is a continuous curve $\gamma : [0,1] \to A$ with $\gamma(0) = a$ and $\gamma(1) = b$ that is entirely in A. If any such curve is null-homotopic, then A is simply connected. A is connected, if it cannot be written as disjoint union of two nonempty, open sets.

   Let $A \subseteq X$ be path connected and $A = U \dot{\cup} V$ a disjoint union of open sets $U, V \neq \emptyset$. Choose $a \in U, b \in V$ and $\gamma(t)$ a continuous path that connects the two points. Then $\gamma^{-1}(U), \gamma^{-1}(V) \subseteq \mathbb{R}$ are disjoint, open sets. Since

   $$[0,1] \subseteq \gamma^{-1}(U) \dot{\cup} \gamma^{-1}(V) = \gamma^{-1}(U \dot{\cup} V) = \gamma^{-1}(A)$$

   is a connected interval, we have w.l.o.g. $[0,1] \subseteq \gamma^{-1}(U)$. But this contradicts $V \ni v = \gamma(1) \notin U$.

   The opposite is false. Consider

   $$A = \underbrace{\{(0,y) \in \mathbb{R}^2 \,|\, -1 \leq y \leq 1\}}_{=:U} \cup \underbrace{\{(x, \sin(x^{-1})) \in \mathbb{R}^2 \,|\, x > 0\}}_{=:V}$$

   equipped with the standard Euclidean topology of $\mathbb{R}^2$, then there is no continuous path from $(0,0) \in U$ to $(\pi^{-1}, 0) \in V$ within A. However, every open neighborhood of $(0,0)$ always contains a point of V, hence A is connected.

   The unit disc $D := \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$ is path connected, i.e. connected, too. If we cut out $B := \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1/4\}$, then $D \backslash B$ is still path connected, but not null-homotopic.

5. $$\int_0^{\frac{\pi}{4}} \log(1 + \tan x)\, dx$$

   Solution:

   $$\begin{aligned} \sqrt{2}\cos\left(\frac{\pi}{4} - x\right) &= \sqrt{2}\cos\left(\frac{\pi}{4}\right)\cos(x) + \sqrt{2}\sin\left(\frac{\pi}{4}\right)\sin(x) \\ &= \cos(x) + \sin(x) = \cos(x) \cdot \frac{\cos(x) + \sin(x)}{\cos(x)} \\ &= \cos(x) \cdot (1 + \tan(x)) \\ \log(1 + \tan(x)) &= -\log(\cos(x)) + \log(\sqrt{2}) + \log\left(\cos\left(\frac{\pi}{4} - x\right)\right) \end{aligned}$$

Substitute $x \in [0, \pi/4]$ by $u = -x + \pi/4 \in [0, \pi/4]$ with $du = -dx$ so

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \log\left(\cos\left(\frac{\pi}{4} - x\right)\right)\,dx &= -\int_{\frac{\pi}{4}}^{0} \log(\cos(u))\,du \\ &= \int_0^{\frac{\pi}{4}} \log(\cos(x))\,dx\end{aligned}$$

hence

$$\int_0^{\frac{\pi}{4}} \log(1 + \tan x)\,dx = \int_0^{\frac{\pi}{4}} \log(\sqrt{2}) = \frac{\pi}{8}\log(2)$$

6. There are currently about $7,808,000,000$ people on earth. If we would enumerate them all, how many of them would have a prime number?

   Solution: A little bit more than the population of the United States of America:

$$\begin{aligned}\pi(x) &\sim \frac{x}{\log(x)} \\ \pi(7,808,000,000) &\approx 342,780,659\end{aligned}$$

7. Let $M = \mathbb{R}^2$ and $G = \mathbb{R}$ and consider the map

$$\psi(\varepsilon, (x, y)) := \left(\frac{x}{1 - \varepsilon x}, \frac{y}{1 - \varepsilon x}\right)$$

   defined on

$$U = \left\{(\varepsilon, (x, y)) \,|\, \varepsilon < \frac{1}{x} \text{ for } x > 0, \text{ or } \varepsilon > \frac{1}{x} \text{ for } x < 0\right\} \subseteq \mathbb{R} \times \mathbb{R}^2$$

   Show that ψ defines a local group action of G on the manifold M. Does it have a global counterpart on $\mathbb{R}^2$?

   Solution: Whenever defined, we get

$$\begin{aligned}\psi(\delta, \psi(\varepsilon, (x, y))) &= \psi\left(\delta, \left(\frac{x}{1 - \varepsilon x}, \frac{y}{1 - \varepsilon x}\right)\right) \\ &= \left(\frac{x/(1 - \varepsilon x)}{1 - \delta x/(1 - \varepsilon x)}, \frac{y/(1 - \varepsilon x)}{1 - \delta x/(1 - \varepsilon x)}\right) \\ &= \left(\frac{x}{1 - \varepsilon x - \delta x}, \frac{y}{1 - \varepsilon x - \delta x}\right) \\ &= \left(\frac{x}{1 - (\delta + \varepsilon)x}, \frac{y}{1 - (\delta + \varepsilon)x}\right) = \psi(\delta + \varepsilon, (x, y))\end{aligned}$$

There is no global counterpart of this local action, because

$$\lim_{\varepsilon \to 1/x} |\psi(\varepsilon, (x,y))| = \infty \text{ for } x \neq 0$$

Note: ψ occurs in the study of the heat equation. Its orbits consists of the straight rays emanating from the origin, and the origin itself. The action is regular on the punctured plane $\mathbb{R}^2 \backslash \{0\}$.

8. Give an example of a ring and a maximal ideal that isn't a prime ideal.

   Solution: If we have a commutative ring R with 1, then an ideal $P \trianglelefteq R$ is prime if R/P is an integral domain, and an ideal $M \trianglelefteq R$ is maximal if R/M is a field. Since all fields are integral domains, all maximal ideals are prime in this case. Hence we consider a ring without 1 and set $R = 2\mathbb{Z}$ and $M := 4\mathbb{Z}$.

   Let $R \neq M \trianglelefteq I \trianglelefteq R$ and $r = 2m \in I \backslash M$. Then m is odd, say $m = 2k+1$, so $I \supseteq IR = 4k\mathbb{Z} + 2\mathbb{Z} = R$, i.e. M is a maximal ideal.

   We have $2m \cdot 2n = 4nm \in M$, but $2m, 2n \notin M$ for n, m odd and neither is a unit, because R has none. This shows that M is not prime.

9. Let $U, V \subseteq \mathbb{C}$ open sets, $\varphi : U \longrightarrow V$ a holomorphic function, and $\gamma : [0,1] \longrightarrow U$ a closed, smooth path. Show that if γ is 0-homologue in U, then $\varphi \circ \gamma$ is 0-homologue in V.

   Solution: If γ is 0-homologue in U, then $\int_\gamma f(z)\,dz = 0$ by Cauchy's integral theorem. Thus

$$\begin{aligned}\int_{\varphi\circ\gamma} g(z)\,dz &= \int_0^1 g(\varphi(\gamma(t))) \cdot (\varphi \circ \gamma)'(t)\,dt \\ &= \int_0^1 (g \circ \varphi)(\gamma(t)) \cdot \varphi'(\gamma(t)) \cdot \gamma'(t)\,dt \\ &= \int_\gamma \underbrace{g \circ \varphi(z) \cdot \varphi'(z)}_{\text{holomorphic in U}}\,dz = 0\end{aligned}$$

   so $\varphi \circ \gamma$ is 0-homologue in V again by Cauchy's integral theorem (converse version).

   It is also possible to calculate it directly without using the backward direction of Cauchy's integral theorem. Let $z \in \mathbb{C}\backslash V$. Then

$$\begin{aligned}2\pi i \cdot \operatorname{ind}_{\varphi\circ\gamma}(z) &= \int_{\varphi\circ\gamma} \frac{1}{\zeta - z}\,d\zeta = \int_0^1 \frac{\varphi'(\gamma(t))\cdot\gamma'(t)}{\varphi(\gamma(t)) - z}\,dt \\ &= \int_\gamma \frac{\varphi'(\zeta)}{\varphi(\zeta)-z}\,d\zeta = 0\end{aligned}$$

The integration kernel $\dfrac{\varphi'(\zeta)}{\varphi(\zeta)-z}$ in the last integral is holomorphic on U since $z \notin V \supseteq \varphi(U)$ and we can use the forward direction of Cauchy's integral theorem.

10. Examine convergence:

$$\prod_{n=2}^\infty \left(1-\frac{1}{n}\right)\;,\;\prod_{n=3}^\infty \left(1-\frac{4}{n^2}\right)$$

Solution: All factors of both products are unequal zero. Set $P_n = \prod_{k=2}^n \left(1-\dfrac{1}{k}\right)$ for $n \geq 2$.

$$\begin{aligned}P_n &= \prod_{k=2}^n \frac{k-1}{k} \stackrel{\text{telescope}}{=} \frac{1}{n} \\ &\Longrightarrow \lim_{n\to\infty} P_n = \lim_{n\to\infty}\frac{1}{n} = 0 \\ &\Longrightarrow \prod_{n=2}^\infty \left(1-\frac{1}{n}\right)\end{aligned}$$

does not converge, since the limit cannot be zero by definition of multiplication. (The logarithm gives a divergent series.)

Set $Q_n = \prod_{k=3}^n \left(1-\dfrac{4}{k^2}\right)$ for $n \geq 3$.

$$\begin{aligned}Q_n &= \prod_{k=3}^n \frac{k^2-4}{k^2} = \prod_{k=3}^n \frac{k+2}{k}\cdot\frac{k-2}{k} = \prod_{k=3}^n \frac{k+2}{k+1}\cdot\frac{k+1}{k}\cdot\frac{k-1}{k}\cdot\frac{k-2}{k-1} \\ &= \prod_{k=3}^n \frac{k+2}{k+1}\cdot\prod_{k=3}^n\frac{k+1}{k}\cdot\prod_{k=3}^n\frac{k-1}{k}\cdot\prod_{k=3}^n\frac{k-2}{k} \\ &\stackrel{\text{telescope}}{=} \frac{n+2}{4}\cdot\frac{n+1}{3}\cdot\frac{2}{n}\cdot\frac{1}{n-1} = \frac{n^2+3n+2}{6n^2-6n}\end{aligned}$$

Hence

$$\prod_{n=3}^{\infty}\left(1-\frac{4}{n^2}\right)=\lim_{n\to\infty}Q_n=\lim_{n\to\infty}\frac{n^2+3n+2}{6n^2-6n}=\frac{1}{6}\,.$$

11. The Heisenberg algebra can be viewed as

$$\mathfrak{H}=\left\{\begin{bmatrix}0&x_1&x_3\\0&0&x_2\\0&0&0\end{bmatrix}\,:\,x_1,x_2,x_3\in\mathbb{R}\right\}.$$

Calculate $\exp(H)$ for a matrix $H\in\mathfrak{H}$.

Solution: Let e_{ij} be the matrix with 1 at position (i,j) and 0 elsewhere, and set $N:=x_1e_{12}+x_2e_{23}, M:=e_{13}$. Then $N^2=x_1x_2M, NM=MN=0$ and $M^2=0$. Thus

$$\begin{aligned}\exp(N+x_3M)&=\sum_{k=0}^{\infty}\frac{(N+x_3M)}{n!}\\&=\operatorname{Id}+(N+x_3M)+\frac{x_1x_2M}{2!}+\frac{0}{3!}+\ldots\\&=\begin{bmatrix}1&x_1&x_3+\dfrac{x_1+x_2}{2}\\0&1&x_2\\0&0&1\end{bmatrix}\in\left\{\begin{bmatrix}1&*&*\\0&1&*\\0&0&1\end{bmatrix}\right\}\end{aligned}$$

the Heisenberg group.

12.
$$\int_{-\infty}^{\infty}\frac{|\sin(\alpha x)|}{1+x^2}\,dx\ ,\ \alpha>0$$

Solution: $|\sin(\alpha x)|$ has the Fourier series

$$|\sin(\alpha x)|=\frac{2}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n-1}\right)\cos(2n\alpha x)$$

so

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{|\sin(\alpha x)|}{1+x^2}\,dx &= 2 + 2\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n-1}\right)\underbrace{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(2n\alpha x)}{1+x^2}\,dx}_{=e^{-2n\alpha}} \\
&= 2 + 2\sum_{n=1}^{\infty}\frac{(e^{-\alpha})^{2n}}{2n+1} - 2\sum_{n=1}^{\infty}\frac{(e^{-\alpha})^{2n}}{2n-1} \\
&= 2\sum_{n=0}^{\infty}\frac{(e^{-\alpha})^{2n}}{2n+1} - 2\sum_{n=0}^{\infty}\frac{(e^{-\alpha})^{2n+2}}{2n+1} \\
&= 2\left(e^{\alpha}-e^{-\alpha}\right)\sum_{n=0}^{\infty}\frac{(e^{-\alpha})^{2n+1}}{2n+1} \\
&= 4\sinh(\alpha)\,\operatorname{artanh}\left(e^{-\alpha}\right)
\end{aligned}$$

13. Show that $(n-1)! \equiv -1 \mod n$ holds if and only if n is prime. Determine the first two primes for which even $(p-1)! \equiv -1 \mod p^2$ holds.

Solution: We may assume $n > 2$. Let $n = p$ be prime. The polynomial $x^p - x = x(x^{p-1}-1)$ has exactly p roots in $\mathbb{Z}_p$. It has only simple roots, since $(x^p-x)' = px^{p-1}-1 \equiv -1 \mod p$, and there are at most p roots. Hence $x^p - x = x(x-1)\cdot\ldots\cdot(x-(p-1))$. Now

$$\begin{aligned}
&f(x) := x^{p-1}-1 = (x-1)\cdot\ldots\cdot(x-(p-1)) \\
&f(p) = p^{p-1}-1 = (p-1)\cdot\ldots\cdot(p-(p-1)) = (p-1)! \equiv -1 \mod p
\end{aligned}$$

Next let $(n-1)! \equiv -1 \mod n$ and $n = a\cdot b$ with $a,b>1$. Then

$$\begin{aligned}
a\,|\,(n-1)! &\Longrightarrow a\cdot c = (n-1)! \wedge n\cdot d = (n-1)!+1 \\
&\Longrightarrow ac = nd-1 = abd-1 \Longrightarrow 0 \equiv -1 \mod a \quad ↯
\end{aligned}$$

and n is prime.

The small faculties are $1, 2, 6, 24, 120, \ldots$ and we see, that $24 = 5! \equiv -1 \mod 5^2$. The next one is a bit harder to find. It is

$$\frac{(13-1)!+1}{13^2} = \frac{479{,}001{,}601}{169} = 2{,}834{,}329$$

Up to now, there is only one more so called Wilson prime known, namely 563. It is unknown whether there are more than that. If, then they are greater than $20{,}000{,}000{,}000{,}000$. The conjecture is, that there are infinitely many Wilson primes.

14. Determine all possible topologies on $X := \{a, b\}$, and which of them are homeomorphic. Give an example of a topological space with more than one element such that all sequences converge.

    Solution: We always have the discrete topology

    $$T_d = \{\emptyset, \{a\}, \{b\}, X\} = \{\{\}, \{a\}, \{b\}, \{a, b\}\}$$

    where all subsets are open, and indiscrete or trivial topology

    $$T_t = \{\emptyset, X\} = \{\{\}, \{a, b\}\}$$

    All topologies have to be close under all unions and all intersections because we have a finite set X. Thus we get also

    $$T_a = \{\emptyset, \{a\}, X\} = \{\{\}, \{a\}, \{a, b\}\}\ ,\ T_b = \{\emptyset, \{b\}, X\} = \{\{\}, \{b\}, \{a, b\}\}$$

    We want to prove, that only T_a and T_b are homeomorph by $f(a) = b\,,\, f(b) = a$. We have $f^{-1}(\emptyset) = \emptyset \in T_b$, $f^{-1}(\{a\}) = \{b\} \in T_b$, and $f^{-1}(X) = X \in T_b$. The same is true for the inverse function, so f and its inverse function are both continuous. Assume there would be a homeomorphism $f : T_t \to T_a$. Then $f^{-1}(\{a\}) \in \{\emptyset, X\}$. But f is bijective, i.e. the pre-image of a singleton has to be a singleton. This contradiction shows that $T_t \not\cong T_a$ and likewise $T_t \not\cong T_b$. Assume there would be a homeomorphism $f : T_a \to T_d$. Then $f^{-1}(\{b\})$ and $f^{-1}(\{a\})$ must both be singletons, and different, which is impossible. Hence $T_a \not\cong T_d$ and likewise $T_b \not\cong T_d$. For the same reason we get $T_t \not\cong T_d$.

    Consider T_a. Each sequence in (X, T_a) converges against b because in every neighborhood of b, which is only $X = \{a, b\}$ are all sequence elements.

15. Explain the difference between $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \ltimes \mathbb{Z}_3$. Is there also a group $\mathbb{Z}_2 \rtimes \mathbb{Z}_3$?

    Solution: All these expressions have $G = \{\mathbb{Z}_2, \mathbb{Z}_3\}$ as common underlying set. It has six elements. $\mathbb{Z}_2 \times \mathbb{Z}_3$ is the direct product with the multiplication

    $$(b, c) \cdot (b', c') = (b \cdot b, c \cdot c')$$

    $\mathbb{Z}_2 \ltimes \mathbb{Z}_3$ indicates that $\mathbb{Z}_3 \trianglelefteq G$ is a normal in the resulting group. It is called semi-direct product and has the multiplication

    $$(b, c) \cdot (b', c') = (b \cdot b', c \cdot \sigma(b)(c'))$$

with a homomorphism $\sigma : \mathbb{Z}_2 \longrightarrow \operatorname{Aut}(\mathbb{Z}_3)$ into the automorphism group of the normal subgroup. It can also be written $\mathbb{Z}_3 \rtimes_\sigma \mathbb{Z}_2$ to indicate the important influence of σ. We know that there are two groups of order 6, $\mathbb{Z}_6$ and S_3. The first is abelian, as is the product $\mathbb{Z}_2 \times \mathbb{Z}_3$. Since $(1,2)$ generates $(\mathbb{Z}_2 \times \mathbb{Z}_3, +)$, it is a cyclic group and we get

$$\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6 \,.$$

S_3 is not abelian and has subgroups $A_3 = \{(1),(123),(132)\}$ and $\{(1),(12)\}$, $\{(1),(13)\}$, $\{(1),(23)\}$ with three, two elements, resp.
Since $(123)(12)(132) = (23)$ the latter three groups are not normal in S_3. On the other hand $(12)(123)(12) = (132)$ so $A_3 \trianglelefteq S_3$ is a normal subgroup. It also shows that $\sigma : \mathbb{Z}_2 \longrightarrow \operatorname{Inn}(\mathbb{Z}_3) \trianglelefteq \operatorname{Aut}(S_3)$ defined by the conjugation with (12), i.e. $\sigma((1)) = (1)$ and $\sigma((12)) = \iota_{(12)}$ defines the required homomorphism, where $\iota_{(12)}$ is the inner automorphism conjugation with (12), and we get

$$G \cong A_3 \rtimes_\sigma \{(1),(12)\} \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$$

Assume $\mathbb{Z}_2 \rtimes_\sigma \mathbb{Z}_3$ would be a group with non-trivial homomorphism $\sigma : \mathbb{Z}_3 \to \operatorname{Aut}(Z_2)$. However, $\operatorname{Aut}(\mathbb{Z}_2) = \{1\}$ because $\alpha(1) = 1$ which also fixes the second element by the requirement that all $\alpha \in \operatorname{Aut}(\mathbb{Z}_2)$ are bijective. Hence $\sigma(\mathbb{Z}_3) = 1$ and σ would be trivial, i.e. the product a direct one. Thus the notation $\mathbb{Z}_2 \rtimes_\sigma \mathbb{Z}_3$ makes no sense.

16. Show that 16 and 33 are Størmer numbers, but no number $2n^2 > 2$ can be one, e.g. 32.

Solution: A Størmer number is a number for which there is a prime p such that $p > 2n$ and $p \,|\, (n^2 + 1)$.

$p = 257 = 16^2 + 1$ is a prime number and $p > 2 \cdot 16 = 32$.
$1090 = 33^2 + 1 = 2 \cdot 5 \cdot 109$ and $p = 109 > 2 \cdot 33 = 66$.
$1025 = 32^2 + 1 = 5^2 \cdot 41$ but $p = 41 < 2 \cdot 25 = 50$.
In general if for $n > 1$

$$p \,|\, (2n^2)^2 + 1 = 4n^4 + 1 = (2n^2 - 2n + 1)(2n^2 + 2n + 1)$$

then $p \,|\, (2n^2 \pm 2n + 1) < 2 \cdot 2n^2 = 4n^2$.

Størmer numbers n are exactly those numbers for which there isn't a linear combination

$$\operatorname{arccot} n = \sum_{k=1}^{n-1} a_k \cdot \operatorname{arccot} k \,,\ a_k \in \mathbb{Z}$$

Størmer numbers are therefore also called arc-cotangent irreducible numbers.

17. Consider a number n which is not a prime and

$$p\,|\,n \Longrightarrow p\,|\,\left(\frac{n}{p}-1\right)$$

E.g. $30 = 2\cdot 3\cdot 5$ is such a number, since $2\,|\,14\,,\ 3\,|\,9\,,\ 5\,|\,5$.

Show that n is square-free (all prime factors have exponent 1), and no semiprime (product of exactly two primes).

Solution: Numbers with those properties are called **Giuga numbers**. Giuga conjectured 1950 that a natural number is prime, if and only if

$$\sum_{k=1}^{n-1} k^{n-1} \equiv -1 \mod n.$$

The equation follows from Fermat's little theorem if n is prime:

$$k^{p-1}\equiv 1 \mod p \Longrightarrow \sum_{k=1}^{n-1} k^{n-1} \equiv (p-1)\cdot 1 = p-1 \equiv -1 \mod p$$

It is not clear whether the other direction holds, i.e. whether there are composite numbers with this property. It is only known that such a number has at least 10,000 decimal digits. A **Carmichael number** n is a composite, square-free number with the additional property

$$a^{n-1}\equiv 1 \mod n \text{ for all coprime } a\ ,\ (a,n)=1.$$

Korselt had shown 1899 that a number n is a Carmichael number, if it is not prime, square-free, and for all its prime divisors $p\,|\,n$ holds $(p-1)\,|\,(n-1)$. This result can be tightened to

$$p\,|\,n \Longrightarrow (p-1)\,|\,\left(\frac{n}{p}-1\right)$$

because $n-1=(n/p)-1+(p-1)(n/p)$, i.e. $n-1\equiv (n/p)-1$ mod $(p-1)$. This shows, that Carmichael numbers and Giuga numbers are closely related. Giuga's conjecture is indeed equivalent to:

No natural number is simultaneously Giuga and Carmichael number.

Another interesting theorem is, that n is a Giuga number, if and only if it is a composite, square-free number and

$$\sum_{p|n} \frac{1}{p} - \prod_{p|n} \frac{1}{p} \in \mathbb{N}.$$

This term equals 1 for all known Giuga numbers:

$$\begin{aligned}&30, 858, 1722, 66198, 2214408306, 24423128562, 432749205173838,\\ &14737133470010574, 550843391309130318,\\ &244197000982499715087866346,\\ &554079914617070801288578559178,\\ &1910667181420507984555759916338506\end{aligned}$$

Proof of the initial statements:

Let n be a Giuga number. Assume $p^2 \,|\, n$. Then $p \,|\, (n/p)$ and $p \nmid (n/p) - 1$ in contradiction to the defining condition.

Next assume $n = p \cdot q$ is a product of two primes with $p < q$. Then $(n/q) - 1 = p - 1 < p < q$. Hence $q \nmid ((n/q) - 1)$, again a contradiction to the defining condition.

18. Prove that path integrals in $\mathbb{R}^n$ over gradient vector fields depend only on starting and endpoint, and not on the path itself.

Solution: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ a gradient vector field, i.e. there is a differentiable function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ with

$$(\operatorname{grad} V)^\tau = f.$$

Let $\gamma : [a, b] \longrightarrow \mathbb{R}^n$ be a smooth curve with $f(a) = x_0, f(b) = x_1$. Then

$$\begin{aligned}\int_\gamma \langle f, ds \rangle &= \int_a^b \langle f(\gamma(t)), \gamma'(t) \rangle \, dt = \int_a^b \langle \operatorname{grad} V(\gamma(t))^\tau, \gamma'(t) \rangle \, dt\\ &= \int_a^b \operatorname{grad} V(\gamma(t)) \cdot \gamma'(t) \, dt = \int_a^b \left[\frac{d}{dt} V(\gamma(t)) \right] \, dt\end{aligned}$$

Now $V \circ \gamma : [a, b] \longrightarrow \mathbb{R}$ is a one-dimensional differentiable function. Thus we can apply the fundamental theorem of calculus and get

$$\int_\gamma \langle f, ds \rangle = V(\gamma(b)) - V(\gamma(a)) = V(x_1) - V(x_0)$$

which is independent of γ.

19. Let $P_0 = 0, P_1 = 1, P_n = 2P_{n-1} + P_{n-2}$ for all $n \in \mathbb{N}, n \geq 2$. Determine a closed form for P_n.

Solution: All sequence elements are greater or equal zero, and the sequence is strictly monotone increasing: $0 < P_{n-1} < 2P_{n-1} + P_{n-2} = P_n$ for $n > 1$. Therefore $2 \leq P_n/P_{n-1} = 2 + P_{n-2}/P_{n-1} < 3$ for $n > 1$. Since $(P_n/P_{n-1})_{n>1}$ is bounded from below and from above, it has at least one convergent subsequence by the theorem of Bolzano-Weierstraß.

If $(P_n/P_{n-1})_{n>1}$ converges, say against L, then

$$L := \lim_{n\to\infty} \frac{P_n}{P_{n-1}} = 2 + \lim_{n\to\infty} \frac{P_{n-2}}{P_{n-1}} = 2 + \frac{1}{L}$$

hence $L^2 - 2L - 1 = 0$ and $L = 1 \pm \sqrt{2}$. As each element of the sequence is greater than 2, we have $L = 1 + \sqrt{2}$.

If we only consider the recursion without initial conditions, then we have two possible limits and the two sequences

$$A_0 = a \wedge A_n := a(1+\sqrt{2})^n$$
$$B_0 = b \wedge B_n := b(1-\sqrt{2})^n$$

are solutions to $P_n = 2P_{n-1} + P_{n-2}$ and therewith all linear comninations $a(1+\sqrt{2})^n + b(1-\sqrt{2})^n$. With $P_0 = 0, P_1 = 1$ we get

$$P_0 = a + b$$
$$P_1 = a(1+\sqrt{2}) + b(1-\sqrt{2})$$
$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{2\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2\sqrt{2} \\ -1/2\sqrt{2} \end{bmatrix}$$

so with our given initial conditions we have

$$P_n = \frac{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}{2\sqrt{2}}\,.$$

It is easy to check that P_n fulfills the recursion. It is called Pell sequence and $1+\sqrt{2}$ the silver ratio. Finally, we have to prove that P_n/P_{n-1}

actually converges.

$$\begin{aligned}\frac{P_n}{P_{n-1}} &= \frac{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n}{\left(1+\sqrt{2}\right)^{n-1}-\left(1-\sqrt{2}\right)^{n-1}}\\ &= \frac{1-\left(\dfrac{1-\sqrt{2}}{1+\sqrt{2}}\right)^n}{\dfrac{1}{1+\sqrt{2}}-\left(\dfrac{(1-\sqrt{2})^{n-1}}{(1+\sqrt{2})^n}\right)} \stackrel{n\to\infty}{\longrightarrow} \frac{1-0}{\dfrac{1}{1+\sqrt{2}}-0} = 1+\sqrt{2}\end{aligned}$$

20. Find the irreducible minimal polynomial for

$$\mathbb{Q}\subseteq\mathbb{Q}\left(\sqrt[3]{\frac{9+\sqrt{69}}{18}}+\sqrt[3]{\frac{9-\sqrt{69}}{18}}\right).$$

Solution: Set $a:=\sqrt[3]{\dfrac{9+\sqrt{69}}{18}}$ and $b:=\sqrt[3]{\dfrac{9-\sqrt{69}}{18}}$. Then

$$\begin{aligned}a^2b &= \sqrt[3]{\frac{(9+\sqrt{69})^2\cdot(9-\sqrt{69})}{18^3}} = \sqrt[3]{\frac{12(9+\sqrt{69})}{18^3}}\\ ab^2 &= \sqrt[3]{\frac{12(9-\sqrt{69})}{18^3}}\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ &= 1+\frac{\sqrt[3]{108+12\sqrt{69}}}{6}+\frac{\sqrt[3]{108-12\sqrt{69}}}{6}\\ &= 1+\frac{\sqrt[3]{12}}{6}\left(\sqrt[3]{9+\sqrt{69}}+\sqrt[3]{9-\sqrt{69}}\right)\\ &= 1+\sqrt[3]{\frac{1}{18}}\left(\sqrt[3]{9+\sqrt{69}}+\sqrt[3]{9-\sqrt{69}}\right)\\ &= 1+a+b\end{aligned}$$

and the minimal polynomial is therefore $x^3-x-1\in\mathbb{Q}[x]$.

$\psi:=\sqrt[3]{\dfrac{9+\sqrt{69}}{18}}+\sqrt[3]{\dfrac{9-\sqrt{69}}{18}}$ is called **plastic number**. The designation plastic number is misleading and does not correspond to van der Laan's intention, because not the material plastic, but the spatial extent (in architecture) was decisive for the name "plastic". The other two solutions of $x^3 - x - 1 = 0$ are

$$-\frac{\psi}{2} \pm i\sqrt{\frac{3-\psi}{4\psi}}$$

which can be proven by long division and Vieta's formula. ψ is the limit of the Padovan sequence, which is defined by

$$P_n := P_{n-2} + P_{n-3}\,,\; P_0 = P_1 = P_2 = 1$$

One can prove that

$$\psi = \sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\ldots}}} \approx 1.324717957244746025960908854\ldots$$

21. Show that the embedding $\mathbb{S}^1 \longrightarrow \mathbb{R}^2 - \{0\}$ is a homotopy equivalence, and that $\mathbb{R} \longrightarrow \mathbb{R}^2 - \{0\}$ defined by $x \mapsto (x,1)$ is none.

    Solution: The homotopy inverse to $\mathbb{S}^1 \subseteq_\iota \mathbb{R}^2 - \{0\}$ is

    $$r\,:\,\mathbb{R}^2 - \{0\} \longrightarrow \mathbb{S}^1\,,\; x \longmapsto \frac{x}{\|x\|}$$

    so $r \circ \iota = id_{\mathbb{S}^1}$ and $\iota \circ r$ is homotope to the identity by

    $$H(x,t) := (1-t)\frac{x}{\|x\|} + tx\,.$$

    A homotopy equivalence induces an isomorphism of the fundamental groups. Now $\pi_1(\mathbb{R}^2 - \{0\}) = \pi_1(\mathbb{S}^1) = \mathbb{Z}$ while $\pi_1(\mathbb{R}) = \{1\}$, so $\mathbb{R}$ and $\mathbb{R}^2 - \{0\}$ cannot be homotopy equivalent.

22. Let $\emptyset \neq X$ be a set, $\mathcal{P}(X)$ its power set. Consider the following mappings

    $$\begin{aligned} f\,:\,X &\longrightarrow \mathcal{P}(X) \\ x &\longmapsto \{x\} \end{aligned}$$

    $$\begin{aligned} g\,:\,\mathcal{P}(X)\times\mathcal{P}(X) &\longrightarrow \mathcal{P}(X) \\ (A,B) &\longmapsto A \cup B \end{aligned}$$

    and decide whether they are injective, surjective, and calculate the fiber (pre-image) of the empty set.

Solution: f is injective because $f(x) = \{x\} = \{y\} = f(y)$ implies $x = y$. f is not surjective since $\emptyset \notin f(X)$. In particular $f^{-1}(\emptyset) = \emptyset$.

Let $x \in X$. Then

$$g((\{x\}, \emptyset)) = \{x\} \cup \emptyset = \{x\} = \emptyset \cup \{x\} = g((\emptyset, \{x\}))$$

which shows that g is not injective. However, g is surjective since $g((A, \emptyset)) = A \cup \emptyset = A$ for any $A \subseteq X$. The fiber of the empty set is $g^{-1}(\emptyset) = \{(\emptyset, \emptyset)\}$.

23. Find the smallest positive integer x that solves

$$\begin{aligned} x &\equiv 2 \mod 3 \\ x &\equiv 3 \mod 4 \\ x &\equiv 2 \mod 5 \end{aligned}$$

Solution: There is a solution x since $3, 4, 5$ are pairwise coprime by the Chinese remainder theorem, and all solutions are congruent modulo $M = 3 \cdot 4 \cdot 5 = 60$. The calculation is

$$\begin{aligned} 7 \cdot 3 + (-1) \cdot \frac{M}{3} &= 1 \Longrightarrow \alpha_1 = -20 \\ 4 \cdot 4 + (-1) \cdot \frac{M}{4} &= 1 \Longrightarrow \alpha_2 = -15 \\ 5 \cdot 5 + (-2) \cdot \frac{M}{5} &= 1 \Longrightarrow \alpha_3 = -24 \end{aligned}$$

which results in

$$x = 2 \cdot \alpha_1 + 3 \cdot \alpha_2 + 2 \cdot \alpha_3 = -133 \equiv 47 \mod M.$$

24. Let $\vec{u}, \vec{v}, \vec{w}$ be three different coplanar vectors of equal length, originating at a point O. Their endpoints define a triangle $\triangle UVW$. How can the barycenter S be found?

Solution: Let H be the point in which the heights intersect. $O = C$ is the circumcenter C per construction. Hence $\vec{OH} = \vec{u} + \vec{v} + \vec{w}$ by Sylvester's triangle theorem. On the other hand are O and H on the Euler straight of the triangle, and the Euler identity says $3\vec{S} = \vec{H} + 2\vec{C}$. Thus $3\vec{OS} = \vec{OH} + 2\vec{OC} = \vec{OH} = \vec{u} + \vec{v} + \vec{w}$ or $\vec{OS} = \frac{1}{3}(\vec{u} + \vec{v} + \vec{w})$.

25. Is a partially differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at some point x_0 also continuous at x_0?

Solution: The answer is no because partial differentials are only the directional differentials in coordinate direction. There is no information about any other direction.

Consider $x_0 = (0,0)$ and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as

$$f(x,y) := \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0 \\ 1, & \text{otherwise} \end{cases}$$

Then

$$\lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0 \Longrightarrow \partial_x f(0,0) = 0$$

and the same is obviously true for the symmetric case $\partial_y f(0,0) = 0$. Thus f is partially differentiable at x_0 but not continuous, e.g.

$$\lim_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n}\right) = \lim_{n\to\infty} 1 \neq 0 = f(0,0) = f(x_0)\,.$$

26. Let $\mathfrak{g}$ be the real Lie algebra generated by

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},\, A_2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\, A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Calculate its center $\mathfrak{Z}(\mathfrak{g}) = \{X \mid [A_i, X] = 0\,(i = 1,2,3)\}$, its commutator subalgebra $[\mathfrak{g},\mathfrak{g}]$, and a Cartan subalgebra $\mathfrak{h}$.

Solution: We observe that

$$\mathfrak{g} = \left\{ X = \sum_{i=1}^{3} x_i A_i \,\middle|\, \beta(X.v,w) + \beta(v,X.w) = 0 \text{ for all } v,w \in \mathbb{R}^3 \right\}$$

with respect to the bilinear form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, i.e. $\mathfrak{g} \cong \mathfrak{so}(3)$. Therefore $\mathfrak{g}$ is simple, which implies that $\mathfrak{Z}(\mathfrak{g}) = \{0\}$, and $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. Since $\dim \mathfrak{g} = 3$ we have $\dim \mathfrak{h} = 1$, which is Abelian and thus nilpotent. We claim that $\mathfrak{h} = \mathbb{R} \cdot A_1$. It remains to show that $\mathfrak{h}$ is self-normalizing. Say $\sum_{i=1}^3 x_i A_i \in N_{\mathfrak{g}}(A_1)$, i.e.

$$\begin{aligned} \mathfrak{h} \ni [X, A_1] &= x_2[A_2, A_1] + x_3[A_3, A_1] \\ &= x_2[e_{13} - e_{21}, e_{22} - e_{33}] + x_3[e_{12} - e_{31}, e_{22} - e_{33}] \\ &= x_2 e_{21} - x_2 e_{13} + x_3 e_{12} - x_3 e_{31} \\ &= -x_2 A_2 + x_3 A_3 \in \mathfrak{h} = \mathbb{R} \cdot A_1 \end{aligned}$$

where e_{ij} are the matrices with 1 at position (i, j) and 0 elsewhere. It follows $x_2 = x_3 = 0$ and $X \in \mathfrak{h}$, hence $\mathfrak{h}$ is self-normalizing.

27. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function and $g : [a, b] \longrightarrow \mathbb{R}$ integrable with $g(x) \geq 0$ for all $x \in [a, b]$. Then there is a $\xi \in [a, b]$ such that
$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$
Solution: f assumes its minimum m and its maximum M on $[a, b]$ since f is continuous. Thus $mg(x) \leq f(x)g(x) \leq Mg(x)$ and by monotony and linearity of the Riemann integral
$$m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx\,.$$
If $\int_a^b g(x)\,dx \neq 0$ then we have fo find $\xi \in [a, b]$ such that
$$f(\xi) = \frac{1}{\int_a^b g(x)\,dx}\int_a^b f(x)g(x)\,dx$$
With $g(x) \geq 0$ we have $\int_a^b g(x)\,dx > 0$ and
$$m = f(x_0) \leq \frac{1}{\int_a^b g(x)\,dx}\int_a^b f(x)g(x)\,dx \leq f(x_1) = M$$
and the intermediate value theorem for continuous functions applies, i.e. there is a $\xi \in [x_0, x_1] \subseteq [a, b]$ with the required property.

If $\int_a^b g(x)\,dx = 0$, then
$$0 = m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx = 0$$
and each element of $\xi \in [a, b]$ satisfies
$$0 = \int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx$$

28. Consider the circle segment above $A = (-1, 0)$ and $B = (1, 0)$ of
$$x^2 + \left(y + \frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}\,.$$

The point $P := \left(\dfrac{1}{\sqrt{3}}, 1-\dfrac{1}{\sqrt{3}}\right)$ lies on this segment. Calculate the height h of the circle segment, and $|AP|+|PB|$.

Solution: Set $C := (0,-\sqrt{3})$. Then $\triangle ABC$ is an equilateral triangle with side length 2. By van Schooten's theorem (a corollary of Ptolemy's theorem for concyclic quadrilaterals) we get $|AP|+|PB| = |PC|$. On the other hand

$$\begin{aligned}|PC| &= |\vec{PC}| = |\vec{PO}+\vec{OC}| = |\vec{OC}-\vec{OP}| \\ &= \left\| \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix} - \begin{bmatrix} 1/\sqrt{3} \\ 1-(1/\sqrt{3}) \end{bmatrix} \right\| \\ &= \sqrt{\frac{1}{3} + \left(-\frac{3}{\sqrt{3}} - 1 + \frac{1}{\sqrt{3}}\right)^2} \\ &= \sqrt{\frac{8+4\sqrt{3}}{3}} \approx 2.231\end{aligned}$$

The height of the segment is the diameter of the circumscribed circle minus the height of equilateral $\triangle ABC$, or the $y-$coordinate of the circle at $x=0$, i.e.

$$h = \frac{4}{\sqrt{3}} - \sqrt{3} = \frac{1}{\sqrt{3}}\,.$$

29. Let $\varphi : V \longrightarrow V$ a linear mapping. Prove

$$\ker(\varphi) \cap \operatorname{im}(\varphi) = \{0\} \Longleftrightarrow \ker(\varphi \circ \varphi) = \ker(\varphi)$$

Solution: $\ker(\varphi) \subseteq \ker(\varphi \circ \varphi)$ is always true since $\varphi(0) = 0$. Let $v \in \ker(\varphi \circ \varphi)$, so $\varphi(v) \in \operatorname{im}(\varphi) \cap \ker$. Hence if that intersection equals $\{0\}$, then $\ker(\varphi \circ \varphi) \subseteq \ker(\varphi)$.

Now assume $\ker(\varphi \circ \varphi) = \ker(\varphi)$ and $v \in \operatorname{im}(\varphi) \cap \ker(\varphi) = \{0\}$. Then there is a $w \in V$ such that $\varphi(w) = v$ and $\varphi(v) = 0$ and $(\varphi \circ \varphi)(w) = 0$. This means that $w \in \ker(\varphi)$ by assumption, i.e. $v = \varphi(w) = 0$, so $\operatorname{im}(\varphi) \cap \ker(\varphi) = \{0\}$.

30. Let A by a cylindric surface (without base or cover) that rotates around the z-axis and stands on the plane $\{z=0\}$, with radius $R > 0$ and height $h > 0$. Give a parameterization and calculate the surface integral

$$\int_A \langle F, n \rangle \, d^2 r$$

for the vector field $F : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $F(x, y, z) = (xz, yz, 123)$.

Solution: One possible parameterization is

$$\begin{aligned}\phi \,:\, [0, 2\pi) \times [0, h] &\longrightarrow A \\ (\varphi, z) &\longmapsto (R\cos\varphi, R\sin\varphi, z)\end{aligned}$$

The height of the cylinder is given by z, and for a fixed z we have a circle parallel to the plane $z = 0$ described by polar coordinates. ϕ is a bijection because every point on A has exactly one pair of parameters (ϕ, z).

We use this parameterization ϕ to calculate the surface integral.

$$\begin{aligned}\int_A \langle F, n\rangle \, d^2r &= \int_0^{2\pi}\int_0^h \langle F\circ\phi, n\rangle \, \|\partial_\varphi\phi \times \partial_z\phi\| \, dz\, d\varphi \\ &= \int_0^{2\pi}\int_0^h \langle F\circ\phi, \partial_\varphi\phi \times \partial_z\phi\rangle \, dz\, d\varphi \\ &= \int_0^{2\pi}\int_0^h \langle \begin{pmatrix} zR\cos\varphi \\ zR\sin\varphi \\ 123 \end{pmatrix}, \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \\ 0 \end{pmatrix}\rangle \, dz\, d\varphi \\ &= \int_0^{2\pi}\int_0^h zR^2\left(\cos^2\varphi + \sin^2\varphi\right) dz\, d\varphi \\ &= \int_0^{2\pi}\int_0^h zR^2 \, dz\, d\varphi = 2\pi R^2 \left[\frac{z^2}{2}\right]_{z=0}^{z=h} = h^2\pi R^2\end{aligned}$$

We could alternatively use Gauß's divergence theorem. This uses closed surfaces, so we have to consider base and cover. Let C be the volume of the cylinder, D_1 its base, and D_2 its cover. Then $\partial Z = A \cup D_1 \cup D_2$. Note that the two integrals over D_1 and D_2 cancel each other since the normal vectors n are parallel to the z-axis, but pointing into opposite directions, and the third component of F is constant.

$$\begin{aligned}\int_{D_1}\langle F, n\rangle \, d^2r + \int_{D_2}\langle F, n\rangle \, d^2r &= \int_{D_1}\langle F, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\rangle \, d^2r + \int_{D_2}\langle F, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\rangle \, d^2r \\ &= \int_{D_1} 123 \, d^2r + \int_{D_2} -123 \, d^2r \\ &= 123(\pi R^2 - \pi R^2) = 0\end{aligned}$$

Therefore

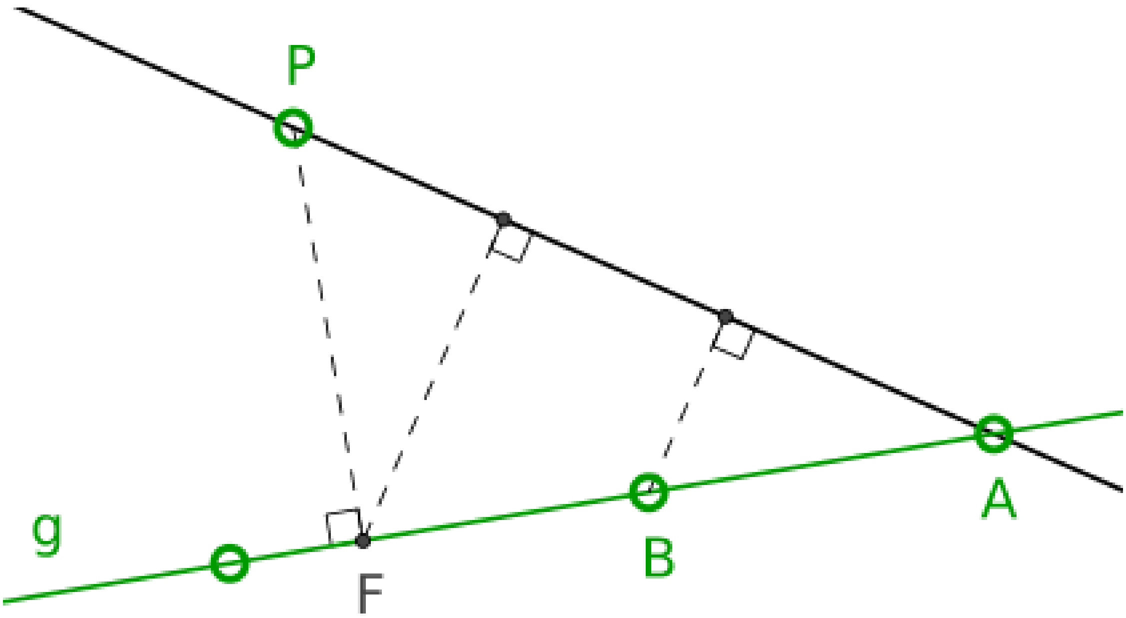
$$\int_{\partial Z}\langle F, n\rangle \, d^2r = \int_A \langle F, n\rangle \, d^2r + \int_{D_1}\langle F, n\rangle \, d^2r + \int_{D_2}\langle F, n\rangle \, d^2r = \int_A \langle F, n\rangle \, d^2r$$

Now we apply Gauß's divergence theorem

$$\begin{aligned}\int_{\partial Z}\langle F,n\rangle\,d^2r &= \int_Z \operatorname{div} F\,d^3r \\ &= \int_0^R\int_0^{2\pi}\int_0^h (\operatorname{div} F)\circ\phi\cdot|\det J_\phi|\,dz\,d\varphi\,d\rho \\ &= \int_0^R\int_0^{2\pi}\int_0^h 2z\rho\,dz\,d\varphi\,d\rho \\ &= 2\cdot\frac{R^2}{2}\cdot\frac{h^2}{2}\cdot 2\pi = h^2\pi R^2\end{aligned}$$

31. Let $\mathcal{P}$ be a finite set of points in a plane, that are not all collinear. Then there is a straight, that contains exactly two points.

    Solution: We consider the pairs (g,P) of a straight g through two points of $\mathcal{P}$ and a point $P\in\mathcal{P}-\{g\}$. Those points exist since not all points are collinear. The number of such pairs is finite because $\mathcal{P}$ is. Hence there is a pair (g,P), such that distance $\operatorname{dist}(g,P)$ is minimal. It remains to show that g doesn't contain a third point from $\mathcal{P}$.



`https://commons.wikimedia.org/wiki/File:Tibor_gallai_proof.svg`

Assume there are three such points. Let F be the basis point of the (minimal) perpendicular from P on g. Now there have to be two points

$A, B \in \mathcal{P}$ which lie on the same side of F by the pigeonhole principle. Say B is closer to F than P. Thus the distance $\operatorname{dist}(\overline{AP}, B)$ of B to the straight through A and P is smaller than the distance $\operatorname{dist}(g, P)$, which is as height in the right triangle $\triangle(APF)$ smaller than the cathetus $\overline{PF}$.

However, this contradicts the choice of the pair (g, P) as minimal distance $\operatorname{dist}(g, P)$, and our assumption that $\mathcal{P} \cap \{g\}$ contains three points.

2 November 2021

1. (a) Let $C \subseteq \mathbb{R}^n$ be compact and $f : C \longrightarrow \mathbb{R}^n$ continuous and injective. Show that the inverse $g = f^{-1} : f(C) \longrightarrow \mathbb{R}^n$ is continuous.

   (b) Let $S := \{x + tv \,|\, t \in (0,1)\}$ with $x, v \in \mathbb{R}^n$, and $f \in C^0(\mathbb{R}^n)$ differentiable for all $y \in S$. Show that there is a $z \in S$ such that
$$f(x+v) - f(x) = \nabla f(z) \cdot v\,.$$

   (c) Let $\gamma : [0,\pi] \longrightarrow \mathbb{R}^3$ be given as
$$\gamma(t) := \begin{pmatrix} \cos(t)\sin(t) \\ \sin^2(t) \\ \cos(t) \end{pmatrix}, \; t \in [0,\pi].$$
   Show that the length $L(\gamma) > \pi$.

Reason: Calculus.

Solution:

(a) Let $(y_k)_{k\in\mathbb{N}} \subseteq f(C)$ a sequence such that $\lim_{k\to\infty} y_k = y \in f(C)$. There is a unique sequence $(x_k)_{k\in\mathbb{N}} \subseteq C$ and $x \in C$ such that
$$f(x_k) = y_k \;(k \in \mathbb{N}) \,\wedge\, f(x) = y$$
since f is injective. Assume g is not continuous. Then there is a $\varepsilon > 0$ and a subsequence $(y_{k_m})_{m\in\mathbb{N}} \subseteq (y_k)_{k\in\mathbb{N}}$ with
$$|x_{k_m} - x| = |g(y_{k_m}) - g(y)| \geq \varepsilon \text{ for all } m \in \mathbb{N} \quad (*).$$
Because C is compact, there is another convergent subsequence $(x_{k_j})_{j\in\mathbb{N}} \subseteq (x_k)_{k\in\mathbb{N}}$ with $\lim_{j\to\infty} x_{k_j} = \tilde{x}$. By continuity of f follows
$$y = \lim_{j\to\infty} y_{k_j} = \lim_{j\to\infty} f(x_{k_j}) = f(\tilde{x}),$$
hence $f(x) = f(\tilde{x})$ and so $x = \tilde{x}$. In particular $\lim_{j\to\infty} x_{k_j} = x$ contradicting $(*)$.

(b) Define $g : [0,1] \longrightarrow \mathbb{R}$ by
$$g(t) := f(x + tv)\,,\; t \in [0,1]$$
Then g is continuous on $[0,1]$ and by assumption differentiable on $(0,1)$. Moreover $g(0) = f(x)$ and $g(1) = f(x+v)$. We get with the chain rule
$$g'(t) = \nabla f(x+tv) \cdot v \text{ for all } t \in (0,1)$$

and with the mean value theorem a $\tau \in (0,1)$ such that

$$g'(\tau) = \frac{g(1)-g(0)}{1-0} = \nabla f(x+\tau v)\cdot v = f(x) - f(x+v)$$

which had to be shown if we set $z := x + \tau v$.

(c)

$$\begin{aligned}|\gamma'(t)| &= \left| \begin{pmatrix} -\sin^2(t)+\cos^2(t) \\ 2\sin(t)\cos(t) \\ -\sin(t) \end{pmatrix} \right| \\ &= \sqrt{(\cos^2(t)-\sin^2(t))^2 + 4\sin^2(t)\cos^2(t) + \sin^2(t)} \\ &= \sqrt{\cos^4(t) + 2\sin^2(t)\cos^2(t) + \sin^4(t) + \sin^2(t)} \\ &= \sqrt{(\cos^2(t)+\sin^2(t))^2 + \sin^2(t)} = \sqrt{1+\sin^2(t)} \geq 1\end{aligned}$$

and in particular $|\gamma'(t)| > 1$ for $t \in (0,\pi)$. Hence

$$L(\gamma) = \int_0^\pi |\gamma'(t)|\,dt > \int_0^\pi dt = \pi\,.$$

2. Let g, h be two skew lines in a three-dimensional projective space $\mathcal{P} = \mathcal{P}(V)$, and P a point that is neither on g nor on h. Prove that there is exactly one straight through P that intersects g and h.

   Reason: Projective Geometry.

   Solution: The plane $\{P, g\}$ has to intersect h in a point Q for dimensional reasons. The straight $\{P, Q\}$ contains P, and intersects g and h because g and $\{P, Q\}$ are contained in a projective plane. This proves existence.

   Assume there were two transverse straights h_1, h_2 through P which both intersect g and h. Then $g \cap h_1 \neq g \cap h_2$ and $h \cap h_1 \neq h \cap h_2$ since $h_1 \neq h_2$. This means that the plane $\{h_1, h_2\}$ contains both lines g, h. However, this implies that g and h intersect, as we are in a projective space, contradicting the assumption that g, h are skew lines.

3. Let $(\mathcal{A}, e)$ be a unital C^*-algebra. A self-adjoint element $a \in \mathcal{A}$ is called positive, if its spectral values are:

$$\sigma(a) := \{\lambda \in \mathbb{C} \,|\, a - \lambda e \text{ is not invertible }\} \subseteq \mathbb{R}^+ := [0,\infty).$$

The set of all positive elements is written $\mathcal{A}_+$. A linear functional $f : \mathcal{A} \longrightarrow \mathbb{C}$ is called positive, if $f(a) \in \mathbb{R}^+$ for all positive $a \in \mathcal{A}_+$.

Prove that a positive functional is continuous.

Reason: C^*-algebras.

Solution: Firstly, we want to show that f is bounded on

$$M := \{a \in \mathcal{A}_+ \,|\, 0 \leq a \leq e\}.$$

If this wasn't the case, then there would be a sequence $(x_n)_{n\in\mathbb{N}} \subseteq M$ such that $\lim_{n\to\infty} f(x_n) = \infty$. Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ be any sequence of l^1 and set $x := \sum_{n=1}^{\infty} a_n x_n$. Note that the series converges absolutely.

$$\sum_{n=1}^{m} a_n x_n \leq x \Longrightarrow f\left(\sum_{n=1}^{m} a_n x_n\right) = \sum_{n=1}^{m} a_n f(x_n) \leq f(x)$$

Since $f(x_n) \geq 0$, $\sum_{n=1}^{\infty} a_n f(x_n)$ converges for any $(a_n)_{n\in\mathbb{N}} \in l^1$.

We can find a subsequence $(f(x_{n_k}))_{k\in\mathbb{N}} \subseteq (f(x_n))_{n\in\mathbb{N}}$ such that $f(x_{n_k}) \geq 2^{n_k}$ because $f(x_n) \stackrel{n\to\infty}{\longrightarrow} \infty$. Define $a \in l^1$ by $a_{n_k} = 2^{-n_k}$ and $a_n = 0$ otherwise. Then $\sum_{k=1}^{\infty} a_{n_k} f(x_{n_k})$ diverges, a contradiction. There is therefore a constant $C > 0$ such that $f(x) \leq C$ for all $x \in M$.

Let x be an arbitrary self-adjoint element with $\|x\| \leq 1$. Then $x = x_+ - x_-$ with positive elements $x_+, x_- \in \mathcal{A}_+$. From $x_\pm \leq |x| \leq e$ follows $x_\pm \in M$ and so $|f(x)| \leq |f(x_+)| + |f(x_-)| \leq 2C$ (Gelfand-Neumark).

If finally $x \in \mathcal{A}$ is arbitrary with $\|x\| \leq 1$, then $\left\|\frac{1}{2}(x \pm x^*)\right\| \leq 1$ and

$$|f(x)| \leq \left|f\left(\frac{1}{2}(x + x^*)\right)\right| + \left|f\left(\frac{1}{2}(x - x^*)\right)\right| \leq 4C$$

hence f is bounded with $\|f\| \leq 4C$ and therewith continuous.

4. Prove that the following groups F_1, F_2 are free groups:

   (a) Consider the functions α, β on $\mathbb{C} \cup \{\infty\}$ defined by the rules

   $$\alpha(x) = x + 2 \text{ and } \beta(x) = \frac{x}{2x+1}.$$

   The symbol ∞ is subject to such formal rules as $1/0 = \infty$ and $\infty/\infty = 1$. Then α, β are bijections with inverses

   $$\alpha^{-1}(x) = x - 2 \text{ and } \beta^{-1}(x) = \frac{x}{1-2x}.$$

   Thus α and β generate a group of permutations F_1 of $\mathbb{C} \cup \{\infty\}$.

(b) Define the group $F_2 := \langle A, B \rangle$ with

$$A := \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } B := \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Reason: Group Theory.

Solution: Let G be a group and $X \subseteq G$ a subset of G. Assume that each element $g \in G$ can be uniquely written in the form $g = x_1^{m_1} x_2^{m_2} \cdots x_s^{m_s}$ where $x_i \in X, s \geq 0, m_i \neq 0$, and $x_i \neq x_{i+1}$. Let F be the free group on X and $\sigma : X \to F$ the associated injection. By the mapping property of free groups, there is a homomorphism $\psi : F \to G$ such that $\psi \circ \sigma : X \to G$ is the inclusion map. Since $G = \langle X \rangle$, we see that ψ is surjective. It is injective by the uniqueness of the normal form. Thus $G \cong F$ is free over X.

(a) Observe that a nonzero power of α maps the interior of the unit circle $|z| = 1$ to the exterior and a nonzero power of β maps the exterior of the unit circle to the interior with 0 removed: the second statement is most easily understood from the equation $\beta(1/x) = 1/(x + 2)$. From this it is easy to see that no nontrivial reduced word in $\{\alpha, \beta\}$ can equal 1. Hence every element of F_1 has a unique expression as a reduced word. It follows from the above that F_1 is free on $\{\alpha, \beta\}$.

(b) Consider the linear fractional transformations $(ad - bc \neq 0)$

$$\begin{aligned} \lambda(a, b, c, d) \,:\, \mathbb{C} \cup \{\infty\} &\longrightarrow \mathbb{C} \cup \{\infty\} \\ x &\longmapsto \frac{ax + b}{cx + d} \end{aligned}$$

Now

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \stackrel{\varphi}{\longmapsto} \lambda(a, b, c, d)$$

is a homomorphism from $\mathrm{GL}(2, \mathbb{C})$ to the group of all linear fractional transformations of $\mathbb{C} \cup \{\infty\}$ in which A maps to α and B maps to β. Since no nontrivial reduced word in $\{\alpha, \beta\}$ can equal 1, the same is true of reduced words in $\{A, B\}$. Consequently the group F_2 is free on $\{A, B\}$.

5. We model the move of a chess piece on a chessboard as timely homogeneous Markov chain with the 64 squares as state space and the position of the piece at a certain (discrete) point in time as state. The transition matrix is given by the assumption, that the next possible state

is equally probable. Determine whether these Markov chains $M(\text{piece})$ are irreducible and aperiodic for (a) king, (b) bishop, (c) pawn, and (d) knight.

Reason: Markov Processes.

Solution: The king can reach every position on the board from any position, i.e. $M(\text{king})$ is irreducible. For any square we have

$$d(s_k) = \gcd\{n \geq 1 \,|\, (P^n)_{k,k} > 0\} = 1$$

i.e. that each state has the period 1 because the king can always get back to its starting position within 2 or 3 moves, so the greatest common divisor of all possible periods is 1. Hence $M(\text{king})$ is aperiodic.

$M(\text{bishop})$ is reducible, since we cannot reach all squares from a given starting position. With the same argument as above, we see that $M(\text{bishop})$ is aperiodic.

$M(\text{pawn})$ is reducible, since we cannot reach all squares from a given starting position, and periodic with $d(s_k) = \infty$ for all k, because a pawn can never return to its starting position.

The knight can always reach all squares from any starting point, so $M(\text{knight})$ is irreducible. For any square we have

$$d(s_k) = \gcd\{n \geq 1 \,|\, (P^n)_{k,k} > 0\} = 2$$

for the greatest common divisor of all periods with positive probability to return to the starting point. So $M(\text{knight})$ is periodic with period 2. The knight can always return in two moves. Since it changes the color of the square with every move, a returning path must always be of an even number of moves.

Summary:

piece	*king*	*bishop*	*pawn*	*knight*
irreducibility	1	0	0	1
period	1	1	∞	2

6. Prove that a n-dimensional manifold X is orientable if and only if

   (a) there is an atlas for which all chart changes respect orientation, i.e. have a positive functional determinant.

   (b) there is a continuous n-form which nowhere vanishes on M.

Reason: Manifolds.

Solution: Orientations of a vector space are elements from either of the two possible equivalence classes of ordered bases, i.e. $\det T \gtrless 0$ where T is the transformation matrix between bases.

An orientation μ of M is a choice of orientations μ_x for every tangent space $T_x(M)$, such that for all $x_0 \in M$ there is an open neighborhood $x_0 \in U \subseteq M$ and differentiable vector fields $\xi_1, \ldots, \xi_n$ on U with

$$[(\xi_1)_x, \ldots, (\xi_n)_x] = \mu_x$$

for all $x \in U$. The manifold M is called orientable, if an orientation for M can be chosen.

Let μ be an orientation on M. A chart (U, φ) with coordinates $x_1, \ldots, x_n$ is called positive oriented, if for all $x \in U$

$$\left[\left. \frac{\partial}{\partial x_1} \right|_x, \ldots, \left. \frac{\partial}{\partial x_n} \right|_x \right] = \mu_x$$

(a) If there is an orientation of X, we can find an atlas, that only contains positive oriented charts. Then all charts (U, φ) with $x \in U$ induce the same orientation on $T_x(M)$, hence they must have a positive functional determinant.

Let conversely be $(U_\alpha, \varphi_\alpha)$ an atlas of M, with positive functional determinant, i.e. $\det D(\varphi_\alpha \circ \varphi_\beta^{-1}) > 0$ on $\varphi_\beta(U_\alpha \cap U_\beta)$. Then all charts $(U_\alpha, \varphi_\alpha)$ with $x \in U_\alpha$ of $T_x(M)$ have the same orientation μ_x. Thus $\mu : x \longmapsto \mu_x$ defines an orientation on M because it is determined by an n-tuple of differential vector fields in a neighborhood of $x \in M$ for every chart.

(b) Let ω_0 be a nowhere vanishing n-form on M, and $(U_\iota, \varphi_\iota)_{\iota \in I}$ an atlas of M. Then there are nowhere vanishing continuous functions h_ι on $B_\iota := \varphi_\iota(U_\iota)$ for every $\iota \in I$, such that

$$(\omega_0)_\iota = h_\iota dx^1 \wedge \ldots \wedge dx^n.$$

We may assume that $h_\iota > 0$ on B_ι by changing the coordinate x^n to x^{-n} if necessary. Hence

$$\begin{aligned} h_\kappa dx^1 \wedge \ldots \wedge dx^n &= (\omega_0)_\kappa \\ &= \det\left(D\left(\varphi_\iota \circ \varphi_\kappa^{-1}\right)\right)(\omega_0)_\iota \\ &= \det\left(D\left(\varphi_\iota \circ \varphi_\kappa^{-1}\right)\right) \cdot h_\iota dx^1 \wedge \ldots \wedge dx^n \end{aligned}$$

Thus $\det\left(D\left(\varphi_\iota \circ \varphi_\kappa^{-1}\right)\right) > 0$, and the atlas is oriented.

Let M conversely be oriented with an oriented atlas (U_ι, φ_ι). Moreover let $(f_\iota)_{\iota\in I}$ a partition of unity for the cover $(U_\iota)_{\iota\in I}$. Then $dx^1 \wedge \ldots \wedge dx^n$ induces a n-form ω_ι on U_ι for every $\iota \in I$. We have $\omega_\iota = d_{\iota\kappa} \cdot \omega_\kappa$ with

$$d_{\iota\kappa} = \left(D\left(\varphi_\iota \circ \varphi_\kappa^{-1}\right)\right) \circ \varphi_\kappa > 0$$

on $U_\iota \cap U_\kappa$. The form $f_\iota \cdot \omega_\iota$ is a n-form on M with compact support $\operatorname{supp}(f_\iota \cdot \omega_\iota) \subseteq U_\iota$. We define $\omega_0 := \sum_{\iota\in I} f_\iota \cdot \omega_\iota$. Let $x \in M$, I_0 the finite set of all $\iota \in I$ with $x \in \operatorname{supp}(f_\iota)$ and $\kappa \in I_0$. Then

$$\begin{aligned}(\omega_0)_x &= \sum_{\iota\in I_0} f_\iota(x) \cdot (\omega_\iota)_x \\ &= \left(\sum_{\iota\in I_0} f_\iota(x) d_{\iota\kappa}(x)\right) \cdot (\omega_\kappa)_x\end{aligned}$$

From $f_\iota(x) > 0$, $\sum_{\iota\in I_0} f_\iota(x) = 1$, and $d_{\iota\kappa} > 0$, we get $(\omega_0)_x \neq 0$.

7. A topological vector space E over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ is normable if and only if it is Hausdorff and possesses a bounded convex neighborhood of $\vec{0}$.

Reason: Kolmogorov's Theorem.

Solution: If E is normable, and if $x \mapsto \|x\|$ is a norm on E that generates te topology, then $U := \{x \,:\, \|x\| \leq 1\}$ is a bounded and convex neighborhood of $\vec{0}$. As a metrizable space, E is also Hausdorff.

Conversely, suppose E is a topological vector space, Hausdorff, and possessing a bounded convex neighborhood $\vec{0} \in U$. Let $V \subseteq U$ be a balanced neighborhood of $\vec{0}$, i.e. $\lambda V \subseteq V \subseteq U$ for all $|\lambda| \leq 1$. Then for the convex hulls holds $\operatorname{conv}(V) \subseteq \operatorname{conv}(U) = U$, where $\operatorname{conv}(V)$ is a balanced convex neighborhood of $\vec{0}$. As any subset of a bounded set is bounded, we may assume (possibly by replacing U with $\operatorname{conv}(V)$) that U is a bounded, convex, balanced neighborhood of $\vec{0}$. For $x \neq \vec{0}$, we define

$$A(x) := \{\lambda \in \mathbb{K} \,:\, x \notin \lambda U\} \ni 0\,,\; A(\vec{0}) := \{0\}.$$

If $x \in E - \{\vec{0}\}$, we assert that $A(x)$ contains nonzero scalars. Indeed, if $\vec{0} \in V$ is a neighborhood, $x \notin V$, and $\alpha \in \mathbb{K} - \{0\}$ such that $U \subseteq \alpha V$ (U is bounded), then $\alpha^{-1}U \subseteq V$, i.e. $x \notin \alpha^{-1}U$, hence $\alpha^{-1} \in A(x)$.

We define now $x \longmapsto \|x\|$ for all $x \in E$ by the formula

$$\|x\| := \sup\{|\lambda| \,:\, \lambda \in A(x)\}.$$

Clearly $\|\vec{0}\| = 0$, $\|x\| > 0$ whenever $x \neq \vec{0}$, and $A(\mu x) = \mu A(x)$, i.e. $\|\mu x\| = |\mu| \cdot \|x\|$. To show that $x \longmapsto \|x\|$ is a norm, it remains to verify that $\|x\| < \infty$ and that the triangle inequality holds.

We first show that

$$\{\lambda \,:\, |\lambda| < \|x\|\} \subseteq A(x) \qquad (*)$$

If $x = \vec{0}$ then the left side of $(*)$ is empty. Assume $x \neq \vec{0}$, and suppose $|\lambda| < \|x\|$. Since $0 \in A(x)$ we can suppose $0 < |\lambda| < \|x\|$. Then, by definition of $\|x\|$, there exists a $\mu \in A(x)$ such that $0 < |\lambda| < |\mu| \leq \|x\|$. So $x \notin \mu U$, and since U is balanced, $\lambda U \subseteq \mu U$, i.e. $x \notin \lambda U$ or $\lambda \in A(x)$, which proves $(*)$.

Let $x \in E$. We claim that $\|x\| < \infty$. As $\lambda \mapsto \lambda x$ is continuous at $\lambda = 0$, and since $0 \cdot x = \vec{0}$, there exists an $\varepsilon > 0$ such that $\lambda x \in U$ whenever $|\lambda| \leq \varepsilon$, i.e. U is absorbent. Thus $1 \notin A(\lambda x)$ and by $(*)$ we have $1 \geq \|\lambda x\| = |\lambda| \cdot \|x\|$, which implies $\|x\| < \infty$.

Let $x, y \in E$. We claim that $\|x+y\| \leq \|x\| + \|y\|$. We may assume that $x, y, x+y \neq \vec{0}$. Given any $\varepsilon > 0$, it will suffice to show that

$$\|x+y\| \leq \|x\| + \|y\| + 2\varepsilon.$$

Let $\alpha := \|x\| + \varepsilon > \|x\|$ and $\beta := \|y\| + \varepsilon > \|y\|$. This means that $\alpha \notin A(x)$ and $\beta \notin A(y)$ by the definition of $\|x\|$, i.e. $x \in \alpha U$ and $y \in \beta U$. As U is convex, it follows that

$$x + y \in \alpha U + \beta U = (\alpha + \beta)U,$$

and thus $\alpha + \beta \notin A(x+y)$, and with $(*)$ that

$$\alpha + \beta = \|x\| + \|y\| + 2\varepsilon \geq \|x+y\|$$

and the triangle inequality holds.

Summarizing, $x \longmapsto \|x\|$ is a norm on E. It remains to show that this norm topology coincides with the given topology. Since both are compatible with the additive group structure, it is sufficient to verify that their neighborhood system at $\vec{0}$ coincide.

Suppose V is any neighborhood of $\vec{0}$ for the given topology. Choose a nonzero scalar λ such that $U \subseteq \lambda V$. If $\|x\| < |\lambda|^{-1}$ then $\lambda^{-1} \notin A(x)$ by the definition of $\|x\|$, i.e. $x \in \lambda^{-1}U \subseteq V$. Thus

$$\{x \,:\, \|x\| < \lambda^{-1}\} \subseteq V$$

which shows that V is a neighborhood of $\vec{0}$ for the norm topology.

Conversely, let V be any neighborhood of $\vec{0}$ for the norm topology. Choose $\varepsilon > 0$ so that $\{x : \|x\| \leq \varepsilon\} \subseteq V$. If $x \in \varepsilon U$, then $\varepsilon \notin A(x)$, therefore $\|x\| \leq \varepsilon$ by $(*)$, hence $\varepsilon U \subseteq V$. Since εU is a neighborhood of $\vec{0}$ for the given topology, so is V.

8. (a) Determine the minimal polynomial of $\pi + e \cdot i$ over the reals.
   (b) Show that $\mathbb{F} := \mathbb{F}_7[T]/(T^3 - 2)$ is a field, and calculate the number of its elements, $(T^2 + 2T + 4)(2T^2 + 5)$, and $(T + 1)^{-1}$.
   (c) Consider $P(X) := X^{7129} + 105X^{103} + 15X + 45 \in \mathbb{F}[X]$ and determine whether it is irreducible in case

$$\mathbb{F} \in \{\mathbb{Q}, \mathbb{R}, \mathbb{F}_2, \mathbb{Q}[T]/(T^{7129} + 105T^{103} + 15T + 45)\}$$

   (d) Determine the matrix of the Frobenius endomorphism in $\mathbb{F}_{25}$ for a suitable basis.

Reason: Galois Theory.

Solution:

(a) $(\pi + e \cdot i)(\pi - e \cdot i) = \pi^2 + e^2 \in \mathbb{R}$ and $(\pi + e \cdot i) + (\pi - e \cdot i) = 2\pi \in \mathbb{R}$ so we get by Vieta's formulas $X^2 - 2\pi X + \pi^2 + e^2 \in \mathbb{R}[X]$.

(b) We have

$$\{a^3 \mid a \in \mathbb{F}_7\} = \{0, 1, 6\} \not\ni 2$$

so $T^3 - 2$ has no roots in $\mathbb{F}_7$ and is thus irreducible, i.e. $\mathbb{F}$ is a field. The equivalence classes of $1, T, T^2$ build a basis, hence $|\mathbb{F}| = |\mathbb{F}_7|^3 = 343$. Now

$$\begin{aligned}(T^2 + 2T + 4)(2T^2 + 5) &= 2T^4 + 4T^3 + 13T^2 + 10T + 20 \\ &= 2T(T^3 - 2) + 4T + 4(T^3 - 2) + 8 + 13T^2 + 10T + 20 \\ &= 13T^2 + 14T + 28 \\ &= 6T^2\end{aligned}$$

Long division yields $(T^3 - 2) : (T + 1) = T^2 - T + 1$ remainder -3, i.e. $0 = T^3 - 2 = (T^2 + 6T + 1)(T + 1) - 3$ or $3 = (T^2 + 6T + 1)(T + 1)$. Now $3^{-1} = 5$ so we have $(T + 1)^{-1} = 5(T^2 + 6T + 1) = 5T^2 + 2T + 5$.

(c) We can use Eisenstein's criterion for $p = 5$ because

$$5|105\,,\ 5|15\,,\ 5|45\,,\ 25 \nmid 45\,,\ 5 \nmid 1$$

and conclude that $P(X)$ is irreducible over $\mathbb{Q}$.

$P(X) \in \mathbb{R}[X]$ is of odd degree and thus has a root in the real number field by the intermediate value theorem, so it cannot be irreducible.

The insertion homomorphism $X \mapsto 1$ on $\mathbb{F}_2[X]$ yields $P(1) = 166 \equiv 0 \mod 2$ so $P(X) \in \mathbb{F}_2[X]$ is reducible.

$\mathbb{F} = \mathbb{Q}[T]/(T^{7129} + 105T^{103} + 15T + 45)$ is a field. It t is the equivalence class of T then $P(t) = 0$ per construction, and $P(X) \in \mathbb{F}[X]$ is reducible.

(d) Because of

$$\{a^2 \,|\, a \in \mathbb{F}_5\} = \{0, 1, 4\} \not\ni 2$$

the polynomial $X^2 - 2 \in \mathbb{F}_5[X]$ is irreducible and

$$\mathbb{F}_{25} \cong \mathbb{F}_5[X]/(X^2 - 2)$$

so we may choose $1, x \subseteq \mathbb{F}_{25}$ as basis where x is the representative of the equivalence class of X. We have $1^5 = 1$ and $x^5 = x^2 \cdot x^2 \cdot x = 2 \cdot 2 \cdot x = 4x$. Hence the required matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

9. Let V and W be finite-dimensional vector spaces over the field $\mathbb{F}$ and $f : V \otimes_{\mathbb{F}} W \longrightarrow \mathbb{F}$ a linear mapping such that

$$\begin{aligned} \forall\, v \in V - \{0\} \quad \exists\, w \in W \,:\, f(v \otimes w) \neq 0 \\ \forall\, w \in W - \{0\} \quad \exists\, v \in V \,:\, f(v \otimes w) \neq 0 \end{aligned}$$

Show that $V \cong_{\mathbb{F}} W$.

Reason: Linear Algebra.

Solution: We get from the first condition that the mapping

$$\begin{aligned} V &\longrightarrow \operatorname{Hom}_{\mathbb{F}}(W, \mathbb{F}) = W^* \\ v &\longmapsto (w \longmapsto f(v \otimes w)) \end{aligned}$$

is injective. Since $\dim W < \infty$ we have

$$\dim_{\mathbb{F}} V \leq \dim_{\mathbb{F}} W^* = \dim_{\mathbb{F}} W.$$

The second condition yields by the analogue argument that $\dim_{\mathbb{F}} W \leq \dim_{\mathbb{F}} V$. Hence $\dim_{\mathbb{F}} V = \dim_{\mathbb{F}} W$ and $V \cong_{\mathbb{F}} W$.

10. Let $R := \mathbb{C}[X,Y]/(Y^2 - X^2)$. Describe $V_{\mathbb{R}}(Y^2 - X^2) \subseteq \mathbb{R}^2$, determine whether $\operatorname{Spec}(R)$ is finite, calculate the Krull-dimension of R, and determine whether R is Artinian.

    Reason: Commutative Algebra.

    Solution:

    (a) $V_{\mathbb{R}}(Y^2 - X^2)$ is the set of zeros of $Y^2 - X^2$, so

    $$\begin{aligned} V_{\mathbb{R}}(Y^2 - X^2) &= \{(x,y) \in \mathbb{R}^2 \,|\, y^2 - x^2 = 0\} \\ &= \{(x,y) \in \mathbb{R}^2 \,|\, y = x\} \cup \{(x,y) \in \mathbb{R}^2 \,|\, y = -x\} \end{aligned}$$

    and we get the diagonals in a Cartesian coordinate system.

    (b) $P_x := ([X - x], [Y - x]) \subseteq R$ is a prime ideal for every $x \in \mathbb{C}$ because

    $$R/P_x \cong_{\text{ring}} \mathbb{C}[X,Y]/(X - x, Y - x) \cong_{\text{ring}} \mathbb{C}$$

    is an integral domain. Furthermore, the well-defined insertion homomorphism $R \longrightarrow \mathbb{C}$ for $(x,x) \in \mathbb{C}^2$ shows that

    $$\forall\, x, y \in \mathbb{C} \,:\, x \neq y \Longrightarrow P_x \neq P_y\,.$$

    Therefore, $\operatorname{Spec}(R)$ cannot be finite.

    (c) Consider the canonical projection $\pi : \mathbb{C}[X,Y] \longrightarrow R$. Then

    $$\begin{aligned} \operatorname{Spec}(R) &\longrightarrow \{Q \in \operatorname{Spec}(\mathbb{C}[X,Y]) \,|\, (Y^2 - X^2) \subseteq Q\} \\ P &\longmapsto \pi^{-1}(P) \end{aligned}$$

    is bijective and compatible with inclusion of ideals. From

    $$(Y^2 - X^2) \subseteq (Y - X) \subsetneq (X,Y)$$

    we get $\dim R \geq 1$. Since $\dim \mathbb{C}[X,Y] = 2$, and $\{0\} \in \operatorname{Spec}(\mathbb{C}[X,Y])$ and $\{0\} \subsetneq (Y^2 - X^2)$ we conversely have $\dim R \leq 1$, hence $\dim R = 1$.

    (d) R is not Artinian because Artinian rings are zero-dimensional. Alternatively, we can also name a decreasing sequence of ideals, e.g. $(X^n)_{n\in\mathbb{N}}$, that doesn't become stationary.

11. (HS-1) Let $a \notin \{-1, 0, 1\}$ be a real number. Solve

    $$\frac{(x^4+1)(x^4+6x^2+1)}{x^2(x^2-1)^2} = \frac{(a^4+1)(a^4+6a^2+1)}{a^2(a^2-1)^2}\,.$$

Reason: Equation.

Solution: A solution x is a root of the polynomial

$$\begin{aligned} P(x) :=\, & a^2(a^2-1)^2(x^4+1)(x^4+6x^2+1) \\ & - x^2(x^2-1)^2(a^4+1)(a^4+6a^2+1) \\ & = (a^6-2a^4+a^2)x^8-(a^8+14a^4+1)x^6+ \\ & 2(a^8+7a^6+7a^2+1)x^4-(a^8+14a^4+1)x^2 \\ & + a^6-2a^4+a^2 \\ & = (x-a)(x+a)[(a^6-2a^4+a^2)x^6-(2a^6+13a^4+1)x^4 \\ & + (a^6+13a^2+2)x^2+(-a^4+2a^2-1)] \\ & = a^2(x-a)(x+a)\left(x-\frac{1}{a}\right)\left(x+\frac{1}{a}\right)\cdot \\ & \cdot (a^4-2a^2+1)x^4-2(a^4+6a^2+1)x^2+(a^4-2a^2+1) \end{aligned}$$

So $P(x)=0$ has the solutions $x=\pm a\,,\,\pm\dfrac{1}{a}$ and the solutions of

$$0=x^4-2\frac{a^4+6a^2+1}{a^4-2a^2+1}x^2+1.$$

The discriminant of this quadratic equation is

$$\begin{aligned} & \frac{(a^4+6a^2+1)^2}{(a^4-2a^2+1)^2}-1=16a^2\,\frac{(a^2+1)^2}{(a^2-1)^4} \\ & \Longrightarrow x^2=\frac{a^4+6a^2+1}{(a^2-1)^2}\pm 4a\,\frac{a^2+1}{(a^2-1)^2}=\frac{(a\pm 1)^4}{(a^2-1)^2} \end{aligned}$$

and all possible solutions are

$$\left\{a,-a,\frac{1}{a},-\frac{1}{a},\frac{a+1}{a-1},-\frac{a+1}{a-1},\frac{a-1}{a+1},-\frac{a-1}{a+1}\right\}$$

Another way is to observe

$$\frac{(x^4+1)(x^4+6x^2+1)}{x^2(x^2-1)^2}=\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)\left(\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2\right)$$

and note that a polynomial of degree 8 has at most 8 roots. However, in this case we would need an argument to show that all these roots are pairwise distinct.

12. (HS-2) Define a sequence $a_1, a_2, \ldots, a_n, \ldots$ of real numbers by

$$a_1 = 1\,,\; a_{n+1} = 2a_n + \sqrt{3a_n^2 + 1} \quad (n \in \mathbb{N})\,.$$

Determine all sequence elements that are integers.

Reason: Sequence.

Solution:

$$\begin{aligned}(a_{n+1} - a_n)^2 = 3a_n^2 + 1 &\Longrightarrow a_{n+1}^2 - 4a_{n+1}a_n + a_n^2 = 1 \quad (n \geq 1)\\ &\Longrightarrow a_n^2 - 4a_n a_{n-1} + a_{n-1}^2 = 1 \quad (n \geq 2)\\ &\Longrightarrow a_{n+1}^2 - a_{n-1}^2 - 4a_n(a_{n+1} - a_{n-1}) = 0\\ &\Longrightarrow (a_{n+1} - a_{n-1}) \cdot (a_{n+1} + a_{n-1} - 4a_n) = 0\end{aligned}$$

We also have $a_{n+1} > 2a_n > a_n > a_{n-1}$ for all $n \geq 2$ so that $a_{n+1} \neq a_{n-1}$, i.e. $a_{n+1} = 4a_n - a_{n-1}$. Since $a_1 = 1$ and $a_2 = 3$ are integers, this equation implies that all subsequent a_{n+1} are integers, too, so the entire sequence is in $\mathbb{Z}$.

13. (HS-3) For $n \in \mathbb{N}$ define

$$f(n) := \sum_{k=1}^{n^2} \frac{n - \left[\sqrt{k-1}\right]}{\sqrt{k} + \sqrt{k-1}}\,.$$

Determine a closed form for $f(n)$ without summation. The bracket means: $[x] = m \in \mathbb{Z}$ if $m \leq x < m + 1$.

Reason: Recursion.

Solution:

$$\begin{aligned}f(n) &= \sum_{m=0}^{n-1} \sum_{k=m^2+1}^{(m+1)^2} \frac{n - \left[\sqrt{k-1}\right]}{\sqrt{k} + \sqrt{k-1}}\\ &= \sum_{m=0}^{n-1} (n-m) \sum_{k=m^2+1}^{(m+1)^2} \left(\sqrt{k} - \sqrt{k-1}\right)\\ &= \sum_{m=0}^{n-1} (n-m) \cdot \left(\sqrt{(m+1)^2} - \sqrt{m^2}\right)\\ &= \sum_{m=0}^{n-1} (n-m) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\end{aligned}$$

14. (HS-4) Solve over the real numbers

$$\begin{array}{lll} (1) & x^4+x^2-2x & \geq 0 \\ (2) & 2x^3+x-1 & < 0 \\ (3) & x^3-x & > 0 \end{array}$$

Reason: Intervals.

Solution: All $x \geq 1$ violate the second equation, and all $x \leq -1$ violate the third, and

$$x^3 - x = x(x+1)(x-1) > 0$$

requires $x \notin [0,1)$ so we are left with $x \in I := (-1,0)$.

$$\begin{aligned} x^4+x^2-2x &> 0+0+0=0 \\ 2x^3+x-1 &< 0+0-1=-1<0 \\ x^3-x = x(x-1)(x+1) &= (-x)(-x+1)(x+1) > 0\cdot 0\cdot 0 = 0 \end{aligned}$$

and all $x \in I$ solve the equation system.

15. (HS-5) Let $f(x) := x^4-(x+1)^4-(x+2)^4+(x+3)^4$. Determine whether there is a smallest function value if $f(x)$ is defined (a) for integers, and (b) for real numbers. Which is it?

Reason: Domains.

Solution:

$$\begin{aligned} f(x) &= x^4-(x^4+4x^3+6x^2+4x+1)-(x^4+8x^3+24x^2+32x+16) \\ &\quad +(x^4+12x^3+54x^2+108x+81) \\ &= 24x^2+72x+64 = 24\left(x+\frac{3}{2}\right)^2+10 \geq 10 \end{aligned}$$

with $f(-3/2) = 10$. So $f(x)$ assumes its minimum value 10 over the reals. However, we have for all integers $x \geq -1$ and all integers $x \leq -2$

$$f(x) \geq 16 \text{ with } f(-1) = f(-2) = 16.$$

Hence $f(x)$ assumes its minimum once over the reals and twice over the integers with two different function values.

3 October 2021

1. Prove that $F\,:\,L^2([0,1]) \longrightarrow (C([0,1]),\|.\|_\infty)$ defined as

$$F(x)(t) := \int_0^1 (t^2+s^2)(x(s))^2\,ds$$

is compact.

Reason: Theorem of Arzelà-Ascoli.

Solution: $F(x)$ are continuous functions

$$F(x)(t) = t^2\int_0^1 (x(s))^2\,ds + \int_0^1 s^2(x(s))^2\,ds = a\cdot t^2+b$$

with $|a|,|b| \leq \|x\|_{L^2([0,1])}$. If $U \subseteq L^2([0,1])$ is bounded, then $F(U) \subseteq C([0,1])$ is bounded, too, and equicontinuous. Therefore $F(U)$ is relative compact in the supremum norm by the theorem of Arzelà-Ascoli. Hence F maps bounded sets on relative compact sets, i.e. F is a compact operator.

2. A project manager has n workers to finish the project. Let x_i be the workload of the $i-$th person, and

$$x \in S := \left\{ x \in \mathbb{R}^n \;|\; \sum_{i=1}^n x_i = 1\,,\, x_i \geq 0 \right\}$$

a possible partition of work. Let X_i be the set of partitions, which person i agrees upon. We may assume that he automatically agrees if $x_i = 0$, that X_i is closed, and that there is always at least one person which agrees to a given partition, i.e. $\bigcup_{i=1}^n X_i = S$.

Prove that there is one partition that all workers agree upon.

Reason: Project management lemma.

Solution: Let $F_i := \{x \in S \,|\, x_i = 0\} \subseteq X_i$ be the $i-$th side of the simplex S. We have to show that $\bigcap_{i=1}^n X_i \neq \emptyset$. Assume the contrary and set $d_i(x) := \operatorname{dist}(x, X_i)$. The distances are continuous functions and $\sum_{i=1}^n d_i(x) > 0$ per assumption. Define

$$f = (f_1,\ldots,f_n)\,:\,S \longrightarrow S$$
$$f_i(x) := \frac{d_i(x)}{\sum_{i=1}^n d_i(x)}$$

Since $\bigcup_{i=1}^n X_i = S$, every $x \in S$ is part of a set X_i, i.e. $f_i(x) = 0$ for some $i \in \{1, \ldots, n\}$. f maps thus onto the boundary $\delta S = \bigcup_{i=1}^n F_i$. We also have $f(F_i) \subseteq X_i$ and so $f(F_i) \subseteq F_i$. Let $g(x)$ be the point, which is the reflexion of $f(x)$ at the center $c := (n^{-1}, \ldots, n^{-1})$. Then

$$c = \lambda(x)g(x) + (1 - \lambda(x))f(x)\,,\; 0 < \lambda(x) < 1\,,\; f(x), g(x) \in \delta S$$

with continuous functions $\lambda(x)$ and $g(x) : S \longrightarrow \delta S$. For $x \in F_i$ we get $g(x) \notin F_i$ because of the reflexion. Hence $g(x) \neq x$ for all $x \in \delta S$. Interior points of S aren't fixed points either, because they are mapped onto the boundary. This means that the continuous function $g(x)$ has no fixed point, in contradiction to Brouwer's fixed point theorem, and the assumption $\bigcap_{i=1}^n X_i = \emptyset$ was wrong.

3. Assume the axiom schema of separation for any predicate $P(x)$

$$\forall A\,:\,\exists M\,:\,\forall x\,:\,(x \in M \Longleftrightarrow x \in A \wedge P(x))$$

Show that $|A| < |\mathcal{P}(A)|$ where $\mathcal{P}(A)$ is the power set of A.

Reason: Cantor's theorem.

Solution: $x \longmapsto \{x\}$ is an injective function from A to $\mathcal{P}(A)$, so $|A| \leq |\mathcal{P}(A)|$. We need to show that there is no surjective function. Assume

$$f\,:\,A \twoheadrightarrow \mathcal{P}(A)$$

is surjective. Set $M := \{x \in A \,|\, x \notin f(a)\}$. Then M is a set by the axiom scheme of separation, and thus $M \in \mathcal{P}(A)$. Since f is onto, there is an element $a \in A$ such that $f(a) = M$. Hence by definition of f and M

$$a \in f(a) = M \Longleftrightarrow a \notin f(a)$$

This shows that the assumption about the existence of a surjective function f is false, and in particular $|A| < |\mathcal{P}(A)|$.

4. Let $\sigma_1, \ldots, \sigma_n$ be homomorphisms from a group G into the multiplicative group $\mathbb{F}^*$ of a field $\mathbb{F}$. Show that they are $\mathbb{F}-$linearly independent if and only if they are pairwise distinct.

Reason: Dedekind's independence theorem.

Solution: If the σ_i are linearly independent, then they are certainly pairwise distinct, so assume the σ_i are all distinct. We proceed by induction on n.

Let $n = 1$ and $c\sigma_1 = 0$. Then $c\sigma_1(G) = 0$ and since $G \neq \emptyset$ there is a $g \in G$ such that $c\sigma_1(g) = 0$. As $\operatorname{im}(\sigma_1) \subseteq \mathbb{F}^*$ which has no zero divisors we conclude $c = 0$ and σ_1 is linearly independent. Assume $n > 1$ and that the statement is true for $n-1$ homomorphisms. Let $\sum_{j=1}^{n} c_j\sigma_j = 0$ for some $c_j \in \mathbb{F}^*$. We know from $\sigma_1 \neq \sigma_n$ that there is an element $g_0 \in G$ such that $\sigma_1(g_0) \neq \sigma_n(g_0)$.

$$
\begin{aligned}
&0 = \sum_{j=1}^{n} c_j\sigma_j(x) \quad \forall\, x \in G \\
&\Longrightarrow 0 = \sum_{j=1}^{n} c_j\sigma_j(g_0x) = c_1\sigma_1(g_0)\sigma_1(x) + \sum_{j=2}^{n} c_j\sigma_j(g_0)\sigma_j(x) \\
&\Longrightarrow 0 = \sigma_1(g_0)\sum_{j=1}^{n} c_j\sigma_j(x) = c_1\sigma_1(g_0)\sigma_1(x) + \sum_{j=2}^{n} c_j\sigma_1(g_0)\sigma_j(x) \\
&\Longrightarrow 0 = \sum_{j=2}^{n} c_j(\sigma_j(g_0) - \sigma_1(g_0))\sigma_j(x) \\
&\Longrightarrow c_j(\sigma_j(g_0) - \sigma_1(g_0)) = 0 \quad \forall\, j > 1 \\
&\Longrightarrow c_n(\sigma_n(g_0) - \sigma_1(g_0)) = 0 \\
&\Longrightarrow c_n = 0 \\
&\Longrightarrow 0 = \sum_{j=1}^{n-1} c_j\sigma_j \\
&\Longrightarrow c_1 = \ldots = c_{n-1} = 0
\end{aligned}
$$

The statement is already true for semigroups G.

5. Prove that general Heisenberg (Lie-)algebras $\mathfrak{H}$ are nilpotent.

   Reason: Engel's theorem.

   Solution: The general $n-$dimensional Heisenberg group ($n \geq 3$) is the linear algebraic group of matrices of the form $\begin{bmatrix} 1 & \vec{a}^\tau & b \\ 0 & \mathbb{I}_{n-2} & \vec{c} \\ 0 & 0 & 1 \end{bmatrix} = \exp\left(\begin{bmatrix} 0 & \vec{a}^\tau & b \\ 0 & 0_{n-2} & \vec{c} \\ 0 & 0 & 0 \end{bmatrix}\right)$. The matrices in the argument of the exponential function form their tangent space. They build a nilpotent associative algebra. To see that it is also nilpotent as a Lie algebra we set E_{ij} to be the matrix with 1 at position (i, j) and zeros elsewhere. Then

$\{E_{12},\ldots,E_{1n},E_{2n},\ldots,E_{(n-1)n}\}$ form a basis of $\mathfrak{H}$.

$$\begin{aligned}
X &= \sum_{j=2}^{n} x_j E_{1j} + \sum_{i=2}^{n-1} y_i E_{in} \\
[X,E_{1k}] &= \sum_{j=2}^{n} x_j [E_{1j},E_{1k}] + \sum_{i=2}^{n-1} y_i [E_{in},E_{1k}] = -y_k E_{1n} \\
[X,E_{kn}] &= \sum_{j=2}^{n} x_j [E_{1j},E_{kn}] + \sum_{i=2}^{n-1} y_i [E_{in},E_{kn}] = x_k E_{1n} \\
[X,E_{1n}] &= 0
\end{aligned}$$

This shows that all linear transformations $\operatorname{ad}(X)$ are nilpotent, hence $\mathfrak{H}$ is a nilpotent Lie algebra by Engel's theorem.

6. Prove that the polynomial $\mathbb{N}_0^2 \stackrel{P}{\longrightarrow} \mathbb{N}_0$ defined as

$$P(x,y) = \frac{1}{2}\left((x+y)^2 + 3x + y\right)$$

is a bijection.

Reason: Theorem of Fueter-Pólya.

Solution:

$$\begin{aligned}
2P(x,y) &= (x+y)(x+y+1) + 2x = x^2 + 2xy + y^2 + 3x + y \\
\nabla_{(a,b)}(2P) &= (2x+2y+3, 2x+2y+1)_{(a,b)} = (2a+2b+3, 2a+2b+1) \\
\nabla_{(a,b)}(2P)(u,v) &= 2au + 2bu + 3u + 2av + 2bv + v
\end{aligned}$$

Hence $2P(x,y)$ is strictly increasing in any direction of the first quadrant and at any point in its domain, i.e. in particular injective.

$2P(0,y) = y^2 + y$ and $2P(x,0) = x^2 + 3x$. Let $N \in \mathbb{N}_0$ and

$$m := \max\{x \in \mathbb{N}_0 \,|\, x^2 + x \leq 2N\}.$$

Then $2N < (m+1)^2 + m + 1 = m^2 + 3m + 2$ or $2N \leq m^2 + 3m + 1$. The right side is always odd, whereas the left is even. Hence we may conclude that

$$m^2 + m \leq 2N \leq m^2 + 3m.$$

Set

$$2N = m^2 + 3m - k \text{ with } k \in \{0, 2, 4, \ldots, 2m\}$$

$$\begin{aligned}2P\left(m-\frac{k}{2},\frac{k}{2}\right) &= \left(m-\frac{k}{2}\right)^2+2\left(m-\frac{k}{2}\right)\frac{k}{2}+\left(\frac{k}{2}\right)^2+3\left(m-\frac{k}{2}\right)+\frac{k}{2}\\ &= m^2+3m-k=2N\end{aligned}$$

and $P(x,y)$ is surjective.

It can be shown in a rather complicated proof, that $P(x,y)$ and $P(y,x)$ are the only quadratic real polynomials that enumerate $\mathbb{N}^2$ in such a way (Theorem of Fueter-Pólya). It is not known whether the requirement quadratic can be dropped.

7. Prove that the spectrum of every element of a complex Banach algebra B with 1 is nonempty. Conclude that if B is a division ring, then $B\cong\mathbb{C}$.

   Reason: Theorem of Gelfand-Mazur.

   Solution: Assume there is an element $a\in B$ such that all $a-\lambda 1$ are invertible ($\lambda\in\mathbb{C}$). Then we have for two distinct numbers $\lambda\neq\mu$

$$\begin{aligned}(a-\lambda 1)^{-1}(\lambda-\mu)(a-\mu 1)^{-1} &= (a-\lambda 1)^{-1}\left[(a-\mu 1)-(a-\lambda 1)\right](a-\mu 1)^{-1}\\ &= [(a-\lambda 1)^{-1}(a-\mu 1)-1](a-\mu 1)^{-1}\\ &= (a-\lambda 1)^{-1}-(a-\mu 1)^{-1}\end{aligned}$$

   Let $f\,:\,B\longrightarrow\mathbb{C}$ be an arbitrary homomorphism from B^*. Then

$$\frac{f((a-\lambda 1)^{-1})-f((a-\mu 1)^{-1})}{\lambda-\mu}=f((a-\lambda 1)^{-1}(a-\mu 1)^{-1})$$

   The right-hand side exists for $\mu\to\lambda$ because f and all algebraic operations including the inversion in B are continuous. Hence $\lambda\stackrel{\varphi}{\longmapsto} f((a-\lambda 1)^{-1})$ is holomorph on $\mathbb{C}$. Furthermore

$$\lim_{|\lambda|\to\infty}\|(a-\lambda 1)^{-1}\|=0$$

   so φ is bounded and vanishes at infinity. Now φ is constant by Liouville's theorem, i.e. identically zero. Since $f\in B^*$ has been chosen arbitrarily, the theorem of Hahn-Banach yields that $(a-\lambda 1)^{-1}=0$ which is impossible for an invertible element.

8. Let $X_1,\ldots,X_n$ be independent random variables, such that almost certain $a_i\leq X_i-E(X_i)\leq b_i$, and let $0<c\in\mathbb{R}$. Prove that

$$\Pr\left(\sum_{i=1}^n(X_i-E(X_i))\geq c\right)\leq\exp\left(\frac{-2c^2}{\sum_{i=1}^n(b_i-a_i)^2}\right).$$

Reason: Hoeffding inequality.

Solution: Let's prove the Markov-Chebyshev inequality first. Let (Ω, Σ, ν) be a measure space, $f : \Omega \longrightarrow \mathbb{R}_0^+$ a measurable function, and $\varepsilon, p > 0$ real numbers. Then

$$\int_\Omega f^p \, d\nu \geq \int_{x \,|\, f(x) \geq \varepsilon} f^p \, d\nu \geq \int_{x \,|\, f(x) \geq \varepsilon} \varepsilon^p \, d\nu = \varepsilon^p \nu \left(x \,|\, f(x) \geq \varepsilon \right)$$

This leads to the exponential version

$$\begin{aligned} \Pr(X \geq a) &= \Pr(\exp(X) \geq \exp(a)) \\ &\leq \inf_{p>0} \frac{1}{\exp(ap)} \int_{\mathbb{R}} \exp(pX) \, d\Pr = \inf_{p>0} \frac{E(\exp(pX))}{\exp(pa)} \end{aligned}$$

With $\nu = \Pr$, $f = |X - E(X)|$ and $p = 2$ we get the simple version

$$\Pr(|X - E(X)| \geq k\sigma) \leq \frac{1}{k^2}$$

where $\sigma^2 = Var(X)$. This can also be seen directly by conditional probabilities

$$\begin{aligned} \sigma^2 &= E((X - E(X))^2) \\ &= E((X - E(X))^2 \,|\, k\sigma \leq |X - E(X)|) \cdot \Pr(k\sigma \leq |X - E(X)|) \\ &+ E((X - E(X))^2 \,|\, k\sigma > |X - E(X)|) \cdot \Pr(k\sigma > |X - E(X)|) \\ &\geq (k\sigma)^2 \cdot \Pr(k\sigma \leq |X - E(X)|) + 0 \cdot \Pr(k\sigma > |X - E(X)|) \\ &= k^2\sigma^2 \Pr(k\sigma \leq |X - E(X)|) \end{aligned}$$

For the matter of convenience we set $Y_i := X_i - E(X_i)$ so $E(Y_i) = 0$. Moreover, consider for $z > 0$ the strictly monotone increasing function $x \mapsto \exp(zx)$ on the real numbers.

We get from the exponential version of the Markov-Chebyshev inequality

$$\Pr\left(\sum_{i=1}^n Y_i \geq c \right) \leq \inf_{z>0} \frac{E\left(\exp\left(z \sum_{i=1}^n Y_i\right)\right)}{\exp(zc)} \leq \frac{\prod_{i=1}^n E(\exp(zY_i))}{\exp(zc)}$$

The real exponential function is convex, so by the given conditions

$$\exp(zY_i) = \exp\left(\frac{b_i - Y_i}{b_i - a_i} z a_i + \frac{Y_i - a_i}{b_i - a_i} z b_i \right) \leq \frac{b_i - Y_i}{b_i - a_i} \exp(z a_i) + \frac{Y_i - a_i}{b_i - a_i} \exp(z b_i)$$

and with $E(Y_i)=0$ and

$$\begin{aligned}
E(\exp(zY_i)) &\leq \frac{b_i}{b_i-a_i}\exp(za_i)-\frac{a_i}{b_i-a_i}\exp(zb_i)\\
&=\exp(-u_i\lambda_i)\left((1-\lambda_i)+\lambda_i\exp(u_i)\right)\\
&\left[1-u_i\lambda_i+\frac{u_i^2\lambda_i^2}{2}-\frac{u_i^3\lambda_i^3}{3!}\pm\ldots\right]\cdot\\
&\cdot\left[1+u_i\lambda_i+\frac{u_i^2\lambda_i}{2}+\frac{u_i^3\lambda_i}{3!}+\ldots\right]\\
&=\left[1+u_i\lambda_i+\frac{u_i^2\lambda_i}{2}+\frac{u_i^3\lambda_i}{3!}+\ldots\right]\\
&-\left[u_i\lambda_i+u_i^2\lambda_i^2+\frac{u_i^3\lambda_i^2}{2}+\ldots\right]\\
&+\left[\frac{u_i^2\lambda_i^2}{2}+\frac{u_i^3\lambda_i^3}{2}+\ldots\right]\\
&=1+\frac{u_i^2}{2}\underbrace{\left(\lambda_i-\lambda_i^2\right)}_{\leq 1/4}+\underbrace{O(\lambda_i u_i^3)}_{<0}\leq 1+\frac{\mathrm{u_i^2}}{8}\leq\exp\left(\frac{\mathrm{u_i^2}}{8}\right)
\end{aligned}$$

with $\lambda_i=-\dfrac{a_i}{b_i-a_i}$, $u_i=z(b_i-a_i)$.

Summing up the results, we have

$$\Pr\left(\sum_{i=1}^n Y_i\geq c\right)\leq\frac{\prod_{i=1}^n\exp(u_i^2/8)}{\exp(zc)}=\exp\left(-zc+\sum_{i=1}^n\frac{u_i^2}{8}\right)$$

which leads by the choice $z:=\dfrac{4c}{\sum_{i=1}^n(b_i-a_i)^2}$ to

$$\begin{aligned}
\Pr\left(\sum_{i=1}^n Y_i\geq c\right)&\leq\exp\left(-\frac{4c^2}{\sum_{i=1}^n(b_i-a_i)^2}+\frac{z^2}{8}\sum_{i=1}^n(b_i-a_i)^2\right)\\
&=\exp\left(\frac{-32c^2+16c^2}{8\sum_{i=1}^n(b_i-a_i)^2}\right)\\
&=\exp\left(\frac{-2c^2}{\sum_{i=1}^n(b_i-a_i)^2}\right)
\end{aligned}$$

9. Let $G\subseteq\mathbb{C}$ be a non-empty, open, connected subset, and f,g holomorphic functions on G. Show that the following statements are equivalent:

   (a) $f(z)=g(z)$ for all $z\in G$.

(b) $\{z \in G \,|\, f(z) = g(z)\}$ has a limit point.

(c) There is a $z \in G$ such that $f^{(n)}(z) = g^{(n)}(z)$ for all $n \in \mathbb{N}_0$.

Reason: Identity theorem.

Solution: Holomorphic function are analytic, i.e. can locally be represented by their Taylor series.

$(a) \Longrightarrow (b)$ is obvious since any point in G is a limit point of G.

$(b) \Longrightarrow (c)$. Let $z_0 \in G$ be a limit point of the set of coincidence points. W.l.o.g. we assume $z_0 = 0$. If (c) wasn't true, then there is a minimal $N \in \mathbb{N}_0$ such that $f^{(N)}(0) \neq g^{(N)}(0)$. We then have in a neighborhood of 0

$$f(z) - g(z) = z^N \underbrace{\sum_{n=0}^{\infty} \frac{f^{(N+n)}(0) - g^{(N+n)}(0)}{(N+n)!} z^n}_{=:h(z)}$$

and $\{z \in G \,|\, h(z) = 0\} = \{z \in G \,|\, f(z) = g(z)\}$ since $h(z)$ is continuous. In particular $0 = h(0) = \dfrac{f^{(N)}(0) - g^{(N)}(0)}{N!}$ in contradiction to the minimality of N.

$(c) \Longrightarrow (a)$. It is sufficient to show that

$$A := \{z \in G \,|\, f^{(n)}(z) = g^{(n)}(z) \;\forall\, n \in \mathbb{N}_0\}$$

is non-empty, open and closed since G is connected. $A \neq \emptyset$ by condition (c). A is also closed because

$$A = \bigcap_{i=0}^{\infty} \{z \in G \,|\, f^{(n)}(z) = g^{(n)}(z)\} = \left(f^{(n)} - g^{(n)}\right)^{-1}(\{0\})$$

it is the union of preimages of a closed set under a continuous function. $f - g$ is an analytic function and as such equal to its Taylor series in a neighborhood of $z \in A$, i.e. identically zero. However, this neighborhood is entirely contained in A, i.e. A is open.

10. Prove that π^2 is irrational.

Reason: Calculus.

Solution: Assume $\pi^2 = \dfrac{p}{q}$ with $p, q \in \mathbb{N}$ and define

$$P_n(x) := \frac{x^n(1-x)^n}{n!} \quad , \quad n!P_n(x) = \sum_{k=n}^{2n} c_k x^k \in \mathbb{Z}[x]$$

Then

$$P_n^{(j)}(0) = \begin{cases} 0 & \text{if } j < n \wedge j > 2n \\ \dfrac{j!c_j}{n!} \in \mathbb{Z} & \text{if } n \leq j \leq 2n \end{cases}$$

and $P_n^{(j)}(x) = (-1)^j P_n^{(j)}(1-x)$ so $P_n^{(j)}(1) = (-1)^j P_n^{(j)}(0) \in \mathbb{Z}$, too. Set

$$Q_n(x) := q^n \left(\pi^{2n} P_n(x) - \pi^{2n-2} P_n''(x) \pm \ldots + (-1)^n \pi^0 P_n^{(2n)}(x)\right)$$

We already know that $Q_n(0), Q_n(1) \in \mathbb{Z}$.

$$\begin{aligned} &\frac{d}{dx}\left(Q_n'(x)\sin(\pi x) - \pi Q_n(x)\cos(\pi x)\right) = \left(Q_n''(x) + \pi^2 Q_n(x)\right)\sin(\pi x) \\ &\quad = q^n \pi^{2n+2} P_n(x)\sin(\pi x) = p^n \pi^2 P_n(x)\sin(\pi x) \\ &\quad \Longrightarrow \\ &p^n \pi \int_0^1 P_n(x)\sin(\pi x)\,dx = \left[\frac{Q_n'(x)\sin(\pi x)}{\pi} - Q_n(x)\cos(\pi x)\right]_0^1 \\ &\quad = Q_n(1) + Q_n(0) \in \mathbb{Z} \end{aligned}$$

On the other hand we have by definition of $P_n(x)$ on $[0,1]$

$$0 < p^n \pi \int_0^1 P_n(x)\sin(\pi x) \leq \frac{\pi p^n}{n!} \stackrel{n\to\infty}{\longrightarrow} 0$$

and $Q_n(0) + Q_n(1)$ cannot be an integer for large enough n.

11. (HS-1) Find all functions f, g such that

$$\begin{gathered} f, g \,:\, \mathbb{R}\backslash\{-1,0,1\} \longrightarrow \mathbb{R} \\ xf(x) = 1 + \frac{1}{x} g\left(\frac{1}{x}\right) \;\text{ and }\; \frac{1}{x^2} f\left(\frac{1}{x}\right) = x^2 g(x) \end{gathered}$$

Extra: Determine a number $r \in \mathbb{R}$ such that $|f(x) - f(x_0)| < 0.001$ whenever $|x - x_0| < r$ and $x_0 = 2$, and explain why there is no such number if we choose $x_0 = 1$ even if we artificially define some function value for $f(1)$.

Reason: Real functions.

Solution: From the second equation we get

$$\begin{aligned} xg(x) = \frac{1}{x^3} f\left(\frac{1}{x}\right) &\Longrightarrow \frac{1}{x} g\left(\frac{1}{x}\right) \\ &\Longrightarrow xf(x) = 1 + x^3 f(x) \\ &\Longrightarrow 1 = f(x)(x - x^3) = f(x)x(1-x)(1+x) \neq 0 \\ &\Longrightarrow f(x) = \frac{1}{x(1-x)(1+x)} = \frac{1}{x - x^3} \\ &\Longrightarrow g(x) = \frac{1}{x^4} f\left(\frac{1}{x}\right) \\ &\Longrightarrow g(x) = \frac{1}{x^4} \cdot \frac{1}{\frac{1}{x} - \frac{1}{x^3}} = \frac{1}{x(x^2-1)} \\ &\qquad = \frac{1}{x(x-1)(x+1)} = -f(x) \end{aligned}$$

One can easily check that the pair $\left(\frac{1}{x-x^3}, \frac{1}{x^3-x}\right)$ satisfies the two initial conditions, i.e. that this pair is a feasible solution.

Extra: Our goal is to achieve

$$|f(x) - f(2)| = \left|f(x) + \frac{1}{6}\right| = \left|\frac{6 + x - x^3}{6(x - x^3)}\right| < 0.001$$

whenever $|x - 2| < r$, i.e. $2 - r < x < 2 + r$ for some real number r. It is only asked for one such number, so we do not need to find a unique, smallest, or greatest one.

Let's start with the upper bound. Keep in mind that $x \sim 2$ and $r \sim 0$.

$$\begin{aligned} \left|\frac{6 + x - x^3}{6(x - x^3)}\right| &= \frac{|6 + x - x^3|}{6 \cdot |x| \cdot |1 - x| \cdot |1 + x|} < \frac{6 + (2 + r) + (r - 2)^3}{6 \cdot (2 - r)(r + 1)(3 - r)} \\ &= \frac{r^3 - 6r^2 + 13r}{6 \cdot (r^3 - 4r^2 + r + 6)} = \frac{r}{6} \cdot \frac{13 - 6r + r^2}{6 + r - 4r^2 + r^3} \\ &< \frac{r}{6} \cdot \frac{18}{1} = 3r < 0.001 \end{aligned}$$

for $r := 0.0001 = 10^{-4}$. Let us check the lower bound with this value.

$$\begin{aligned} \left|\frac{6 + x - x^3}{6(x - x^3)}\right| &> \frac{6 + (2 - r) - (2 + r)^3}{6 \cdot (2 + r)(1 - r)(3 + r)} = -\frac{r}{6} \cdot \frac{13 + 6r + r^2}{6 - r - 4r^2 - r^3} \\ &> -\frac{r}{6} \cdot \frac{18}{1} = -3r = -0.0003 > -0.001 \end{aligned}$$

Finally, let us define $f(1)=c\in\mathbb{R}$ for some real number $c\in\mathbb{R}$. Then for $n>10$

$$f(x)=\frac{1}{x(1-x)(1+x)}\begin{cases} >0 \stackrel{n\to\infty}{\longrightarrow} \infty & \text{if } x=1-\dfrac{1}{n} \\ =\dfrac{1}{c-c^3}=\text{const.} & \text{if } x=1 \\ <0 \stackrel{n\to\infty}{\longrightarrow} -\infty & \text{if } x=1+\dfrac{1}{n} \end{cases}$$

Hence the distance $|f(x)-f(1)|$ between the function values becomes arbitrary large at some location in $1-r<1-\dfrac{1}{n}<x<1+\dfrac{1}{n}<1+r$.

12. (HS-2) Solve the following equation system in $\mathbb{R}^3$

$$x^2+y^2+z^2=1 \quad\wedge\quad x+2y+3z=\sqrt{14}$$

Extra: Give an alternative solution in case you have the additional information that the solution is unique.

Reason: Non-linear equations.

Solution: Assume we have a solution (x,y,z), then

$$\begin{aligned} 0&=(\sqrt{14}-2y-3z)^2+y^2+z^2-1\\ &=10z^2+12yz-6z\sqrt{14}-4y\sqrt{14}+5y^2+13\\ &=10\left(z+\frac{3}{5}y-\frac{3}{10}\sqrt{14}\right)^2+\frac{7}{5}y^2+\frac{2}{5}-\frac{2}{5}y\sqrt{14}\\ &=10\left(z+\frac{3}{5}y-\frac{3}{10}\sqrt{14}\right)^2+\frac{7}{5}\left(y-\frac{1}{7}\sqrt{14}\right)^2\\ &\Longrightarrow\\ y&=\sqrt{\frac{2}{7}}\wedge z=\frac{3}{10}\sqrt{14}-\frac{3}{5}y=\frac{3}{2}\sqrt{\frac{2}{7}}=\frac{3}{\sqrt{14}}\\ &\Longrightarrow\\ x&=14-2\sqrt{\frac{2}{7}}-\frac{9}{\sqrt{14}}=\frac{1}{\sqrt{14}} \end{aligned}$$

It is easy to check that conversely the triple $\left(\dfrac{1}{\sqrt{14}},\dfrac{2}{\sqrt{14}},\dfrac{3}{\sqrt{14}}\right)$ satisfies the conditions of the statement.

Extra: Given that the equation system has a unique solution, we conclude that we have the equations of a sphere and a plane which intersect at exactly one point. This makes the plane a tangent space to the sphere. The tangent space of the sphere $x^2+y^2+z^2=1=1$ at $\vec{p}=(x_0,y_0,z_0)$ are all perpendicular vectors, i.e.

$$\vec{p}+\left\{\begin{bmatrix}v_x\\v_y\\v_z\end{bmatrix}\,:\,v_xx_0+v_yy_0+v_zz_0=0\right\}=\vec{p}+\alpha\begin{bmatrix}z_0\\0\\-x_0\end{bmatrix}+\beta\begin{bmatrix}0\\z_0\\-y_0\end{bmatrix}$$

This means for a point (x,y,z) on the plane that

$$\begin{aligned} x=x_0+\alpha z_0\,,\;y=y_0+\beta z_0\,,\;z&=z_0-x_0\alpha-y_0\beta\\ z_0z&=z_0^2-x_0(x-x_0)-y_0(y-y_0)\\ x_0x+y_0y+z_0z&=z_0^2+x_0^2+y_0^2=1 \end{aligned}$$

Hence we get by comparison of coefficients with the given equation of the plane

$$x_0=\frac{1}{\sqrt{14}}\,,\;y_0=\frac{2}{\sqrt{14}}\,,\;z_0=\frac{3}{\sqrt{14}}$$

13. (HS-3) If $(x_n)_{n\in\mathbb{N}}\subseteq\mathbb{R}_{>0}$ is a monotone decreasing sequence of positive real numbers such that for every $n\in\mathbb{N}$

$$\frac{x_1}{1}+\frac{x_4}{2}+\frac{x_9}{3}+\ldots+\frac{x_{n^2}}{n}\leq 1$$

prove that for every $n\in\mathbb{N}$

$$\frac{x_1}{1}+\frac{x_2}{2}+\frac{x_3}{3}+\ldots+\frac{x_n}{n}\leq 3$$

Extra: Prove that both sequences converge to 0.

Reason: Sequences.

Solution: For every natural number n there is a number $k\in\mathbb{N}$ such that $k^2\leq n<(k+1)^2$. Hence

$$\begin{aligned} \sum_{i=1}^{n}\frac{x_i}{i}\leq\sum_{i=2}^{k+1}\sum_{j=(i-1)^2}^{i^2-1}\frac{x_j}{j}&\leq\sum_{i=2}^{k+1}(2i-1)\,\frac{x_{(i-1)^2}}{(i-1)^2}\\ &=\sum_{i=1}^{k}(2i+1)\frac{x_{i^2}}{i^2}\leq 3\sum_{i=1}^{k}\frac{x_{i^2}}{i}\leq 3\cdot 1=3 \end{aligned}$$

by the given condition.

Extra: Assume the sequence $(x_n)_{n\in\mathbb{N}}$ becomes stationary at one point, say $x_n = a$ for all $n \geq N$. Then for any $M > N$

$$\sum_{i=1}^{M} \frac{x_i}{i} = \underbrace{\sum_{i=1}^{N-1} \frac{x_i}{i}}_{=:C_1} + \sum_{i=N}^{M} \frac{a}{i} = C_1 + a \cdot \underbrace{\sum_{i=N}^{M} \frac{1}{i}}_{=:C_M} \leq 3$$

for all M by assumption. However $\lim_{M\to\infty} C_M = \infty$, which cannot both hold. Hence the monotone decreasing sequences are strictly monotone decreasing, i.e.

$$1 \geq x_1 > x_2 > x_3 > \ldots > x_n > \ldots > 0.$$

If we now cut the interval $[0,1]$ in half, then the right half must contain infinitely many sequence members. Then we choose this half and cut it again into half. The right part has to contain infinitely many sequence members again. Going on with these nested intervals, we get interval lengths that converge to zero. If we pick one sequence member from each interval, we get a subsequence which converges to a real number (because $\mathbb{R}$ is complete). By strict monotony the sequence itself has to converge to the same number, say $L \in [0,1]$.

Assume $\lim_{n\to\infty} x_n = L > 0$. Then

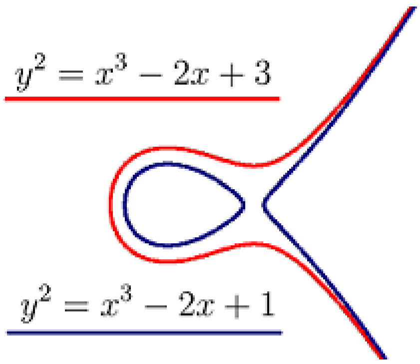
$$3 \geq \lim_{n\to\infty} \left(\frac{x_1}{1} + \ldots + \frac{x_n}{n} \right) \geq L \cdot \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} = \infty$$

which cannot hold, hence our assumption was wrong and $L = 0$.

14. (HS-4) Solve the following equation system for real numbers:

$$\begin{aligned} (1) \qquad x + xy + xy^2 &= -21 \\ (2) \qquad y + xy + x^2y &= 14 \\ (3) \qquad x + y &= -1 \end{aligned}$$

Extra:



Consider the two elliptic curves and observe that one has two connection components and the other one has only one. Determine the constant $c \in [1,3]$ in $y^2 = x^3 - 2x + c$ where this behavior exactly changes. What is the left most point of this curve?

Reason: Non-linear equations.

Solution: Adding the (1) and (2) and using (3) gets

$$\begin{aligned}(x+y)+2xy+xy(x+y)&=-7\\ \Longrightarrow xy&=-6\\ \Longrightarrow x-6y&=-15\\ \Longrightarrow \begin{bmatrix}x\\y\end{bmatrix}&=\begin{bmatrix}1&-6\\1&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}-15\\-1\end{bmatrix}=\frac{1}{7}\begin{bmatrix}1&6\\-1&1\end{bmatrix}\cdot\begin{bmatrix}-15\\-1\end{bmatrix}=\begin{bmatrix}-3\\2\end{bmatrix}\end{aligned}$$

The pair $(-3,2)$ conversely satisfies the required conditions, and is thus the unique solution.

Extra: The curves are symmetric to the $x-$axis $y=0$. Moreover, the behavior changes at the point where the two most right extremal points coincide. Hence we have to find the points, where

$$\frac{dy}{dx}=0$$

Now $2\cdot y(x)\cdot y'(x) = 3x^2-2$ so for our points hold $x=\pm\sqrt{\frac{2}{3}}$. The negative values lead to the extremal points on the left, and the positive value is the one which we are interested in. We also know that for symmetry reasons, that

$$\begin{aligned}y^2\left(\sqrt{\frac{2}{3}}\right)=0=\sqrt{\frac{2}{3}}^3-2\sqrt{\frac{2}{3}}+c&=-\frac{4}{3}\sqrt{\frac{2}{3}}+c\\ \Longrightarrow c=\frac{4}{3}\sqrt{\frac{2}{3}}&=\sqrt{\frac{32}{27}}\approx 1.089\end{aligned}$$

To determine the left most point, we search for the points where $y=0$, i.e. $0=x^3-2x+\sqrt{\frac{32}{27}}=x^3-2x+\frac{4}{3}\sqrt{\frac{2}{3}}$. We know from the previous calculation that $x=\sqrt{\dfrac{2}{3}}$ is a solution. Long division shows

$$\begin{aligned}\left(x^3-2x+\frac{4}{3}\sqrt{\frac{2}{3}}\right):\left(x-\sqrt{\frac{2}{3}}\right)&=x^2+\sqrt{\frac{2}{3}}x-\frac{4}{3}\\&=\left(x+2\sqrt{\frac{2}{3}}\right)\left(x-\sqrt{\frac{2}{3}}\right)\\x^3-2x+\frac{4}{3}\sqrt{\frac{2}{3}}&=\left(x+2\sqrt{\frac{2}{3}}\right)\left(x-\sqrt{\frac{2}{3}}\right)^2\end{aligned}$$

The left most point is thus $-2\sqrt{\dfrac{2}{3}}\approx -1.633$ and the double root at $x=\sqrt{\dfrac{2}{3}}$ confirms the previous result.

15. (HS-5) Find all real numbers $m\in\mathbb{R}$, such that for all real numbers $x\in\mathbb{R}$ holds

$$f(x,m):=x^2+(m+2)x+8m+1>0 \qquad (*)$$

and determine the value of m for which the minimum of $f(x,m)$ is maximal. What is the maximum?

Extra: The set of all intersection points of two perpendicular tangents is called orthoptic of the parabola. Prove that it is the directrix, the straight parallel to the tangent at the extremum on the opposite side of the focus.

Reason: Parametric equation.

Solution: The quadratic equation $x^2+(m+2)x+8m+1=0$ has the solutions

$$-\frac{m+2}{2}\pm\frac{1}{2}\sqrt{m^2-28m}$$

If $m\leq 0$ then $m^2-28m\geq 0$ and the parabola has at least one intersection with the $x-$axis, i.e. $(*)$ cannot be greater than zero for all real numbers.

If $m\geq 28$ then the discriminant is again not negative and the parabola has at least one intersection with the $x-$axis again, i.e. $(*)$ cannot be greater than zero for all real numbers.

Hence the only possible numbers are $0 < m < 28$ in which case the discriminant is negative and the parabola does not intersect the $x-$axis:

$$\begin{aligned}x^2+(m+2)x+8m+1 &= \left(x+\frac{m+2}{2}\right)^2-\frac{(m+2)^2}{4}+8m+1\\ &\geq -\frac{1}{4}\left(m^2-28m\right)=-\frac{1}{4}m(m-28)>0\end{aligned}$$

for all $x \in \mathbb{R}$ and $0 < m < 28$. The minimum of $f(x,m)$ for each parameter is determined by

$$\frac{d}{dx}f(x,m)=0=2x+m+2 \Longrightarrow x=-\frac{m+2}{2}$$

so we want to maximize

$$\begin{aligned}f\left(-\frac{m+2}{2},m\right) &= \left(\frac{m+2}{2}\right)^2-(m+2)\frac{m+2}{2}+8m+1\\ &= -\frac{1}{4}(m+2)^2+8m+1=-\frac{1}{4}m^2+7m\end{aligned}$$

which is maximal at $-\frac{1}{2}m+7$, i.e. $m=14$ and $f(-8,14)=49$.

Extra: A parabola is defined as the set of points, that has equal distance to its focus and its directrix. We may assume that our parabola has the equation $y=ax^2$, $a \neq 0$. Then its focus is $F=\left(0,\frac{1}{4a}\right)$ and its directrix L thus $y=-\frac{1}{4a}$. This means

$$P=\{(x,y)\in\mathbb{R}^2 \,|\, y=ax^2\}=\{p\in\mathbb{R}^2\,|\,d(F,p)=d(F,L)\}.$$

The slope of P is given by the first derivative $m=2ax$. Hence

$$P=\left\{\left(\frac{m}{2a},\frac{m^2}{4a}\right)\,\middle|\, m\in\mathbb{R}\right\}$$

and the tangent T with slope m has the equation $y=mx-\frac{m^2}{4a}$. Let $(x_0,y_0)\notin P$ be a point on T. Then $y_0=mx_0-\frac{m^2}{4a} \Longleftrightarrow 0 = m^2-4ax_0m+4ay_0$ which has two solutions, corresponding to the two possible tangents from (x_0,y_0). Now if the tangents meet in a right angle at (x_0,y_0), then the product of their slopes is -1. This equals the product of the solutions of the quadratic equation, i.e. by Vieta's formulas

$$-1 = 4ay_0 \Longrightarrow y_0 = -\frac{1}{4a}$$

so the set of intersection points is L, what had to be shown.

4 September 2021

1. Let f be a function defined on $(0, \infty)$ such that $f(x) > 0$ for all $x > 0$. Suppose that f has the following properties:

   (a) $\log f(x)$ is a convex function.

   (b) $f(x+1) = x \cdot f(x)$ for all $x > 0$.

   (c) $f(1) = 1$.

   Then $f(x) = \lim\limits_{n\to\infty} \dfrac{n!n^x}{x(x+1)\cdots(x+n)} =: \Gamma(x)$ for all $x > 0$.

   Reason: Bohr-Mollerup theorem.

   Solution: The given conditions $(b),(c)$ allows to conclude

$$\begin{aligned} f(x+n) &= f(x+n-1+1) \\ &= (x+n-1)f(x+n-2+1) \\ &= (x+n-1)(x+n-2)f(x+n-3+1) \\ &= \\ &\vdots \\ &= \\ &= (x+n-1)(x-n-2)\cdots(x+1)xf(x) \end{aligned}$$

   This implies in particular that $f(N+1) = N!$ for all $N \in \mathbb{N}$ and if we can show $f(x) = \Gamma(x)$ for all $0 < x \leq 1$ then we conclude for all $N < y = x + N \leq N + 1$

$$\begin{aligned}
f(y) &= f(x+N) = (x+N-1)(x-N-2)\cdots(x+1)\cdot x\cdot\Gamma(x)\\
&= (x+N-1)(x-N-2)\cdots(x+1)\cdot x\cdot\lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)}\\
&= \lim_{n\to\infty}\frac{(x+N-1)(x-N-2)\cdots(x+1)\cdot x\cdot n!n^x}{x\cdot(x+1)\cdots(x+n)}\\
&= \lim_{n\to\infty}\frac{n!n^x}{(x+N)(x+N+1)\cdots(x+n)}\\
&= \lim_{n\to\infty}\frac{n!n^{y-N}}{y(y+1)\cdots(y-N+n)}\\
&= \lim_{n\to\infty}\frac{n!n^y}{y\cdot(y+1)\cdots(y+n)}\cdot\frac{(y-N+n+1)\cdots(y+n)}{n^N}\\
&= \lim_{n\to\infty}\frac{n!n^y}{y\cdot(y+1)\cdots(y+n)}\cdot\left(\frac{y}{n}-\frac{N}{n}+\frac{n}{n}+\frac{1}{n}\right)\cdots\left(\frac{y}{n}+\frac{n}{n}\right)\\
&= \lim_{n\to\infty}\frac{n!n^y}{y\cdot(y+1)\cdots(y+n)}\cdot 1^N\\
&= \Gamma(y)
\end{aligned}$$

Hence we may assume that $0 < x \leq 1$. Let $n > 2$ be some integer. Since $\log f(x)$ is convex, we have that the function lies beneath the secant of any two points at $a, b > 0$.

$$\log f(x) \leq \log f(a) + \frac{\log f(b) - \log f(a)}{b-a}\cdot(x-a) \quad \text{for all } x \in (a,b)$$

$$\Longleftrightarrow$$

$$\frac{\log f(x) - \log f(a)}{x-a} \leq \frac{\log f(b)-\log f(a)}{b-a} \quad \text{for all } a < x < b$$

Thus for $n < n+x \leq n+1$

$$\underbrace{\frac{\log f(n+x)-\log f(n)}{(n+x)-n}}_{=\frac{\log f(n+x)-\log(n-1)!}{x}} \leq \underbrace{\frac{\log f(n+1)-\log f(n)}{(n+1)-n}}_{=\log n}$$

and for $n-1 < n < n+x$

$$\underbrace{\frac{\log f(n)-\log f(n-1)}{n-(n-1)}}_{=\log(n-1)} \leq \underbrace{\frac{\log f(n+x)-\log f(n-1)}{n+x-(n-1)}}_{=\frac{\log f(n+x)-\log(n-2)!}{x+1}}$$

Thus

$$\begin{gathered}(x+1)\log(n-1)+\log(n-2)! \leq \log f(n+x) \leq \log(n-1)!+x\log n\\ \log\left((n-1)^x(n-1)!\right) \leq \log f(n+x) \leq \log\left(n^x(n-1)!\right)\\ \underbrace{\frac{(n-1)^x(n-1)!}{x(x+1)\cdots(x+n-1)}}_{=:L(n)} \leq f(x) \leq \underbrace{\frac{n^x(n-1)!}{x(x+1)\cdots(x+n-1)}}_{=:R(n)}\end{gathered}$$

Since $f(x)$ in the sandwich is independent of n we may write

$$L(n+1)=\frac{n^x n!}{x(x+1)\cdots(x+n)} \leq f(x) \leq R(n)=\frac{n^x n!}{x(x+1)\cdots(x+n)}\cdot\frac{x+n}{n}$$

and letting $n \to \infty$ we get $\Gamma(x) \leq f(x) \leq \Gamma(x)$ which we had to show.

2. Let $T=(x_1,x_2,\ldots,x_m)$ be a sequence of not necessarily distinct reals. For any positive b, define

$$T_b := \{(x_i,x_j)\,|\,1 \leq i,j \leq m, |x_i-x_j| \leq b\}.$$

Show that for any sequence T and for every integer $r>1$,

$$|T_r| < (2r-1)|T_1|.$$

Reason: Combinatorics.

Solution: We apply induction on $|T|=m$. The result is trivial for $m=1$. Assuming it holds for $m-1$, we prove it for $m>1$. Given a sequence $T=(x_1,\ldots,x_m)$ let $t+1$ be the maximum number of points of T in a closed interval of length 2 centered at a member of T. Let x_i be any rightmost point of T so that there are $t+1$ members of T in the interval $[x_i-1,x_i+1]$ and define $T' := T-\{x_i\}$. The number of members of T' in the interval $[x_i-1,x_i+1]$ is clearly t and hence x_i appears in precisely $2t+1$ ordered pairs of T_1. Thus

$$|T_1| = 2t+1+|T'_1|.$$

The interval $[x_i-r,x_i+r]$ is the union of the $2r-1$ smaller intervals

$$[x_i-r,x_i-r+1),\ldots,[x_i-2,x_i-1),[x_i-1,x_i+1],(x_i+1,x_i+2],\ldots,(x_i+r-1,x_i+r].$$

By the choice of x_i, each of these smaller intervals can contain at most $t+1$ members of T, and each of the last $r-1$ ones, which lie to the right of x_i, can contain at most t members of T. Altogether there are thus at most $(r-1)(t+1)+rt$ members of T' in $[x_i - r, x_i + r]$ and hence

$$|T_r| \leq 2(r-1)(t+1) + 2rt + 1 + |T_r'| = (2r-1)(2t+1) + |T_r'|.$$

By induction hypothesis $|T_r'| < (2r-1)|T_1'|$ and hence $|T_r| < (2r-1)|T_1|$.

3. Let X, Y be two independent identically distributed real random variables. For a positive b, define $p_b := \operatorname{prob}(|X-Y| \leq b)$. Then for every integer r, $p_r \leq (2r-1)p_1$. Thus

$$\operatorname{prob}(|X-Y| \leq 2) \leq 3 \cdot \operatorname{prob}(|X-Y| \leq 1)$$

Reason: Stochastic.

Solution: Fix an integer m, and let $S := (x_1, \ldots, x_m)$ be a random sequence of m elements, where each x_i is chosen, randomly and independently, according to the distribution of X. By the previous statement

$$|S_r| < (2r-1)|S_1|.$$

Therefore, the expectation of $|S_r|$ is smaller than that of $(2r-1)|S_1|$. However, by the linearity of expectation it follows that the expectation of $|S_b|$ is precisely $m + m(m-1)p_b$ for every $b > 0$. Thus

$$m + m(m-1)p_r < (2r-1)(m + (m-1)p_1),$$

implying that for every integer m,

$$p_r < (2r-1)p_1 + \frac{2r-2}{m-1} \stackrel{m\to\infty}{\longrightarrow} (2r-1)p_1$$

It can be proven that even the strict inequality

$$\operatorname{prob}(|X-Y| \leq 2) < 3 \cdot \operatorname{prob}(|X-Y| \leq 1)$$

holds, in which case we speak of the $1-2-3$ theorem.

4. Let $\mathbb{F}$ be a field and G a finite group, such that $\operatorname{char}\mathbb{F} \nmid |G|$. Prove that $\mathbb{F}G$ is semisimple, and show that this is not true if $\operatorname{char}\mathbb{F} \mid |G|$.

Reason: Theorem of Maschke.

Solution: Let $W \subseteq V$ be finite-dimensional $\mathbb{F}G$-modules. Pick an idempotent $e \in \operatorname{End}_{\mathbb{F}}(V)$ with $eV = W$ and define

$$\overline{e} := \frac{1}{|G|} \sum_{g\in G} geg^{-1}$$

where the elements of G are considered as endomorphisms of V. Then

$$h\overline{e} = \frac{1}{|G|}\sum_{g\in G} hgeg^{-1} = \frac{1}{|G|}\sum_{g\in G}(hg)e(hg)^{-1}h = \overline{e}h$$

for all $h \in G$ and thus $\overline{e} \in \operatorname{End}_{\mathbb{F}G}(V)$. Since W is a submodule if V, the endomorphism $\overline{e}$ still satisfies $\overline{e}V \subseteq W$ and $\overline{e}|_W = \operatorname{id}_W$. Hence $\overline{e}$ is an idempotent with $\overline{e}V = W$, and we have

$$V = W \oplus (\operatorname{id}_V - \overline{e})\, V$$

i.e. every submodule splits and V is semisimple, and so is $\mathbb{F}G$.

Let $x := \sum_{g\in G} g \in \mathbb{F}G$ satisfies $gx = x$ for all $g \in G$ and $x^2 = |G|x = 0$. Thus $\mathbb{F}Gx = \mathbb{F}x$ is a submodule of $\mathbb{F}G$ which contains no idempotent. In particular, $\mathbb{F}x$ is not projective, and hence $\mathbb{F}G$ is not semisimple.

5. A group G together with a topology, such that the mapping on $G \times G$ (equipped with the product topology) to G given by $(x, y) \longmapsto xy^{-1}$ is continuous, is called a topological group (e.g. a Lie group). Prove

   (a) G is a topological group if and only if inversion and multiplication are continuous.

   (b) The mappings $x \longmapsto xg$ and $x \longmapsto gx$ are homeomorphisms for each $g \in G$.

   (c) Each open subgroup $U \leq G$ is closed, and each closed subgroup $U \leq G$ of finite index is open. If G is compact, then each open subgroup is of finite index.

   (d) Let $H \leq G$ be a subgroup equipped with the subspace topology, $K \trianglelefteq G$ a normal subgroup, and G/K equipped with the quotient space topology. Then H and G/K are again topological groups and the projection $\pi : G \twoheadrightarrow G/K$ is open.

   (e) G is Hausdorff if and only if $\{1\}$ is a closed set in G. G/K is Hausdorff for a normal subgroup $K \trianglelefteq G$, if and only if K is closed in G. If G is totally disconnected, then G is Hausdorff.

   (f) Let G be a compact topological group and $\{X_j\ (j \in I)\} \subseteq G$ a family of closed subsets such that for all $i, j \in I$ there is a $k \in I$ with $X_k \subseteq X_i \cap X_j$. Then we have for any closed subset $Y \subseteq G$

$$Y \cdot \left(\bigcap_{i\in I} X_i\right) = \bigcap_{i\in I} Y X_i$$

Reason: Topological Groups.

Solution:

(a) If inversion and multiplication are continuous, then their composition is continuous, too. Now let

$$\varphi : G \times G \longrightarrow G\,,\ \varphi(x,y) := xy^{-1}$$

be continuous. Inversion ι is the composition of the continuous functions

$$G \stackrel{x\to(1,x)}{\longrightarrow} \{1\} \times G \stackrel{\varphi|_{\{1\}\times G}}{\longrightarrow} G$$

and thus continuous, too. For the multiplication we get the composition of

$$G \times G \stackrel{(\mathrm{id},\iota)}{\longrightarrow} G \times G \stackrel{\varphi}{\longrightarrow} G$$

continuous functions again.

(b) It is sufficient to prove it for $\mu_g(x) := xg$. Now

$$\mu_g \,:\, G \stackrel{x\to(g,x)}{\longrightarrow} \{g\} \times G \stackrel{\varphi|_{\{g\}\times G}}{\longrightarrow} G$$

is continuous as it is the composition of continuous functions. By $\mu_g^{-1} = \mu_{g^{-1}}$ we see that the inverse function is continuous, too.

(c) Let $U \leq G$ be a open subgroup, and $g \in G - U$. Then $gU \subseteq G - U$ is open because left multiplication is a homeomorphism, and

$$G - U = \bigcup_{g\in G-U} g = \bigcup_{g\in G-U} gU$$

is a union of open sets, i.e. open, i.e. U is closed.
Next let $A \leq G$ be a closed subgroup of finite index. Then

$$G - A = \bigcup_{g\in G-A} g = \bigcup_{g\in G-A} gA = \bigcup_{g=1}^{n} gA$$

is a finite union of closed sets, hence $G - A$ is closed and A is open.
If G is compact, and $U \leq G$ an open subgroup, then $\{gU \in G/U\}$ define an open, disjoint cover of G which has a finite subcover, i.e. G/U is of finite index.

(d) H is a topological group since the subspace topology is defined that way. Let $\overline{V} \subseteq G/K$ be an open set. Then $V := \pi^{-1}(\overline{V})$ is open by definition of the quotient topology, and $\pi(V) = VK = \overline{V}$ since π is surjective. Now let $V \subseteq G$ be open. Then Vk is open for each $k \in K$ since right multiplication is a homeomorphism and thus open, hence

$$VK = \bigcup_{k\in K} Vk = \pi(V) \subseteq G/K$$

is open. So π is continuous and open. Set

$$\varphi\,:\, G/K \times G/K \longrightarrow G/K\,,\; \varphi(gK, hK) = gh^{-1}K$$

Let $\overline{V} \subseteq G/K$ be an open set, and $(gK, hK) \in \varphi^{-1}(\overline{V})$. Since $(g, h) \longmapsto \pi(gh^{-1}) = gh^{-1}K$ is continuous, there are open neighborhoods $V_g, V_h \subseteq G$ of g, h such that $V_gV_h^{-1} \subseteq \pi^{-1}(\overline{V})$. Since π is open, $\pi(V_g) \times \pi(V_h) = V_gK \times V_hK \subseteq G/K \times G/K$ is an open neighborhood of $(gK, hK) \in G/K \times G/K$. This proves that φ is continuous since $V_gK \times V_hK \subseteq \varphi^{-1}(\overline{V})$.

(e) An equivalent definition of a Hausdorff space is, that it is a topological space in which all singleton sets are the intersection of their closed neighborhoods. In particular $\{1\} \subseteq G$ is closed if G is Hausdorff.
Now let $\{1\}$ be closed. Since right multiplication is closed, all sets $\{ab^{-1}\} = \mu_{ab^{-1}}(1)$ are closed, too. Thus there are disjoint open neighborhoods $V_{\{1\}}, V_{\{ab^{-1}\}}$ in case $a \neq b$. Since right multiplication is open, $V_b = V_{\{1\}}b$ and $V_a = V_{\{ab^{-1}\}}b$ are disjoint open neighborhoods of a and b, i.e. G is a Hausdorff space. If G/K is Hausdorff, then $\{\overline{1}\} \subseteq G/K$ is closed, and so is $K = \pi^{-1}(\{\overline{1}\}) \subseteq G$ since π is continuous. If conversely $K \subseteq G$ is closed, then $G - K$ is open. Since π is open,

$$\pi(G-K) = \pi\left(\bigcup_{g\in G-K} g\right) = \bigcup_{g\notin K} \pi(g) = \bigcup_{g\notin K} gK = \bigcup_{\overline{g}\neq 1} \overline{g} = G/K - \{\overline{1}\}$$

is open, too, and $\{\overline{1}\} \subseteq G/K$ is closed, i.e. G/K is Hausdorff. Finally, G is totally disconnected, if the empty set and all singleton sets are the only connection components. But these are always closed, so $\{1\}$ is closed and G is Hausdorff.

(f) Clearly

$$Z := Y \cdot \left(\bigcap_{i \in I} X_i \right) \subseteq \bigcap_{i \in I} Y X_i$$

Assume there is an element $g \notin Z$, then $Y^{-1}g \cap \left(\bigcap_{i \in I} X_i g \right) = \emptyset$ since otherwise there would be an element $h = y^{-1}g$ for some $y \in Y$ and $h \in X_i$ for all $i \in I$, i.e. $g = yh \in Z$. Since G is compact and $Y^{-1}g, X_i \, (i \in I)$ are all closed with empty intersection, their complements build an open cover of G, from which finitely many will do. In any case, their complements intersect to the empty set, hence $Y^{-1}g \cap (\cap_{i=1}^n X_i) = \emptyset$. Now by our hypothesis we can recursively find a $k \in I$ such that $X_k \in X_i$ for all $i = 1, \ldots, n$. If $g = yx \in YX_k$, then $y^{-1}g \in Y^{-1}g \cap X_k \subseteq Y^{-1}g \cap (\cap_{i=1}^n X_i)$, a contradiction. Thus $g \notin YX_k$, i.e. $g \notin \bigcap_{i \in I} YX_i$ what had to be shown.

6. Let $(X_n, d_n)_{n \in \mathbb{N}_0}$ be a sequence of complete metric spaces, and let $(f_n)_{n \in \mathbb{N}_0}$ be a sequence of continuous functions $f_n : X_{n+1} \longrightarrow X_n$ such that the image $f_n(X_{n+1}) \subseteq (X_n, d_n)$ is dense for all $n \in \mathbb{N}_0$. Then

$$M_0 := \{ v_0 \in X_0 \mid \exists \, (v_n)_{n \in \mathbb{N}} \, \forall \, n \in \mathbb{N} \, : \, v_n \in X_n \, \wedge \, f_{n-1}(v_n) = v_{n-1} \}$$

and

$$M_0 \subseteq M := \bigcap_{n=0}^{\infty} (f_0 \circ f_1 \circ \ldots \circ f_n)(X_{n+1})$$

are dense in (X_0, d_0). In particular $M \neq \emptyset$ in case $X_0 \neq \emptyset$.

Reason: Mittag-Lefler theorem.

Solution: Let $x \in X_0$ and $\varepsilon > 0$. We want to show that there is a $v_0 \in M_0$ such that $d_0(x, y) \leq \varepsilon$. We begin by constructing inductively a sequence $(y_n)_{n \in \mathbb{N}_0}$ with the properties

$$y_n \in X_n \, \wedge \, d_n(y_n, f_n(y_{n+1})) \leq \frac{\varepsilon}{2^{n+1}} \quad (1)$$

$$d_k((f_k \circ f_{k+1} \circ \ldots \circ f_{n-1})(y_n), (f_k \circ f_{k+1} \circ \ldots \circ f_n)(y_{n+1})) \leq \frac{\varepsilon}{2^{n+1}} \quad (2)$$

for all $n \in \mathbb{N}_0$ and $0 \leq k < n$. We set $y_0 := x$ and find $y_1 \in X_1$ with $d_0(y_0, f_0(y_1)) < \varepsilon/2$ by the density of $f_0(X_1) \subseteq X_0$. This satisfies both conditions in case $n = 0$.
Now assume we have constructed the points $y_0, \ldots, y_m$ for some $m \in \mathbb{N}$ such that the conditions hold for $0 \leq n < m$ and $0 \leq k < n$. Since

$f_m(X_{m+1}) \subseteq X_m$ is dense, there is a sequence $(z_j)_{j\in\mathbb{N}} \subseteq X_{m+1}$ with $\lim_{j\to\infty} f_m(z_j) = y_m$. By continuity of $f_0, f_1, \ldots$ we also have for all $k = 0, \ldots, m-1$

$$\lim_{j\to\infty} d_k((f_k \circ \ldots \circ f_{m-1})(y_m), (f_k \circ \ldots \circ f_m)(z_j)) = 0$$

Hence there is a $j_0 \in \mathbb{N}$ such that with $y_{m+1} := z_{j_0}$ both conditions hold even for $n = m$ and $0 \leq k < m$. We have thus constructed the required sequence.

For all $k, j \in \mathbb{N}_0$ define

$$u_{k,0} := y_k \;\wedge\; u_{k,j} := (f_k \circ \ldots \circ f_{k+j-1})(y_{k+j}).$$

By induction, condition (2), and the triangle inequality

$$\begin{aligned} d_k(u_{k,j}, u_{k,j+p}) &\leq \sum_{m=1}^{p} d_k(u_{k,j+m-1}, u_{k,j+m}) \\ &\leq \frac{\varepsilon}{2^{k+j}} \sum_{m=1}^{p} \frac{1}{2^m} \\ &< \frac{\varepsilon}{2^{k+j}} \stackrel{j\to\infty}{\longrightarrow} 0 \quad (3) \end{aligned}$$

So all sequences $(u_{k,j})_{j\in\mathbb{N}_0} \subseteq (X_k, d_k)$ for $k \in \mathbb{N}_0$ are Cauchy sequences in a complete metric spaces, i.e. they converge:

$$\lim_{j\to\infty} u_{k,j} =: v_k \in X_k$$

and thus for all $k \in \mathbb{N}_0$

$$\begin{aligned} \lim_{j\to\infty} f_k(u_{k+1,j}) &= \lim_{j\to\infty} f_k(f_{k+1} \circ \ldots \circ f_{k+j})(y_{k+j+1}) \\ &= \lim_{j\to\infty} u_{k,j+1} \\ &= v_k \\ &= f_k(\lim_{j\to\infty} u_{k+1,j}) \\ &= f_k(v_{k+1}) \end{aligned}$$

In particular $v_0 \in M_0$ and by continuity of the metric d_0 and (3) for $k = j = 0$

$$d_0(x, v_0) = d_0(y_0, v_0) = \lim_{j\to\infty} d_0(u_{0,0}, u_{0,j}) \leq \limsup_{j\to\infty} \varepsilon \sum_{m=1}^{j} \frac{1}{r^m} = \varepsilon$$

7. Prove for all $x > -1$

$$x - (1+x)\log(1+x) \leq -\frac{3x^2}{2(x+3)}$$

Reason: Logarithmic inequality.

Solution:

$$\begin{aligned}
f(x) &:= x - (1+x)\log(1+x) \\
f'(x) &= 1 - \log(1+x) - (1+x)\cdot\frac{1}{1+x} = -\log(1+x) \\
f''(x) &= -\frac{1}{1+x} \\
g(x) &:= -\frac{3x^2}{2(x+3)} \\
g'(x) &= -\frac{6x\cdot 2(x+3) - 3x^2\cdot 2}{4(x+3)^2} = -\frac{3x(x+6)}{2(x+3)^2} \\
g''(x) &= -\frac{(6x+18)(2(x+3)^2) - (3x^2+18x)(4(x+3))}{4(x+3)^4} \\
&= -\frac{12(x+3)^2 - 12x^2 - 72x}{4(x+3)^3} = -\frac{27}{(x+3)^3}
\end{aligned}$$

It is $f(0) = f'(0) = g(0) = g'(0) = 0$ and for $x > -1$

$$\begin{aligned}
\frac{1}{g''(x)} &= -\frac{(x+3)^3}{27} = -\frac{1}{27}\cdot(x+3)(x^2+6x+9) \\
&= -\frac{1}{27}(x^2(x+9) + 27x + 27) < -\frac{1}{27}(27x+27) \\
&= -1 - x = \frac{1}{f''(x)} < 0 \\
f''(x) &< g''(x) < 0
\end{aligned}$$

For $x \geq 0$ is

$$f(x) = \int_0^x \int_0^t f''(s)\,ds\,dt < \int_0^x \int_0^t g''(s)\,ds\,dt = g(x)$$

which is equally true for $-1 < x < 0$ with exchanged integral limits.

8. Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space, B, T, σ positive real numbers, and $n \in \mathbb{N}$. For independently distributed random variables $X_1, \ldots, X_n$:

$\Omega \longrightarrow \mathbb{R}$ with expectation values $E(X_k) = 0$ and $E(X_k^2) \leq \sigma^2$, and boundary $\|X_k\|_\infty \leq B$ for all $k = 1, \ldots, n$ prove

$$P\left(\frac{1}{n}\sum_{k=1}^{n} X_k \geq \sqrt{\frac{2\sigma^2 T}{n}} + \frac{2BT}{3n}\right) \leq e^{-T}.$$

Reason: Bernstein inequality.

Solution: Set $\varepsilon := \dfrac{\sqrt{18Tn\sigma^2 + T^2B^2} + TB}{3n} = \sqrt{\dfrac{2T\sigma^2}{n} + \dfrac{T^2B^2}{9n^2}} + \dfrac{TB}{3n}$.

Assume $\sqrt{\alpha + \beta^2} + \beta > \sqrt{\alpha} + 2\beta$ for $\alpha, \beta > 0$. Then $\alpha + \beta^2 > \alpha + \beta^2 + 2\beta\sqrt{\alpha}$ which isn't possible for positive numbers. Thus we have

$$\varepsilon \leq \sqrt{\frac{2\sigma^2 T}{n}} + \frac{2TB}{3n}$$

Rearrangement of the definition of ε for T is

$$\begin{aligned}
(3n\varepsilon - TB)^2 &= 9n^2\varepsilon^2 - 6n\varepsilon TB + T^2B^2 = 18Tn\sigma^2 + T^2B^2 \\
3n\varepsilon^2 &= T(6\sigma^2 + 2\varepsilon B) \\
T &= \frac{3n\varepsilon^2}{6\sigma^2 + 2\varepsilon B}
\end{aligned}$$

The Markov inequality says that for a monotone increasing function $f : \mathbb{R} \longrightarrow [0, \infty)$ and a constant $a \in \mathbb{R}$

$$f(a) \cdot P(X \geq a) \leq E(f(X))$$

and in particular for $f(a) > 0$

$$P(X \geq a) \leq \frac{E(f(X))}{f(a)}.$$

Applied to $X := n^{-1}\sum_{i=1}^{n} X_i$ and $f(\varepsilon) := e^{tn\varepsilon}$ for some $t > 0$ which will be specified later in the proof, we get

$$P\left(X \geq \sqrt{\frac{2\sigma^2 T}{n}} + \frac{2TB}{3n}\right) \leq P(X \geq \varepsilon) \leq e^{-tn\varepsilon} E\left(\prod_{i=1}^{n} \exp(tX_i)\right)$$

The random variables are independent, so we may change the order of products and expectation values. $\exp(tX_i)$ is bounded by the integrable upper bound e^{tB}. Hence we can change the order of summation and expectation value, too. With $X_i^0=1$ and $E(X_i)=0$ we have

$$\begin{aligned}
e^{-tn\varepsilon}E\left(\prod_{i=1}^n \exp(tX_i)\right) &= e^{-tn\varepsilon}\prod_{i=1}^n E\left(\sum_{k=0}^\infty \frac{t^k}{k!}X_i^k\right)\\
&= e^{-tn\varepsilon}\prod_{i=1}^n \left(1+\sum_{k=2}^\infty \frac{t^k}{k!}E(X_i^k)\right)\\
&\stackrel{(1)}{\leq} e^{-tn\varepsilon}\prod_{i=1}^n \left(1+\sum_{k=2}^\infty \frac{t^k}{k!}\sigma^2 B^{k-2}\right)\\
&= e^{-tn\varepsilon}\left(1+\frac{\sigma^2}{B^2}\left(e^{tB}-tB-1\right)\right)^n\\
&\stackrel{(2)}{\leq} e^{-tn\varepsilon}\cdot\exp\left(\frac{n\sigma^2}{B^2}\left(e^{tB}-tB-1\right)\right)
\end{aligned}$$

$$\begin{aligned}
&(1)\quad E(X_i^k)\leq E(X_i^2)B^{k-2}\leq \sigma^2 B^{k-2}\\
&(2)\quad (1+x)^n=\sum_{k=0}^n \frac{n!}{(n-k)!}\cdot\frac{x^k}{k!}\leq \sum_{k=0}^n \frac{n^k x^k}{k!}\leq e^{nx}
\end{aligned}$$

Now we choose $t:=\dfrac{1}{B}\log\left(1+\dfrac{\varepsilon B}{\sigma^2}\right)>0$ and get

$$\begin{aligned}
&e^{-tn\varepsilon}\cdot\exp\left(\frac{n\sigma^2}{B^2}\left(e^{tB}-tB-1\right)\right)\\
&\quad=\exp\left(-\frac{\varepsilon n}{B}\log\left(1+\frac{\varepsilon B}{\sigma^2}\right)+\frac{n\sigma^2}{B^2}\left(1+\frac{\varepsilon B}{\sigma^2}-\log\left(1+\frac{\varepsilon B}{\sigma^2}\right)-1\right)\right)\\
&\quad=\exp\left(\frac{n\sigma^2}{B^2}\left(-\frac{\varepsilon B}{\sigma^2}\log\left(1+\frac{\varepsilon B}{\sigma^2}\right)+\frac{\varepsilon B}{\sigma^2}-\log\left(1+\frac{\varepsilon B}{\sigma^2}\right)\right)\right)
\end{aligned}$$

Now we get by the previous Lemma for $x:=\dfrac{\varepsilon B}{\sigma^2}>0$

$$x-(1+x)\log(1+x)\leq -\frac{3x^2}{2x+6}$$

in total

$$P\left(X\geq\sqrt{\frac{2\sigma^2 T}{n}}+\frac{2TB}{3n}\right)\leq \exp\left(-\frac{n\sigma^2}{B^2}\cdot\frac{\frac{3\varepsilon^2B^2}{\sigma^4}}{\frac{2\varepsilon B}{\sigma^2}+6}\right)$$

$$=\exp\left(-\frac{3\varepsilon^2 n}{2\varepsilon B+6\sigma^2}\right)=e^{-T}$$

9. Let $\mathbb{P}$ be the set of all primes, $p \in \mathbb{P}$, and $n \in \mathbb{N}$ a positive integer. $\operatorname{ord}_p(N)$ denotes the number of primes p which occur as divisor in $\{1, 2, \ldots, N\}$ counted by multiplicity. E.g. $N = 24 = 4!$ and $p = 3$ yields in $\{3 = 3^1, 6 = 3^1 \cdot 2, 9 = 3^2, 12 = 3^1 \cdot 4, 15 = 3^1 \cdot 5, 18 = 3^2 \cdot 2, 21 = 3^1 \cdot 7, 24 = 3^1 \cdot 8\}$

$$\operatorname{ord}_3(24) = 1 + 1 + 2 + 1 + 1 + 2 + 1 + 1 = 10$$

Prove

(a) $\operatorname{ord}_p(n) = \sum_{k \geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor$

(b) $2 \mid \binom{2n}{n}$ and $p \mid \binom{2n}{n}$ for all $n < p \leq 2n$

(c) $p \geq 3 \wedge 2n/3 < p \leq n \Longrightarrow p \nmid \binom{2n}{n}$

(d) $p^r \mid \binom{2n}{n} \Longrightarrow p^r \leq 2n$

(e) $\frac{2^{2n-1}}{n} \leq \binom{2n}{n} \leq 2^{2n-1}$

(f) $\prod_{p \leq n} p < 4^n$

Reason: Primes.

Solution:

(a) The number of numbers in $\{1, 2, \ldots, n\}$ that are divisible by p is $\lfloor n/p \rfloor$. Among them are $\lfloor n/p^2 \rfloor$ many divisible by p^2, $\lfloor n/p^3 \rfloor$ divisible by p^3 etc.

(b) $\binom{2n}{n} = \binom{2n-1}{n-1} + \binom{2n-1}{n} = 2\binom{2n-1}{n-1} \Longrightarrow 2 \mid \binom{2n}{n}$

$\binom{2n}{n} = \frac{2n \cdot \ldots \cdot (n+1)}{1 \cdot \ldots \cdot n}$ and a prime $n < p \leq 2n$ in the numerator doesn't cancel.

(c) From $p > 3$ we get $p^2 > 2n$ for $1 \leq n \leq 4$ and from $p > 2n/3$ we get $p^2 > 4n^2/9 > 20n/9 > 2n$ for all $n \geq 5$. Thus

$$\begin{aligned} \operatorname{ord}_p \binom{2n}{n} &= \operatorname{ord}_p \left(\frac{(2n)!}{(n!)^2} \right) = \operatorname{ord}_p((2n)!) - 2 \operatorname{ord}_p(n!) \\ &= \sum_{k \geq 1} \left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \sum_{k \geq 1} \left\lfloor \frac{n}{p^k} \right\rfloor = \left\lfloor \frac{2n}{p} \right\rfloor - 2 \left\lfloor \frac{n}{p} \right\rfloor \\ &= 2 - 2 \cdot 1 = 0 \end{aligned}$$

which means that $p \nmid \binom{2n}{n}$

(d) Every term of the sum

$$\operatorname{ord}_p \binom{2n}{n} = \sum_{k\geq 1} \left\{ \left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right\}$$

is either 0 or 1 because for all real numbers x it is $\lfloor 2x \rfloor - 2\lfloor x \rfloor \in \{0,1\}$. Thus $\lfloor 2n/p^k \rfloor = 0$ for

$$k > r_p := \left\lfloor \log_p(2n) \right\rfloor$$

implies $\operatorname{ord}_p \binom{2n}{n} \leq r_p$, i.e. $p^r \leq p^{r_p} \leq 2n$.

(e)

$$(1+1)^{2n-1} = \binom{2n-1}{0} + \ldots + \underbrace{\binom{2n-1}{n-1} + \binom{2n-1}{n}}_{\leq \binom{2n}{n} = 2^{2n-1}} + \ldots + \binom{2n-1}{2n-1}$$

and $\binom{2n-1}{n-1} = \binom{2n-1}{n} \geq \dfrac{2^{2n-1}}{2n}$, i.e. $\binom{2n}{n} = 2\binom{2n-1}{n} \geq \dfrac{2^{2n-1}}{n}$.

(f) Set $P(n) := \prod_{p\leq n} p$. The statement is obviously true for $n = 1, 2$, so we may assume that $P(k) < 4^k$ for all $k < n$ and $n \geq 3$. If n is even, then by induction hypothesis $P(n) = P(2m) = P(2m-1) = P(n-1) < 4^{n-1} < 4^n$. So let $n = 2m-1$. we have seen that

$$\forall\ m < p \leq 2m\ :\ 2\cdot p \,|\, \binom{2m}{m} \Rightarrow 2\left(\prod_{m<p\leq 2m} p \right) \,|\, \binom{2m}{m} \leq 2^{2m-1}$$

$$\Longrightarrow \prod_{m<p\leq 2m} p \leq 2^{2m-2} = 4^{m-1}$$

$$\Longrightarrow P(n) = P(2m-1) = P(m) \cdot \prod_{m<p<2m} p < 4^m \cdot 4^{m-1} = 4^n$$

10. Let K be compact and $C(K) := \{f : K \to \mathbb{R} \text{ or } \mathbb{C} \,|\, f \text{ is continuous}.$ A $n-$dimensional subspace $M \subseteq C(K)$ is called Haar space, if all $f \in M - \{0\}$ have at most $n-1$ zeros. Linear independent functions $S := \{\varphi_1, \ldots, \varphi_n\} \subseteq C(K)$ are called a Chebyshev- or Haar-system, if $\operatorname{span}(S)$ is a Haar space. We denote the (compact) unit circle $\mathbb{T} := \{e^{2\pi i t} \,|\, t \in [0,1)\}$. Let $K \subseteq \mathbb{R}$ be compact or $K = \mathbb{T}$, $f \in C(K)$.

    We call a point $\xi \in K$ with $f(\xi) = 0$ a simple zero of f if ξ is either on the boundary of K or f changes sign in ξ. If $f(\xi - t)f(\xi + t) > 0$ in a neighborhood of ξ, then we speak of a double zero.

(a) A subspace $M \subseteq C(K)$ with $\dim M = n$ is a Haar space if and only if each $f \in M - \{0\}$ that has $j \in \mathbb{N}_0$ simple zeros and $k \in \mathbb{N}_0$ double zeros holds
$$j + 2k < n.$$
So each element $f \in M - \{0\}$ has at most $n - 1$ different zeros.

(b) The space of all real-valued trigonometric polynomials on $[0, 1)$ of degree at most n is a Haar space of dimension $2n + 1$.

(c) Let $n \in \mathbb{N}_0,\ p \in T_n$, and $x \in \mathbb{T}$. Then
$$|p'(x)| \leq 2\pi n \sqrt{\|p\|_\infty^2 - |p(x)|^2}.$$

Remark: Consider the linear differential operator $D(p) = p'$ on T_n. From $|D(\sin(2\pi nx))| = |2\pi n \cos(2\pi nx)|$ we conclude that
$$2\pi n \leq \|D\| = \sup_{\|p\|_\infty \leq 1} \|p'\|_\infty = \sup_{\|p\|_\infty = 1} \sup_{x \in \mathbb{T}} |p'(x)| < \infty$$
because T_n is finite-dimensional.

Reason: Szegö's inequality.

Solution:

(a) We only have to show $j + 2k < n$ holds for $f \neq 0$ in a Haar space. Assume $\xi_1, \ldots, \xi_j \in K$ are the simple zeros of f and $\eta_1, \ldots, \eta_k \in K$ the double zeros, and that
$$j + 2k \geq n$$
Set $A_0(f) := \{\xi_1, \ldots, \xi_j, \eta_1, \ldots, \eta_k\}$. Then $\#A_0(f) = j + k \leq n - 1$ by definition of a Haar space. In particular $k \geq 1$. We choose an interval $[\eta_i - \delta_i, \eta_i + \delta_i] \subseteq K$ with $\delta_i > 0$ around each double zero of f, such that no further zeros are contained, and set
$$\begin{aligned} c_i &:= \operatorname{sgn} f(\eta_i - \delta_i) = \operatorname{sgn} f(\eta_i + \delta_i) \\ C &:= \min_{1 \leq i \leq k} \{|f(\eta_i - \delta_i)|\, ,\, |f(\eta_i + \delta_i)|\} \end{aligned}$$
We construct an interpolation function $q \in M$ by adding arbitrary points $\theta_1, \ldots, \theta_{n-j-k} \in K - A_0(f)$ such that
$$q(\eta_i) = c_i\,(1 \leq i \leq k),\ q(\xi_i) = 0\,(1 \leq i \leq j),\ q(\theta_i) = 0\,(1 \leq i \leq n{-}j{-}k).$$

The functions $f_\alpha := f - \alpha q \in M$ for $0 < \alpha < \dfrac{C}{\|q\|_\infty}$ have function values

$$f_\alpha(\xi_i) = 0\,(1 \leq i \leq j),\; f_\alpha(\eta_i) = -\alpha c_i \neq 0\,(1 \leq i \leq k),$$

and sign changes

$$c_i = \operatorname{sgn} f_\alpha(\eta_i - \delta_i) = -\operatorname{sgn} f_\alpha(\eta_i) = \operatorname{sgn} f_\alpha(\eta_i + \delta_i).$$

Thus we have two zeros of f_α in each interval $[\eta_i - \delta_i, \eta_i + \delta_i]$, and f_α has at least $j + 2k \geq n$ zeros, contradicting the Haar condition for f_α.

(b) Set $z := e^{2\pi i x} \in \mathbb{C}$. Each trigonometric polynomial $f \in T_n - \{0\}$ has the form

$$f(x) = \sum_{k=-n}^{n} c_k e^{2\pi i k x} = \sum_{k=-n}^{n} c_k z^k = z^{-n} \sum_{k=0}^{2n+1} c_{k-n} z^k$$

with coefficient vector $(c_{-n}, \ldots, c_n) \neq 0$. The last sum is a polynomial $q \neq 0$ of degree $2n + 1$. It has at most $2n$ complex zeros which may be in $\mathbb{T}$. Hence $f(x)$ has at most $2n$ zeros in $[0, 1)$.

(c) Set $q(x) := \dfrac{p(x)}{\|p\|_\infty}$. Then for all $\|q\|_\infty = 1$

$$|p'(x)| \leq 2\pi n \sqrt{\|p\|_\infty^2 - |p(x)|^2} \Longleftrightarrow |q'(x)| \leq 2\pi n \sqrt{1 - |q(x)|^2}.$$

We will first show that there are no $p \in T_n, x_0 \in \mathbb{T}$ such that

$$\|p\|_\infty < 1\;,\; |p'(x_0)| = 2\pi n \sqrt{1 - |p(x_0)|^2}$$

Assume there is. We may assume w.l.o.g. that the condition holds at $x_0 = 0$, for otherwise, we shift the periodic function, and we assume $p'(x_0) = p'(0) \geq 0$ by choice of sign.

We choose an $\alpha \in \left(-\dfrac{1}{4n}, \dfrac{1}{4n}\right)$ with $p(0) = \sin(2\pi n \alpha)$ which is possible since $|p(0)| \leq \|p\|_\infty < 1$.

Now define $q(x) := \sin(2\pi n(x + \alpha)) - p(x) \in T_n$. If $q(x) \equiv 0$, then $1 > \|p\|_\infty = \|\sin(2\pi n(x + \alpha))\|_\infty = 1$ which is a contradiction. Hence $q(x) \neq 0$.

$$\begin{aligned} q(0) = q'(0) &= 2\pi n \cos(2\pi n \alpha) - p'(0) \\ &= 2\pi n \cos(2\pi n \alpha) - 2\pi n \sqrt{1 - |p(0)|^2} = 0 \end{aligned}$$

i.e. we have a double zero at $x = 0$. The points

$$x_k := \alpha + \frac{2k+1}{4n}\,,\; 0 \leq k \leq 2n-1$$

are pairwise distinct in $(0,1)$. These points are extreme values of $q(x) - p(x)) = \sin(2\pi n(x-\alpha))$, so for $0 \leq k \leq 2n-1$

$$\operatorname{sgn}(q(x_k) - p(x_k)) = \operatorname{sgn}\left(\sin\left(\frac{\pi}{2}\cdot(2k+1)\right)\right) = (-1)^k$$

Since $|p(x_k)| < 1$, we have $\operatorname{sgn}(q(x_k) - p(x_k)) = \operatorname{sgn}(q(x_k))$ which by the mean value theorem means that $q(x) \in T_n - \{0\}$ has at least $2n-1$ zeros in $(0,1)$, which together with the double zero at $x = 0$ gives $2n+1$ zeros counted by multiplicities, and contradicting the Haar condition. We could also conclude by the previous part that

$$2n - 1 + 2 \cdot 1 = 2n + 1 < 2n + 1$$

which is also a contradiction.

Thus we have proven, that for all $p \in T_n - \{0\}$ that for all $x \in [0,1)$

$$\|p\|_\infty \geq 1 \;\vee\; |p'(x)| \neq 2\pi n\sqrt{1 - |p(x)|^2}$$

Let $p \in T_n - \{0\}$ and $\lambda \in \left[0, \dfrac{1}{\|p\|_\infty}\right) \subseteq [0,1)$, so $\|\lambda p\|_\infty < 1$. Let

$$f(\lambda, x) := |\lambda p'(x)| - 2\pi n\sqrt{1 - |\lambda p(x)|^2} \neq 0$$

because $\lambda p \in T_n$. By continuity of $f(\cdot, x)$, and $f(0,x) = -2\pi n < 0$ we get from the intermediate value theorem that $f(\lambda, x) < 0$ for all $\lambda \in \left[0, \dfrac{1}{\|p\|_\infty}\right)$ and

$$\lim_{\lambda \to \|p\|_\infty^{-1}} f(\lambda, x) \leq 0$$

i.e.

$$\left|\frac{p'(x)}{\|p\|_\infty}\right| \leq 2\pi n\sqrt{1 - \left|\frac{p(x)}{\|p\|_\infty}\right|^2}$$

what had to be shown.

11. (HS-1) Let S be a set of real numbers which is closed under multiplication (that is, if a and b are in S, then so is ab). Let T and U be disjoint subsets of S whose union is S. Given that the product of any three (not necessarily distinct) elements of T is in T and that the product of any three elements of U is in U, show that at least one of the two subsets T, U is closed under multiplication.

    Reason: Logic.

    Solution: Suppose on the contrary that there exist $t_1, t_2 \in T$ with $t_1t_2 \in U$ and $u_1, u_2 \in U$ with $u_1u_2 \in T$. Then $(t_1t_2)u_1u_2 \in U$ while $t_1t_2(u_1u_2) \in T$, a contradiction.

12. (HS-2) Suppose we have a necklace of n beads. Each bead is labeled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy
$$\sum_{i=1}^{k} x_i \leq k-1 \quad (k = 1, \ldots, n).$$

    Reason: Cycles.

    Solution: Let $S_k = x_1 + \ldots + x_k - \dfrac{k(n-1)}{n}$, so that $S_n = S_0 = 0$. These form a cyclic sequence that doesn't change when you rotate the necklace, except that the entire sequence gets translated by a constant. In particular, it makes sense to choose x_i for which S_i is maximal and make that one x_n; this way $S_i \leq 0$ for all i, and thus $x_1 + \ldots + x_i \leq i \cdot \dfrac{n-1}{n}$. However, the right side may be replaced by $i-1$ because the left side is an integer.

13. (HS-3) Let $d := d_1d_2 \ldots d_9$ be a number with not necessarily distinct nine decimal digits. A number $e := e_1e_2 \ldots e_9$ is such that each of the nine digit numbers formed by replacing just one of the digits d_j by the corresponding digit e_j is divisible by 7 for all $1 \leq j \leq 9$. A number $f := f_1f_2 \ldots f_9$ is formed the same way by starting with e, i.e. each of the nine numbers formed by replacing a e_k by f_k is divisible by 7. Example: If $d = 20210901$ then $e_6 \in \{0, 7\}$ since $7 \,|\, 20210001$ and $7 \,|\, 20210701$. Show that, for each j, $d_j - f_j$ is divisible by 7.

    Reason: Numbers.

    Solution: We are given that for all $1 \leq j \leq 9$
$$(e_j - d_j)10^{9-j} + d \equiv 0 \equiv (f_j - e_j)10^{9-j} + e \mod 7 \quad (*)$$

Thus $\sum_{j=1}^{9}(e_j - d_j)10^{9-j} + d = e - d + 9d \equiv e + d \equiv 0 \mod 7$. Now add the first and second relation from $(*)$ for any particular value j and get

$$0 \equiv (f_j - d_j)10^{9-j} + e + d \equiv (f_j - d_j)10^{9-j} \mod 7$$

Because 7 is prime and $7 \nmid 10^{9-j}$ this implies $7 \,|\, (d_j - f_j)$.

14. (HS-4) An ellipse, whose semi-axes have lengths a and b, rolls without slipping on the curve $y = c\sin(x/a)$. How are a, b, c related, given that the ellipse completes one revolution when it traverses one period of the curve?

    Reason: Analytical geometry.

    Solution: Without slipping means that the perimeter of the ellipse equals the length of one period of the sine curve, which translates to the integral equation

$$\int_0^{2\pi} \sqrt{(-a\sin(\theta))^2 + (b\cos(\theta))^2}\, d\theta = \int_0^{2\pi a} \sqrt{1 + \frac{c^2}{a^2}\cos^2\left(\frac{x}{a}\right)}\, dx$$

    Let $\theta = \dfrac{x}{a}$ in the second integral, $1 = \sin^2\theta + \cos^2\theta$, then

$$\int_0^{2\pi} \sqrt{a^2\sin^2(\theta) + b^2\cos^2(\theta)}\, d\theta = \int_0^{2\pi} \sqrt{a^2\sin^2\theta + (a^2 + c^2)\cos^2\theta}\, d\theta$$

    Since the left side is an increasing function in b, and the right side doesn't explicitly depend on b, we must have equality if and only if $b^2 = a^2 + c^2$.

15. (HS-5) For a partition π of $N := \{1, 2, \ldots, 9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π_1 and π_2, there are two distinct numbers x and y in N such that $\pi_j(x) = \pi_j(y)$ for $j = 1, 2$.

    Reason: Sets.

    Solution: For a given partition π_1, no more than three different values of $\pi_1(x)$ are possible, since four would require one part each of size at least $1, 2, 3, 4$, and that's already more than 9 elements. If no such x, y exist, each pair $(\pi_1(x), \pi_2(x))$ occurs for at most one element of x, since there are only $3 \cdot 3$ possible pairs, and each must occur exactly once. In particular, each value of $\pi_1(x)$ must occur 3 times. However, any given value of $\pi_1(x)$ occurs $C \cdot \pi(x)$ times, where C is the number of distinct partitions of that size. Thus $\pi_1(x)$ can occur 3 times only if it equals 1 or 3, but we have three distinct values for which it occurs, a contradiction.

5 August 2021

1. Let (X, ρ) be a metric space, and suppose that there exists a sequence $(f_i)_i$ of real-valued continuous functions on X with the property that a Cauchy sequence $(x_n)_n$ is convergent whenever each of the sequences $(f_i(x_n))_i$ is bounded. Then X can be remetrized (with equivalent metrics) so as to be complete.

   Reason: Metric spaces.

   Solution: Define a new distance function in X by

   $$\sigma(x, y) = \rho(x, y) + \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, |f_i(x) - f_i(y)|\}$$

   which is a metric because the triangle axiom is satisfied by each term. The other axioms are obvious.

   For any $\varepsilon > 0$ and $x \in X$ there is an integer N such that $2^{-N} < \varepsilon$ and a positive number $\delta < \varepsilon$ such that

   $$\rho(x, y) < \delta \Longrightarrow |f_i(x) - f_i(y)| < \varepsilon \quad (i = 1, 2, \ldots, N)$$

   If $\rho(x, y) < \delta$, then

   $$\sigma(x, y) < \varepsilon + \sum_{i=1}^{N} \frac{1}{2^i} \min\{1, |f_i(x) - f_i(y)|\} + \frac{1}{2^N} < 3\varepsilon.$$

   Therefore $\sigma(x, x_n) \to 0$ whenever $\rho(x, x_n) \to 0$. The converse follows from the inequality $\rho(x, y) \leq \sigma(x, y)$. Thus σ and ρ are equivalent metrics.

   To show that (X, σ) is complete, let $(x_n)_n$ be a Cauchy sequence relative to σ. Then for any natural number k there is a natural number N such that $\sigma(x_n, x_m) < 2^{-k}$ for all $n, m \geq N$. For all $n, m \geq N$, we have

   $$1 > 2^k \sigma(x_n, x_m) \geq \min\{1, |f_k(x_n) - f_k(x_m)|\},$$

   and therefore $|f_k(x_n) - f_k(x_m)| < 1$. Let $M(k) := \max\{|f_k(x_1)|, \ldots, |f_k(x_N)|\}$.

   $$\begin{aligned} |f_k(x_n)| &= |f_k(x_n) - f_k(x_N) + f_k(x_N)| \\ &\leq |f_k(x_n) - f_k(x_N)| + |f_k(x_N)| \\ &< 1 + M(k) \end{aligned}$$

   and the sequence $(f_k(x_n))_n$ is bounded, for each k. Since $\rho(x, y) \leq \sigma(x, y)$, the sequence $(x_n)_n$ is also a Cauchy sequence relative to ρ, and by hypothesis, convergent.

2. Let $X = C([0,1])$ be the topological space of real-valued continuous functions,
$$\rho(f,g) := \sup_{x\in[0,1]} |f(x) - g(x)|$$
the uniform metric induced by the L^∞ norm,
$$\sigma(f,g) := \int_0^1 |f(x) - g(x)|\,dx$$
the L^1 induced metric, and for $n \in \mathbb{N}$
$$E_n := \{\, f \in X \mid \exists\, x \in [0, 1 - 1/n]\ \forall\, h \in (0, 1-x)\ :\ |f(x+h) - f(x)| \leq nh \}\,.$$
Show that

(a) (X, ρ) is complete.

(b) $(X, \rho) \ncong (X, \sigma)$ is not complete.

(c) $E_n \subseteq (X, \rho)$ is closed.

Reason: Topology.

Solution:

(a) ρ is called uniform metric, because convergence in this metric implies uniform convergence. Let $(f_n) \subseteq (X, \rho)$ be a Cauchy sequence, say $\rho(f_i, f_j) < \varepsilon$ for all $i, j > N(\varepsilon)$. Then
$$|f_i(x) - f_j(x)| < \varepsilon \text{ for all } i, j > N(\varepsilon) \text{ and } x \in [0,1].$$
Hence $(f_n(x)) \subseteq \mathbb{R}$ is a Cauchy sequence for all $x \in [a, b]$, and therefore converging to a limit $f(x)$. Letting $j \to \infty$ we see that $|f_i(x) - f(x)| < \varepsilon$ for all $i > N(\varepsilon)$ and all $x \in [0, 1]$. Thus (f_i) converges uniformly on $[0, 1]$. Let $x_0 \in [0, 1]$ and $\rho(f_M, f) < \varepsilon/3$. Since f_M is continuous at x_0, there is a $\delta > 0$ such that
$$|f_M(x) - f_M(x_0)| < \varepsilon/3 \text{ for all } |x - x_0| < \delta.$$
Hence
$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_M(x)| + |f_M(x) - f_M(x_0)| + |f_M(x_0) - f(x_0)| \\ &\leq \rho(f, f_M) + \varepsilon/3 + \rho(f_M, f) < \varepsilon, \end{aligned}$$
i.e. $f(x)$ is continuous at any $x_0 \in [0, 1]$ and $f_n \to f$ in C. This shows that (X, ρ) is complete.

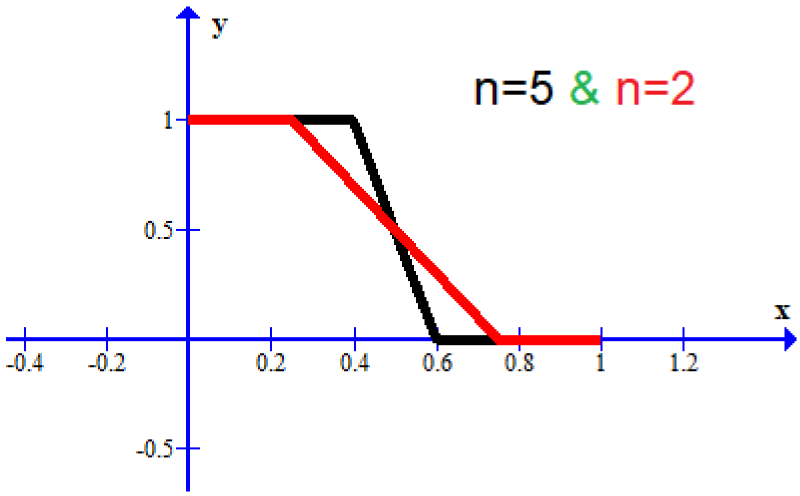
(b) Consider $f_n(x) := \max\{1 - nx, 0\}$ and let $f \equiv 0$. Then we get for $n > 1$

$$\sigma(f_n, f) = \int_0^1 |\max\{1-nx,0\}|\,dx = \int_0^{1/n} |1-nx|\,dx = \frac{1}{n} - \frac{n}{2n^2} = \frac{1}{2n}$$

whereas $\rho(f_n, f) = 1$. Thus $f_n \to f$ in (X, σ) but not in (X, ρ), hence these spaces are not homeomorphic.

To see that (X, σ) is not complete, let

$$f_n(x) := \begin{cases} \min\left\{1\,,\,\frac{1}{2} - n\left(x - \frac{1}{2}\right)\right\} & \text{on } \left[0, \frac{1}{2}\right] \\ \max\left\{0\,,\,\frac{1}{2} - n\left(x - \frac{1}{2}\right)\right\} & \text{on } \left[\frac{1}{2}, 1\right] \end{cases}$$



$$\sigma(f_n - f_m) = \frac{1}{4} \cdot \left|\frac{1}{n} - \frac{1}{m}\right|$$

and (f_n) is a Cauchy sequence. Suppose $\sigma(f_n, f) \to 0$ as $n \to \infty$ for some $f \in C([0,1])$. Then

$$\begin{aligned} \sigma(f_n, f) &= \int_0^1 |f_n(x) - f(x)|\,dx \\ &\geq \int_0^{\frac{1}{2}-\frac{1}{2n}} |f_n(x) - f(x)|\,dx + \int_{\frac{1}{2}+\frac{1}{2n}}^1 |f_n(x) - f(x)|\,dx \\ &\geq \int_0^{\frac{1}{2}-\frac{1}{2n}} |1 - f(x)|\,dx + \int_{\frac{1}{2}+\frac{1}{2n}}^1 |f(x)|\,dx \end{aligned}$$

Letting $n \to \infty$ it follows that

$$\int_0^{\frac{1}{2}} |1 - f(x)| \, dx = \int_{\frac{1}{2}}^1 |f(x)| \, dx = 0.$$

Since $f(x)$ is continuous, we must have $f(x) = 1$ on $[0, 1/2]$ and $f(x) = 0$ on $[1/2, 1]$, which is impossible. Therefore (X, σ) cannot be complete.

(c) Let $(f_k) \subseteq E_n$ be a sequence that converges to $f \in (X, \rho)$. This is possible because $(X, \rho) \supseteq E_n$ is complete. Be definition of E_n we have a corresponding sequence $(x_k) \subseteq [0, 1 - (1/n)]$ and

$$|f_k(x_k + h) - f_k(x_k)| \leq nh \text{ for all } 0 < h < 1 - x_k \, .$$

We may assume also that $\lim_{k\to\infty} x_k = x \in [0, 1-(1/n)]$, for some suitable x. This condition can be achieved by choosing an appropriate subsequence of (f_k) and because $[0, 1 - (1/n)]$ is compact. If $0 < h < 1 - x$, the inequality $0 < h < 1 - x_k$ holds for all sufficiently large k, and then

$$\begin{aligned} |f(x+h) - f(x)| &\leq |f(x+h) - f(x_k+h)| + |f(x_k+h) - f_k(x_k+h)| \\ &\quad + |f_k(x_k+h) - f_k(x_k)| + |f_k(x_k) - f(x_k)| \\ &\quad + |f(x_k) - f(x)| \\ &\leq |f(x+h) - f(x_k+h)| + \rho(f, f_k) + nh + \rho(f_k, f) \\ &\quad + |f(x_k) - f(x)| \end{aligned}$$

Letting $k \to \infty$, and using the fact that f is continuous at x and $x + h$, it follows that

$$|f(x+h) - f(x)| \leq nh \text{ for all } 0 < h < 1 - x.$$

Therefore $f \in E_n$ and E_n is closed.

3. Show that the set of real algebraic numbers is infinite, and denumerable.

Reason: Countability.

Solution: Let us define the weight of a polynomial $f(x) = \sum_{i=0}^n a_i x^i$ to be the number $n + \sum_{i=0}^n |a_i|$. There are only a finite number of polynomials having a given weight. Arrange these in some order, say lexicographically (first in order of n, then in order of a_0, and so on). Every non-constant polynomial has a weight at least equal to 2. Taking the polynomials of weight 2 in order, then those of weight 3, and so on, we obtain a sequence $f_1, f_2, f_3, \ldots$ in which every polynomial has at most a finite number of real zeros. Number the zeros of f_1 in order, then those of f_2, and so on, passing over any that have already been numbered. In this way we obtain a definite enumeration of all real algebraic numbers. The sequence is infinite because it includes all rational numbers.

4. A topology $\mathcal{T}$ on a vector space L over a non-discrete topological field K defines a topological vector space, i.e. addition and scalar multiplication are continuous, if and only if $\mathcal{T}$ is translation-invariant (all mappings $x \mapsto x + x_0$ are homeomorphisms) and possesses a $0-$neighborhood base $\mathcal{B}$ with the following properties:

   (a) For each $V \in \mathcal{B}$, there exists $U \in \mathcal{B}$ such that $U + U \subseteq V$.

   (b) Every $V \in \mathcal{B}$ is radial (i.e. there exists a $\lambda_0 \in K$ such that whenever $|\lambda| \geq |\lambda_0|$ we have $F \subseteq \lambda V$ for each finite subset $F \subseteq L$) and circled ($\lambda V \subseteq V$ whenever $|\lambda| \leq 1$).

   (c) There exists $\lambda \in K$, $0 < |\lambda| < 1$, such that $V \in \mathcal{B}$ implies $\lambda V \in \mathcal{B}$.

   If K is an Archimedean valuated field, e.g. $K \in \{\mathbb{R}, \mathbb{C}\}$, then the last condition is dispensable.

   Reason: Topological vector spaces.

   Solution: First let $(L, \mathcal{T})$ be a topological vector space. We note that for each $x_0 \in L$ and each $\lambda_0 \in K - \{0\}$, the mapping $x \longmapsto \lambda_0 x + x_0$ is a homeomorphism of L onto itself. It is clearly onto L and, by continuity of scalar multiplication and addition, continuous with continuous inverse $y \longmapsto \lambda_0^{-1}(y - x_0)$. Given a $0-$neighborhood W in L, there exists a $0-$neighborhood U and a real number $\varepsilon > 0$ such that $\lambda U \subseteq W$ whenever $|\lambda| < \varepsilon$ since scalar multiplication is continuous; hence since K is non-discrete, $V := \cup\{\lambda U \,|\, |\lambda| < \varepsilon\}$ is a $0-$neighborhood which is contained in W, and obviously circled. This the family $\mathcal{B}$ of all circled $0-$neighborhoods in L is a base at 0. The continuity at $\lambda = 0$ of $(\lambda, x_0) \longrightarrow \lambda x_0$ for each $x_0 \in L$ implies that every $V \in \mathcal{B}$ is radial. It is obvious from continuity of addition that $\mathcal{B}$ satisfies the first condition. Given $V \in \mathcal{B}$, and since K is non-discrete, there is a $\lambda \in K, 0 < |\lambda| < 1$, such that λV is a $0-$neighborhood by our initial statement, and which is circled. Again by our initial statement, we observe that the topology in translation-invariant.

   Conversely let $\mathcal{T}$ be a translation-invariant topology on L possessing a

0−neighborhood base $\mathcal{B}$ with the three properties as stated. We need to show continuity of addition and scalar multiplication. It is clear that $\{x_0 + V \,|\, V \in \mathcal{B}\}$ is a neighborhood base of $x_0 \in L$, hence if $V \in \mathcal{B}$ is given, and $U \in \mathcal{B}$ can be selected such that $U + U \subseteq V$, then $x - x_0, y - y_0 \in U$ implies that $x + y \in x_0 + y_0 + V$, so addition is continuous. Now let $\lambda_0 \in K, x_0 \in L$ be any fixed points. If $V \in \mathcal{B}$ is given, then there is a $U \in \mathcal{B}$ such that $U + U \subseteq V$. By radiality of U, there is a real number $\varepsilon > 0$ such that $(\lambda - \lambda_0)x_0 \in U$ whenever $|\lambda - \lambda_0| < \varepsilon$. Let $\mu \in K$ satisfy the third condition. Then there exists a natural number $n \in \mathbb{N}$ such that $|\mu^{-n}| = \infty\mu|^{-n} > |\lambda_0| + \varepsilon$. Set $W := \mu^n U \in \mathcal{B}$. Now since U is circled, the relations $x - x_0 \in W$ and $|\lambda - \lambda_0| < \varepsilon$ imply $\lambda(x - x_0) \in U$, and hence with

$$\lambda x = \lambda_0 x_0 + (\lambda - \lambda_0)x_0 + \lambda(x - x_0)$$

that $\lambda x \in \lambda_0 x_0 + U + U \subseteq \lambda_0 x_0 + V$, i.e. scalar multiplication is continuous.

Finally, if K is an Archimedean valuated field, then $|2| > 1$ for $2 = 1 + 1 \in K$ and thus $|2^n| = |2|^n > \lambda_0 + \varepsilon$ for a suitable $n \in \mathbb{N}$. By repeated application of the second condition, we can select a $W_1 \in \mathcal{B}$ such that

$$2^n W_1 \subseteq W_1 + \ldots + W_1 \subseteq U$$

Since W_1 and hence $2^n W_1$ are circled, W_1 can be substituted for W in the preceding proof, so the third condition is dispensable in this case.

5. Let $L \stackrel{\mu}{\longrightarrow} M$ be locally convex topological vector spaces, $\mathcal{P}$ a family of semi-norms ($\|\alpha x\|_p = |\alpha| \cdot \|y\|_p$ and $\|x+y\|_p \leq \|x\|_p + \|y\|_p$) generating the topology of L and μ algebraically homomorph, i.e. linear. Then μ is continuous if and only if for each continuous semi-norm q on M, there exists a finite subset $\{p_j \,|\, j = 1, \ldots, n\} \subseteq \mathcal{P}$ and a number $c > 0$ such that $\|\mu(x)\|_q < c \cdot \sup_j p_j(x)$ for all $x \in L$.

   Reason: Continuity of Linear Maps.

   Solution: Necessity. Let V be the 0−neighborhood $\{y \,|\, \|y\|_q < 1\}$, where q is a given continuous semi-norm on M. Since μ is continuous and $\mathcal{P}$ generates the topology of L, there exist 0−neighborhoods $U_j = \{x \,|\, \|x\|_{p_j} < \varepsilon_j\}$ where $\varepsilon_j > 0$ and $p_j \in \mathcal{P}$ for $j = 1, \ldots, n$, such that $\mu\left(\cap_{j=1}^n U_j\right) \subseteq V$. Hence, letting $\varepsilon := \min\{\varepsilon_1, \ldots, \varepsilon_n\}$, the relation $\sup\{p_1(x), \ldots, p_n(x)\} < \varepsilon$ implies $\mu(x) \in V$, thus $\|\mu(x)\|_q < \varepsilon^{-1} \sup\{p_1(x), \ldots, p_n(x)\}$ for all $x \in L$.

   Sufficiency. If V is a given convex circled 0−neighborhood in M, its gauge function $q : x \longmapsto \|q(x)\| = \inf\{\lambda > 0 \,|\, x \in \lambda M\}$ is a continuous semi-norm. Thus if $\|\mu(x)\|_q < c \cdot \sup\{p_1(x), \ldots, p_n(x)\}$, where $c > 0$ and $p_j \in \mathcal{P}$ for $j = 1, \ldots, n$, it follows that $\mu(U) \subseteq V$ for the $0-$neighborhood $U = \{x \,|\, cp_j(x) < 1\,,\, j = 1, \ldots, n\}$ in L.

An important corollary is:

If $(L, \|.\|_L) \stackrel{\mu}{\longrightarrow} (M, \|.\|_M)$ is a linear map between normed spaces, i.e. a linear operator, then μ is continuous if and only if μ is bounded: $\|\mu(x)\|_M < c\|x\|_L$ for some $c > 0$ and all $x \in L$.

6. Let $\mathbb{F}_q$ be a finite field of characteristic p. Show that it's multiplicative group $\mathbb{F}_q^* = \mathbb{F}_q - \{0\}$ is cyclic.

Reason: Finite fields.

Solution: If $n \in \mathbb{N}$ and φ denotes the Euler φ-function, then

$$n = \sum_{d\,|\,n} \varphi(d).$$

If $d\,|\,n$, let C_d be the unique subgroup of $\mathbb{Z}_n$ of order d, and let $\Phi(d)$ be the set of generators of C_d. Since all elements of $\mathbb{Z}_n$ generate one of the C_d, the group $\mathbb{Z}_n$ is the disjoint union of the $\Phi(d)$ and we have

$$n = \operatorname{card}(\mathbb{Z}_n) = \sum_{d\,|\,n} \operatorname{card}(\Phi(d)) = \sum_{d\,|\,n} \varphi(d).$$

Let H be any finite group of order n. Suppose that, for all divisors d of n, the set of $x \in H$ such that $x^d = 1$ has at most d elements. Then H is cyclic.

Let $d\,|\,n$ and $x \in H$ of order d. Then all elements $y \in H$ with $y^d = 1$ are at most d many by the hypothesis. They form a group that contains x, hence

$$\langle x \rangle = \left\{ y \in H \,|\, y^d = 1 \right\} \cong C_d$$

In particular, all elements of H of order d are generators of $\langle x \rangle$ and these are in number $\varphi(d)$. Hence, the number of elements of H of order d is 0 or $\varphi(d)$. If it were zero for a value of d, the formula $n = \sum_{d\,|\,n} \varphi(d)$ would show that the number of elements H is less that n, contrary to hypothesis. In particular, there exists an element $x \in H$ of order n and $H = \langle x \rangle$, i.e. H is a cyclic group.

Finally, we set $H = \mathbb{F}_q^*$ and $n = q - 1$. The polynomial $X^d - 1 \in \mathbb{F}_q[X]$ has at most d solutions in $\mathbb{F}_q$, so we can apply what we just have proven and conclude that $H = \mathbb{F}_q^*$ is cyclic.

7. Let $\mathbb{F}_q$ be a finite field of characteristic p and $f_\alpha \in \mathbb{F}_q[X_1,\ldots,X_n]$ polynomials such that $\sum_\alpha \deg f_\alpha < n$, and $V \subseteq \mathbb{F}_q^n$ be the set of their common zeros. Then
$$p \,|\, \operatorname{card}(V)$$
Reason: Polynomials over finite fields.

Solution: We first observe that for an integer $n \geq 0$
$$S_n(X) := \sum_{x\in\mathbb{F}_q} x^n = \begin{cases} -1 & \text{if } n \geq 1 \wedge (q-1)\,|\,n \\ 0 & \text{otherwise} \end{cases}$$
with the convention $0^0 = 1$. The case $n = 0$ is obvious since $q \equiv 0\,(p)$ so we may assume $n \geq 1$. The multiplicative group of $\mathbb{F}_q$ is cyclic of order $q-1$, so in case n is divisible by $q-1$, we have $S_n(X) = 0^n + \sum_{x\neq 0} (x^{q-1})^e = 0 + \sum_{x\neq 0} 1^e = q-1 = -1$ for some $e \in \mathbb{N}$. Next assume $(q-1) \nmid n$. Then there exists a $y \in \mathbb{F}_q - \{0\}$ with $y^n \neq 1$ and
$$\begin{aligned} S_n(X) &= \sum_{x\in\mathbb{F}_q} x^n = \sum_{x\neq 0} x^n = \sum_{x\neq 0} y^n x^n = y^n \sum_{x\neq 0} x^n = y^n S_n(X) \\ &\Longrightarrow (1-y^n)S_n(X) = 0 \\ &\Longrightarrow S_n(X) = 0 \end{aligned}$$
Let $P(X) := \prod_\alpha (1-f_\alpha)^{q-1}$ and $x \in \mathbb{F}_q^n$. If $x \in V$ then all $f_\alpha(x) = 0$ and $P(x) = 1$; if $x \notin V$, we have one of the f_α with $f_\alpha \neq 0$ but $f_\alpha^{q-1} = 1$, hence $P(x) = 0$. Thus $P(X)$ is the characteristic function of V. If, for every polynomial f, we put $S(f) := \sum_{x\in\mathbb{F}_q^n} f(x)$, we have
$$\operatorname{card}(V) \equiv S(P)\,(p)$$
and we have to show that $S(P) = 0$. The hypothesis $\sum_\alpha \deg f_\alpha < n$ implies that $\deg P < n(q-1)$. Thus $P(X)$ is a linear combination of monomials $X^v = X_1^{u_1}\cdots X_n^{u_n}$ with $\sum u_j < n(q-1)$. This means that at least one $0 \leq u_j < q-1$ and $S(X_j^{u_j}) = 0$ by our initial observation. Now
$$S(X^v) = \sum_{x\in\mathbb{F}_q^n} X^v(x) = \sum_{x\in\mathbb{F}_q^n} x_1^{u_1}\cdots x_n^{u_n} = 0 \Longrightarrow S(P) = 0$$

8. A linear fractional transformation $S : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ defined by $S(z) = \dfrac{az+b}{cz+d}$ is called a Möbius transformation if $ad - bc \neq 0$. Here $S(\infty) =$

a/c and $S(-d/c) = \infty$. Show that Möbius transformations form a group by composition, and that there is a unique Möbius transformation $S(z)$ which takes (z_1, z_2, z_3) to $(1, 0, \infty)$. Which one?

Reason: Möbius transformation.

Solution: Associativity is guaranteed by taking composition as multiplication, multiplicative closure and the identity element are obvious, so only the existence of an inverse has to be shown. Set

$$S^{-1}(z) := \frac{dz - b}{-cz + a}$$

$$S(S^{-1}(z)) = \frac{a\dfrac{dz-b}{-cz+a} + b}{c\dfrac{dz-b}{-cz+a} + d} = \frac{adz - ab - bcz + ab}{cdz - cb - cdz + ad} = z$$

$$S^{-1}(S(z)) = \frac{d\dfrac{az+b}{cz+d} - b}{-c\dfrac{az+b}{cz+d} + a} = \frac{adz + bd - bcz - bd}{-acz - bc + acz + ad} = z$$

Let $S \not\equiv 1$ and $S(z) = z$. Then $0 = cz^2 + (d-a)z - b$ which has at most two different solutions. If $a, b, c \in \mathbb{C}_\infty$ are three different points such that $S(a) = T(a), S(b) = T(b), S(c) = T(c)$ for two Möbius transformations S, T. Then $T^{-1} \circ S$ has three fixed points, i.e. $T^{-1}S \equiv 1$ and $S = T$. Hence, a Möbius map is uniquely determined by its action on any three distinct given points in $\mathbb{C}_\infty$. Set

$$S(z) := \begin{cases} \dfrac{\dfrac{z-z_2}{z-z_3}}{\dfrac{z_1-z_2}{z_1-z_3}} & \text{if } z_1, z_2, z_3 \in \mathbb{C} \\ \dfrac{z-z_2}{z-z_3} & \text{if } z_1 = \infty \\ \dfrac{z_1-z_3}{z-z_3} & \text{if } z_2 = \infty \\ \dfrac{z-z_2}{z_1-z_2} & \text{if } z_3 = \infty \end{cases}$$

Then $S(z_1) = 1\,,\ S(z_2) = 0\,,\ S(z_3) = \infty$.

9. Let G be a connected open set and let $f : G \longrightarrow \mathbb{C}$ be an analytic function. Show that the following statements are equivalent:

   (a) $f \equiv 0$

   (b) There is a point $a \in G$ such that $f^{(n)}(a) = 0$ for each $n \geq 0$.

   (c) $\{z \in G \,|\, f(z) = 0\}$ has a limit point in G.

   Reason: Function theory.

   Solution: It is sufficient to show that $(c) \Rightarrow (b)$ and $(b) \Rightarrow (a)$.

   Let $a \in G$ be a limit point of $Z := \{z \in G \,|\, f(z) = 0\}$, and let $R > 0$ be such that $B(a; R) \subseteq G$. Since f is continuous, it follows $f(a) = 0$. Suppose there is an integer $n \geq 1$ such that $f(a) = f^{'}(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$. Expanding f in power series about a gives that
   $$f(z) = \sum_{k=n}^{\infty} a_k(z-a)^k$$
   for $|z - a| < R$. If
   $$g(z) := \sum_{k=n}^{\infty} a_k(z-a)^{k-n}$$
   then g is analytic in $B(a; R)$, $f(z) = (z - a)^n g(z)$, and $g(a) = a_n \neq 0$. Since g is continuous in $B(a; R)$ we can find an $0 < r < R$, such that $g(z) \neq 0$ for $|z - a| < r$. But since a is a limit point of Z there is a point $b \in Z$ with $0 < |b - a| < r$. This gives $0 = (b - a)^n g(b)$ and so $g(b) = 0$, a contradiction. Hence no such integer n can be found, which proves (b).

   Let $A := \{z \in G \,|\, f^{(n)}(z) = 0 \text{ for all } n \geq 0\}$. From the hypothesis (b) we have that $A \neq \emptyset$. We will show that A is both open and closed in G; by connectedness of G it will follow that $A = G$ and so $f \equiv 0$.
   To see that $A$is closed, let $z \in \overline{A}$ and let (a_k) be a sequence in A such that $z = \lim a_k$. Since $f^{(n)}$ is continuous it follows that $f^{(n)}(z) = \lim f^{(n)}(a_k) = 0$. So $z \in A$ and A is closed.
   To see that A is open, let $a \in A$ and let $R > 0$ be such that $B(a; R) \subseteq G$. Then $f(z) = \sum a_n(z-a)^n$ for $|z-a| < R$ where $a_n = (n!)^{-1} f^{(n)}(a) = 0$ for each $n \geq 0$. Hence $f(z) = 0$ for all $z \in B(a; R)$ and, consequently, $B(a; R) \subseteq A$. Thus A is open and this completes the proof.

10. Suppose f and g are meromorphic in a neighborhood of $\overline{B}(a; R)$ with no zeros (Z) or poles (P) on the circle $\gamma = \{z \in \mathbb{C} \,|\, |z - a| = R\}$. If

Z_f, Z_g, P_f, P_g are the numbers of zeros, resp. poles, of f and g inside γ counted according to their multiplicities and if

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

on γ, then

$$Z_f - P_f = Z_g - P_g \ .$$

Reason: Rouché's theorem.

Solution: From the hypothesis

$$\left| \frac{f(z)}{g(z)+1} \right| < \left| \frac{f(z)}{g(z)} \right| + 1$$

on γ. If $\lambda := f(z)/g(z)$ and if λ is a positive real number then this inequality becomes $\lambda + 1 < \lambda + 1$, a contradiction. Hence the meromorphic function f/g maps γ onto $\Omega := \mathbb{C} - [0, \infty)$. If L is a branch of the logarithm on Ω then $L(f(z)(g(z))$ is a well-defined primitive for $(f/g)'(f/g)^{-1}$ in a neighborhood of γ. Thus

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \int_\gamma (f/g)'(f/g)^{-1} \\ &= \frac{1}{2\pi i} \int_\gamma \left(\frac{f'}{f} - \frac{g'}{g} \right) \\ &= (Z_f - P_f) - (Z_g - P_g). \end{aligned}$$

11. (HS-1) A gardener holds a water hose horizontally and wants to water a bed 6 m away. The water exits the hose at a speed of 8 m/s . Calculate the minimum height the gardener needs to hold the hose for the water to reach the bed, the speed at which the water droplets hit the bed, and the angle at which the water droplets hit the bed.

Reason: Projectile motion.

Solution: It is a uniformly motion, i.e. $x_0 = v_x \cdot t_0$. We have also an acceleration towards earth the whole time, i.e.

$$y_0 = \frac{g}{2} \cdot t_0^2 = \frac{g}{2} \cdot \left(\frac{x_0}{v_x} \right)^2 = \frac{9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}}{2} \cdot \left(\frac{6\,\mathrm{m}}{8\,\frac{\mathrm{m}}{\mathrm{s}}} \right)^2 = 2.76\,\mathrm{m}$$

For the y-component of the velocity we have $v_y = g \cdot t_0$ so

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_x^2 + g^2 \cdot \left(\frac{x_0}{v_x}\right)^2}$$
$$= \sqrt{64\,\frac{\mathrm{m}^2}{\mathrm{s}^2} + 96.04\,\frac{\mathrm{m}^2}{\mathrm{s}^4} \cdot \left(\frac{6\,\mathrm{m}}{8\,\frac{\mathrm{m}}{\mathrm{s}}}\right)^2} = 10.86\,\frac{\mathrm{m}}{\mathrm{s}}$$

Finally we have for the angle

$$\tan\alpha = \frac{v_y}{v_x} = \frac{g \cdot \frac{x_0}{v_x}}{v_x} = \frac{g \cdot x_0}{v_x^2} = 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2} \cdot \frac{6\,\mathrm{m}}{64\,\frac{\mathrm{m}^2}{\mathrm{s}^2}} = 0.91875$$

which results in $\alpha \approx 42.6°$

12. (HS-2) A faucet delivers a volume flow of $V' = 6\,\frac{l}{\mathrm{min}}$. The connected garden hose has an inner diameter of $d_1 = 18\,\mathrm{mm}$, the nozzle a cross-section of $d_2 = 5\,\mathrm{mm}$. Calculate the mass flow in the garden hose, the speed of the water in the garden hose, and the speed of the water at the nozzle. It is observed that the water jet widens after the nozzle. Why?

    Reason: Fluid dynamics.

    Solution: Water is an incompressible fluid and has a density of $\rho = 1,000\,\frac{\mathrm{kg}}{\mathrm{m}^3}$ under standard conditions. We have by hypothesis a flow of volume
$$V' = 6\,\frac{l}{\mathrm{min}} = \frac{6 \cdot 10^{-3}\,\mathrm{m}^3}{60\,\mathrm{s}} = 10^{-4}\,\frac{\mathrm{m}^3}{\mathrm{s}}$$
    From $m = \rho \cdot V$ for the mass of incompressible fluids we get

$$m' = \rho V' = 1000\,\frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 10^{-4}\,\frac{\mathrm{m}^3}{\mathrm{s}} = 0.1\,\frac{\mathrm{kg}}{\mathrm{s}}$$

    The cross-section of the hose has a radius $r_1 = d_1/2 = 9\,\mathrm{mm} = 0.009\,\mathrm{m}$ and so an area of $A_1 = \pi r_1^2 = 2.545 \cdot 10^{-4}\,\mathrm{m}^2$. This results in a velocity of

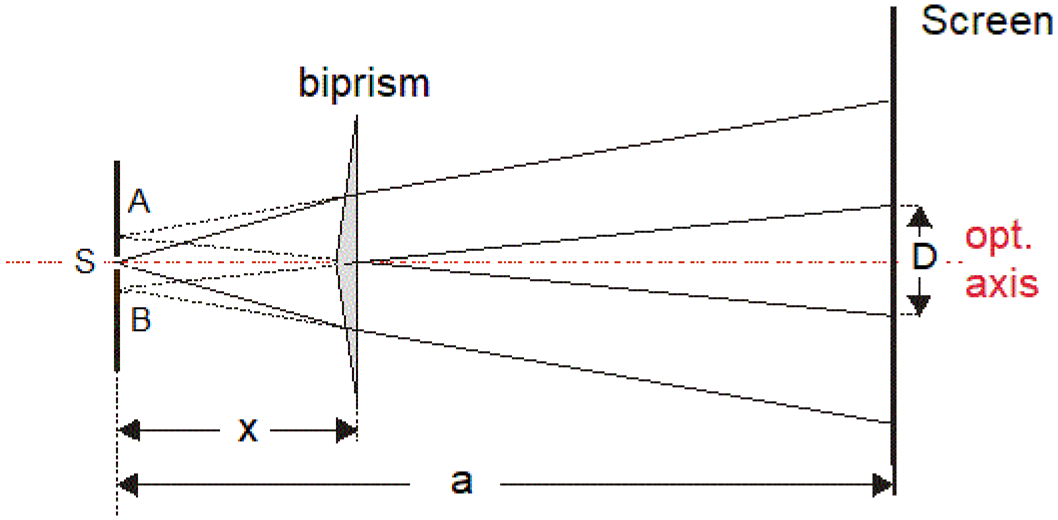
$$v_1 = \frac{V'}{A_1} = \frac{10^{-4}\,\frac{\mathrm{m}^3}{\mathrm{s}}}{2.545 \cdot 10^{-4}\,\mathrm{m}^2} = 0.393\,\frac{\mathrm{m}}{\mathrm{s}}.$$

The cross-section of the nozzle is $A_2 = \pi r_2^2 = \pi \frac{d_2^2}{4} = 0.2 \cdot 10^{-4}\,\mathrm{m}^2$, hence the velocity at the nozzle is

$$v_2 = \frac{V'}{A_2} = \frac{10^{-4}\,\frac{\mathrm{m}^3}{\mathrm{s}}}{0.2 \cdot 10^{-4}\,\mathrm{m}^2} = 5\,\frac{\mathrm{m}}{\mathrm{s}}.$$

The velocity of the water is slowed down due to friction and air resistance. However, mass and volume stay the same, such that velocity times cross-section is constant. That is why the beam widens.

13. (HS-3) As a result of the refraction, the light bundle emanating from the slit S produces two bundles which overlap in the screen area of width D and appear to arise from two virtual slit images A and B. Since the two virtual slit images originate from the same slit, the light emanating from them is coherent and can interfere in the area of overlap.



Calculate the wavelength if monochromatic light is used from the quantities given in the sketch and the distance Δy between two adjacent interference strips? Assume that the dimensions parallel to the optical axis can be viewed as large compared to those perpendicular to the optical axis.

Reason: Optics.

Solution: Let b be the distance between the two virtual slit images A, B. The intercept theorem yields

$$\frac{b}{D} = \frac{x}{a-x} \Longrightarrow b = \frac{Dx}{a-x}$$

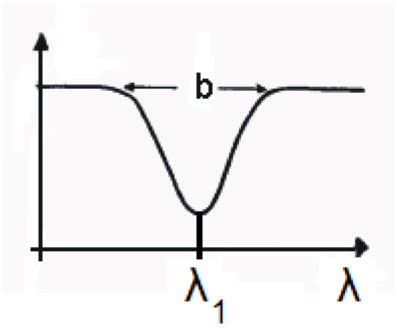
For $a \gg b$ we may assume that the beams that run from A and B in direction P on the screen are approximately parallel. For the interference at the k-th maximum, and small angles, we have

$$\Delta s_k = b \cdot \sin \alpha_k = k \cdot \lambda \Longrightarrow \sin \alpha_k = \frac{k\lambda}{b} \approx \tan \alpha_k = \frac{y_k}{a}$$

By the same arguments we get $\dfrac{(k+1)\lambda}{b} = \dfrac{y_{k+1}}{a}$ and thus

$$\lambda = \frac{b}{a}(y_{k+1} - y_k) = \frac{b}{a} \cdot \Delta y \ .$$

14. (HS-4) A galaxy is 42 MLy away and oriented in space, such that its rotation axis is perpendicular to the line of sight. The α line of hydrogen is measured to occur at $\lambda_1 = 658.003\,\text{nm}$ instead of $\lambda_0 = 656.28\,\text{nm}$



widened to $b = 0.438\,\text{nm}$. Assume that the main cause of the widening is the rotation of the stars around the center of the galaxy. Assume further that the different wavelength is only due to the radial motion of the galaxy compared to our solar system.

What is the maximal rotational velocity of the observed stars, and what is the maximal velocity the galaxy is moving and in which direction as seen from our solar system?

Reason: Astronomy.

Solution: If we consider the rotational velocity v, we have $v = \dfrac{\Delta\lambda}{\lambda_1} \cdot c$ and with $\Delta\lambda = 0.5\,b$

$$v = \frac{0.219\,\text{nm}}{658.003\,\text{nm}} \cdot 3 \cdot 10^8\,\frac{\text{m}}{\text{s}} = 99778.5\,\frac{\text{m}}{\text{s}} \approx 100\,\frac{\text{km}}{\text{s}}$$

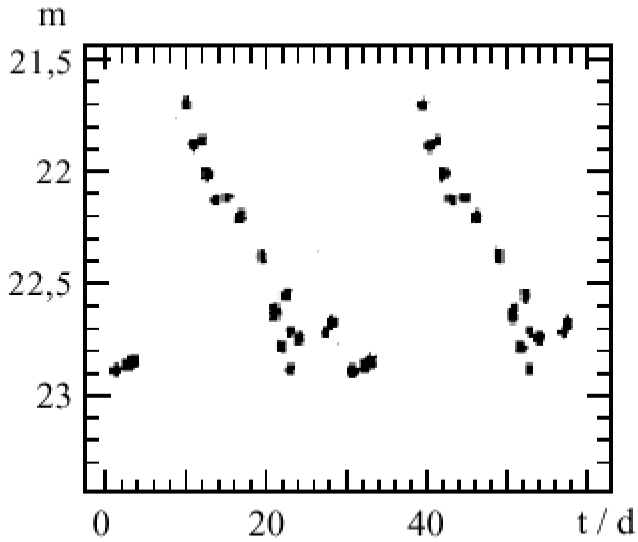
The calculation of the galaxy's relative motion to us is

$$v = \frac{\Delta\lambda}{\lambda_1} \cdot c = \frac{\lambda_1 - \lambda_0}{\lambda_1} \cdot c = \frac{1.723\,\text{nm}}{656.28\,\text{nm}} \cdot 3 \cdot 10^8\,\frac{\text{m}}{\text{s}} = 787,076\,\frac{\text{m}}{\text{s}} \approx 790\,\frac{\text{km}}{\text{s}} \,.$$

Since $\lambda_1 > \lambda_0$ the galaxy is moving at around $790\,\frac{\text{km}}{\text{s}}$ away from us.

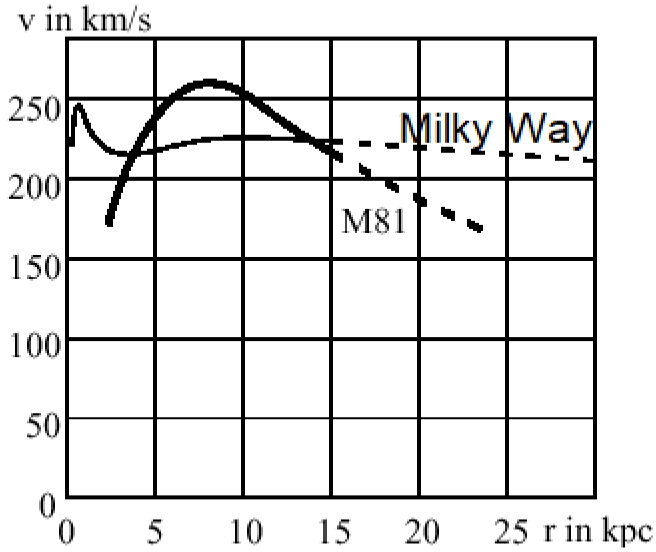
15. (HS-5) The spiral galaxy M81 near Ursa Major can already be viewed by a small telescope. It has an apparent magnitude of $M = 6.9$. The angle to the celestial pole is about 21°. Is it possible to observe M81 the entire year, if you live in Toronto?

    The following diagram shows data-points of light from the cepheid C27 in M81.



Calculate our distance from M81 in lightyears. (Use an average value of magnitude 22.3 at a pulsation rate of 30 per day and the relation $M = -1.67 - 2.54 \cdot \log_{10} p$.)

The second diagram is a comparison between M81 and Milky Way. It shows the radial orbit velocity v of the stars in relation to their distance r from the galaxy center. Optical wavelengths are hardly to observe from around $16\,kpc$ on, so radio wavelengths are used.



Verify that if a celestial body orbits a center of great mass, then we can calculate the central mass approximately by $M = \dfrac{v^2 \cdot r}{G}$. Show by choosing two data-points that the rotation curve of M81 is approximately $v \sim \dfrac{1}{\sqrt{r}}$ for $r = 10\,\text{kpc}$. What does that mean for the mass distribution in M81 ? Estimate the mass of M81 within the optical spectrum in units of sun masses.

The rotation curves of M81 and the Milky Way differ a lot for great distances from the center. What does that mean for the mass distribution in our Milky Way?

The wavelength of the α-line of hydrogen from the optical center of M81 is measured to be $\lambda_1 = 656.38\,\text{nm}$ in comparison to $\lambda_0 = 656.47\,\text{nm}$. Can we apply Hubble's law to M81?

Reason: M81.

Solution:

(a) An observer in the Northern hemisphere can see all stars (or galaxies) whose angular distance from the celestial pole is less than its geographical latitude. Toronto is at 43° 39' 40,86" N, 79° 22' 59,11" W, which is significantly greater than 21°, hence M81 can be seen at any time of the year.

(b) We read an average of magnitude $m = 22.3$ at a pulsation rate of 30 per day. With the given relation we calculate $M = -5.42$. The distance is given by

$$m - M = 5 \cdot \log_{10}\left(\frac{d}{10\,\mathrm{pc}}\right) \Longrightarrow d = 10^{(m-M)/5} \cdot 10\,\mathrm{pc}$$
$$d = 10^{(22.3+5.42)/5} \cdot 10\,\mathrm{pc} = 350,000 \cdot 10\,\mathrm{pc}$$
$$\approx 10,798,258 \cdot 10^{16}\,\mathrm{m} \approx 11.4\,\mathrm{MLy}$$

(c) $F_G = F_C \Longrightarrow G \cdot \dfrac{m \cdot M}{r^2} = \dfrac{m \cdot v^2}{r} \Longrightarrow M = \dfrac{v^2 \cdot r}{G}$

(d) The diagram gives us two data-points $r_1 = 10\,\mathrm{kpc}$, $\mathrm{v}_1 = 250\,\mathrm{kms}^{-1}$ and $r_2 = 20\,\mathrm{kpc}$, $\mathrm{v}_2 = 185\,\mathrm{kms}^{-1}$. Hence

$$\frac{v_1}{v_2} = \frac{250}{185} = 1.35 \text{ and } \frac{\sqrt{r_2}}{\sqrt{r_1}} = \sqrt{2} = 1.41 \Longrightarrow v \sim \frac{1}{\sqrt{r}}.$$

This is almost the proportion we have for a single large central mass, which in return means that almost the entire mass of M81 is within 10 kpc of range. The optical limit is the point $r = 16\,\mathrm{kpc}$, $\mathrm{v} = 210\,\mathrm{kms}^{-1}$.

$$m_{M81} = \frac{\left(210\,\dfrac{\mathrm{km}}{\mathrm{s}}\right)^2 \cdot 16\,\mathrm{kpc}}{6.673 \cdot 10^{-11}\,\dfrac{\mathrm{m}^3}{\mathrm{kg\,s}^2}} = \frac{44,100 \cdot 10^6 \cdot 16,000 \cdot 3.0857 \cdot 10^{16}\,\mathrm{m}^3\mathrm{s}^{-2}}{6.673 \cdot 10^{-11}\,\dfrac{\mathrm{m}^3}{\mathrm{kg\,s}^2}}$$
$$= 3.2628 \cdot 10^{41}\,\mathrm{kg} \approx 164 \cdot 10^9\,\mathrm{m}_{\mathrm{sun}}$$

(e) The orbital velocity of the Milky Way is almost constant for large distances from its center. So there must be considerable (non-luminous) masses at these distances.

(f) Since the α-line of hydrogen is blue-shifted, $\lambda_0 > \lambda_1$, Hubble's law does not apply. M81 is approaching the Milky Way.

6 July 2021

1. Suppose that G is a finite group such that for each subgroup H of G there exists a homomorphism $\varphi : G \longrightarrow H$ such that $\varphi(h) = h$ for all $h \in H$. Show that G is a product of groups of prime order.

   Reason: Group Theory.

   Solution: We proceed by induction on $|G|$. The base case $|G| = 1$ is trivial (empty product), as are $|G| = 2, 3$. Suppose that $|G| > 3$ and that the statement is true for all smaller groups. Choose a subgroup H of G of prime order p. Such a subgroup exists by the first Sylow theorem. By assumption, there is a homomorphism $\varphi : H \longrightarrow H$ such that $\varphi(h) = h$ for all $h \in H$. Let $K := \ker \varphi$. By induction hypothesis, K is a product of groups of prime order. Let $\sigma : G \longrightarrow K$ be a homomorphism which is constant on K, i.e. $\sigma(k) = k$ for all $k \in K$, which exists by assumption. Now we define

   $$\alpha : G \longrightarrow K \times H \ , \ \alpha(g) := (\sigma(g), \varphi(g))$$

   Since σ restricted to K equals the identity, the kernel of α is trivial, i.e. α is injective and thus $|G| = |K| \cdot |H|$. But then α is an isomorphism, K a product of subgroups of prime order by induction hypothesis, and H was of prime order p.

2. Let G be a finite group that operates on a set X. Then the number of orbits is

   $$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

   where $X^g = \{ x \in X \ : \ g.x = x \}$ are the fixed points in X.

   Reason: Burnside's Lemma (Frobenius-Cauchy Lemma).

   Solution: Long version.

   $G_x = \{ g \in G \,|\, g.x = x \} \leqslant G$ is the stabilizer of x.
   $G(x) = \{ g.x \,|\, g \in G \} \subseteq X$ is the orbit of x under G.

   Step 1: Stabilizer-Orbit Formula: $|G| = |G_x| \cdot |G(x)|$

   Consider the relation $R = \{ (g, y) \in G \times X \,|\, y = g.x \}$. For each $g \in G$ there is exactly one $y = g.x \in X$, hence $|R| = |G|$. On the other hand, we have for $y \in G(x)$, say $y = g_0.x$, exactly $|G_x|$ many elements $h \in G$ with $h.x = y$, because these are exactly all elements $h = g_0 g$ with

$g \in G_x$. In case $y \notin G(x)$ there is no element $(g,y) \in R$. Therefore

$$|R| = |G| = \sum_{y \in G(x)} |G_x| = |G(x)| \cdot |G_x|$$

Step 2: $\displaystyle\sum_{g \in G} |X^g| = \sum_{x \in X} |G_x|$

This time we use the double count argument on the relation $S = \{\,(g,x) \in G \times X \,|\, g.x = x\,\}$. For a fixed element $h \in G$ the set $\{\,(h,x) \,|\, x \in X^h\,\}$ is the set of pairs in S which have h as their first coordinate. On the other hand we have for a given $z \in X$ the set of pairs in S with second coordinate z the set $\{\,(g,z) \,|\, g \in G_z\,\}$. Hence

$$\sum_{h \in G} |X^h| = |S| = \sum_{z \in X} |G_z|$$

Step 3: $\displaystyle |X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$

We use the stabilizer-orbit formula and sort the summands on the RHS with equal stabilzers; especially all elements $y \in G(x)$ have stabilizers G_x of equal size:

If $g \in G_x$ and $y = g_0.x \in G(x)$ then $g.x = x$ and thus $(g_0 g.x = g_0.x = y)$, i.e. $(g_0 g g_0^{-1}).(g_0.x) = g_0.x = y$. If g runs through the stabilizer G_x, then $g_0 G_x g_0^{-1}$ runs through the stabilizer of $y = g_0.x$. But both sets are of equal size $|G_x|$. With the previous steps, especially with $|G_x| = |G|/|G(x)| = |G|$, we get

$$\begin{aligned}\sum_{g \in G} |X^g| = \sum_{x \in X} |G_x| &= \sum_{A \in X/G} \sum_{x \in A} |G_x| \\ &= \sum_{A \in X/G} |A| \cdot \frac{|G|}{|A|} = \sum_{A \in X/G} |G| = |X/G| \cdot |G|\end{aligned}$$

Note: William Burnside wrote this formula down around 1900. Historians of mathematics, however, found this formula already from Cauchy (1845) and Frobenius (1887). Therefore the formula is sometimes referred to as the Lemma which is not from Burnside.

3. Prove that there is a Lie algebra monomorphism $\mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$ if $\mathfrak{g}$ is a semisimple Lie algebra. Is this also a necessary condition?

   Reason: Adjoint Representation and Center.

Solution: A semisimple Lie algebra has no Abelian ideals. Its center, however, is an Abelian ideal. Thus we have

$$\begin{aligned}\mathfrak{Z}(\mathfrak{g}) &= \{\, Z \in \mathfrak{g} \,|\, [X,Z] = 0 \;\forall\, X \in \mathfrak{g} \,\} \\ &= \bigcap_{X\in\mathfrak{g}} \ker \mathfrak{ad} X = \ker \mathfrak{ad} = \{\,0\,\}\end{aligned}$$

This means that $\mathfrak{ad} \,:\, \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is a monomorphism of Lie algebras and

$$\mathfrak{g} \cong \mathfrak{ad}(\mathfrak{g}) \cong \mathfrak{Der}(\mathfrak{g}) \subseteq \mathfrak{gl}(\mathfrak{g})$$

The adjoint representation cannot be onto, since the center of $\mathfrak{gl}(\mathfrak{g})$ are all multiples of the identity matrix.

If we consider the non Abelian two dimensional Lie algebra defined by $[X,Y] = Y$, which is the Borel subalgebra of the simple Lie algebra $\mathfrak{sl}(2)$, or the Lie algebra of matrices $\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$, then we have a solvable and therewith no semisimple Lie algebra which has only a trivial center, too. Hence the condition of semisimplicity is not necessary.

4. If $n > 1$ is a square-free natural number, prove for all $k > 1$

$$\sum_{d|n} \sigma(d^{k-1})\,\varphi(d) = n^k$$

Remark: φ is Euler's phi-function and $\sigma(m)$ the sum of divisors of m.

Reason: Number Theory.

Solution: Let $n = p_1p_2\dots p_r > 1$ a square-free number. The function $f(m) = \sigma(m^t)\varphi(m)$ with $t \geq 1$ is build of multiplicative functions and as such multiplicative, too. This means for coprime numbers a, b we have $f(ab) = f(a)f(b)$. The function $F(n) = \sum_{d|n} f(d) = \sum_{d|n} \sigma(d^{k-1})\varphi(d)$ is also multiplicative:

$$F(ab) = \sum_{d|ab} f(ab) = \sum_{d|a}\sum_{e|b} f(a)f(b) = \sum_{d|a} f(a) \sum_{e|b} f(b) = F(a)F(b)$$

Thus it is sufficient to show $F(p) = p^k$ since as n is square-free and $F(n) = F(p_1\dots p_r) = p_1^k \dots p_r^k = n^k$

$$F(p) = \sum_{d|p} \sigma(d^{k-1})\varphi(d) = 1 + \sigma(p^{k-1})\varphi(p) = 1 + \frac{p^k - 1}{p-1} \cdot (p-1) = p^k$$

5. Show that

$$M := \{\, x \in \mathbb{R}^3 \,|\, x_1+x_2+x_3 = 0\,,\ x_1^2+2x_2^2+x_3^2-2x_2(x_1+x_3) = 9 \,\} \subseteq \mathbb{R}^3$$

is a manifold, and determine the tangent space T_pM and the normal space N_pM at $p = (2,-1,-1) \in M$.

Reason: Manifolds.

Solution: We consider the function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \stackrel{f}{\longrightarrow} \begin{bmatrix} x_1+x_2+x_3 \\ x_1^2+2x_2^2+x_3^2-2x_2(x_1+x_3)-9 \end{bmatrix}$$

such that $M = f^{-1}(\{(0,0)\})$. Its Jacobi matrix is

$$J_xf = \begin{bmatrix} 1 & 1 & 1 \\ 2(x_1-x_2) & 2(2x_2-x_1-x_3) & 2(x_3-x_2) \end{bmatrix}$$

$\operatorname{rk} J_xf = 1$ if $x_1 - x_2 = 2x_2 - x_1 - x_3 = x_3 - x_2$ or $x_1 = x_2 = x_3$. Since $f(t,t,t) = (3t,-9) \neq 0$, $(0,0)$ is a regular value of f and M a submanifold of dimension $3-1-1 = 1$.

For $p = (2,-1,-1)$ we have $J_pf = \begin{bmatrix} 1 & 1 & 1 \\ 6 & -6 & 0 \end{bmatrix}$, hence $T_pM = \ker D_pf = \mathbb{R}\cdot\begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$ and $N_pM = (T_pM)^{\perp} = \mathbb{R}\cdot\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \mathbb{R}\cdot\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, which is the row space of J_pf .

6. Two persons P and Q play the following game:

P starts by selecting exactly one real value for a, b, or c in the equation

$$x^3 + ax^2 + bx + c = 0$$

Then Q does the same for one of the remaining coefficients, before P finally chooses the last value. P wins if and only if the equation has three different real roots. Is there a winning strategy for one of the players?

Reason: Mean Value Theorem.

Solution: P has the following winning strategy:

P chooses $c = 1$ in his first move. In case Q sets a value for a, then P

finally sets $b < -a-2$; whereas in case Q sets a value for b, P finally sets $a < -b-2$. We now have to show that the equation has three distinct real roots. Let $f(x) = x^3 + ax^2 + bx + 1$. Since $\lim_{x\to\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$ there is a real number $k > 1$ such that

$$f(k) > 0\,,\; f(0) = 1\,,\; f(-k) < 0\,,\; f(1) = a + b + 2 < 0$$

By the mean value theorem, there have to be roots $f(\xi_j) = 0$ with

$$-k < \xi_1 < 0 < \xi_2 < 1 < \xi_3 < k$$

7. What are the composition factors of $\mathrm{GL}(2, \mathbb{F}_{19})$?

   Reason: Group Theory.

   Solution: We know that

$$\mathrm{GL}(2, \mathbb{F}_{19})/\mathrm{SL}(2, \mathbb{F}_{19}) \cong \mathbb{F}_{19}^{\times} \cong \mathbb{Z}_{18} \cong \mathbb{Z}_2 \times \mathbb{Z}_9$$

   $\mathbb{Z}_9$ has the composition factor $\mathbb{Z}_3$ twice and $\mathbb{Z}_2$ is simple. We know further that

$$\mathrm{SL}(2, \mathbb{F}_{19})/\mathrm{Z}(\mathrm{SL}(2, \mathbb{F}_{19})) \cong \mathrm{PSL}(2, \mathbb{F}_{19})$$

   which is simple, too. With Iwasawa's criterion for simplicity, it can be shown that all groups $\mathrm{PSL}(\mathrm{m}, \mathbb{F}_{\mathrm{p^k}})$ are simple, except of $\mathrm{PSL}(2, \mathbb{F}_2)$ and $\mathrm{PSL}(2, \mathbb{F}_3)$.

$$|\mathrm{PSL}(\mathrm{m}, \mathbb{F}_{\mathrm{p^k}})\| = \mathrm{d}^{-1}\mathrm{q}^{\frac{\mathrm{m(m-1)}}{2}} \prod_{\mathrm{j=2}}^{\mathrm{m}} (\mathrm{q^j} - 1)\,,\; \mathrm{d} := \gcd(\mathrm{m}, \mathrm{q}-1)\,,\; \mathrm{q} = \mathrm{p^k}$$

   Thus in our case we have $|\mathrm{PSL}(2, \mathbb{F}_{19})| = \dfrac{19}{2}(19^2-1) = 3420$. $\mathrm{PSL}(2, \mathbb{F}_{19})$ is the Chevalley group $A_1(19)$. The remaining composition factors are provided by

$$\mathrm{Z}(\mathrm{SL}(2, \mathbb{F}_{19}))/\{\,\mathbf{1}\,\} \cong \mathrm{Z}(\mathrm{SL}(2, \mathbb{F}_{19})) \cong \mathbb{Z}_2$$

   such that we have the following list:

$$\mathbb{Z}/2\mathbb{Z}\text{ (twice)}\;,\;\mathbb{Z}/3\mathbb{Z}\text{ (twice)}\;,\;\mathrm{PSL}(2, \mathbb{F}_{19})$$

8. A group G of order 70 has always a normal subgroup of order 5.

   Reason: Group Theory.

Solution: According to Sylow's first theorem, there is at least one subgroup $U \leqslant G$ of order 5. According to his third theorem the number s of such subgroups satisfies

$$s \equiv 1 \mod 5\,,\; s\,|\,|G| = 70$$

If $s \cdot \alpha = 70 = 5 \cdot 14$ and $5 \nmid s$, then $5\,|\,\alpha$, say $\alpha = 5\beta$ and so $s \cdot \beta = 14$ and $s\,|\,14 = 2 \cdot 7$. However, $2 \equiv 2 \not\equiv 1 \mod 5$, $7 \equiv 2 \not\equiv 1 \mod 5$ and $14 \equiv 4 \not\equiv 1 \mod 5$, a contradiction except for $s = 1$. As all gUg^{-1} are Sylow 5-subgroups, too, they are already contained in the only one U, which means that U is a normal subgroup.

9. Let X, Y, Z be topological spaces, X covering compact (not necessarily Hausdorff), and Z Hausdorff. Let $g : X \longrightarrow Y$ be continuous, and $h : X \longrightarrow Z$ surjective and continuous. Show that the following statements are equivalent:

   (a) $g(x) = g(x')$ for all $x, x' \in X$ with $h(x) = h(x')$.

   (b) There is a continuous function $f : Z \longrightarrow Y$ with $g = f \circ h$.

   (c) There is a unique continuous function $f : Z \longrightarrow Y$ with $g = f \circ h$.

   Reason: Topology.

   Solution: $(a) \Longrightarrow (b)$: Since h is surjective, we have for any $z \in Z$ an element $x \in X$ such that $z = h(x) = h(x')$, hence $g(x) = g(x')$ by assumption. Therefore there is a well-defined function $f : Z \longrightarrow Y$ with $f(z) := f(h(x)) = g(x)$ for all $z \in Z$, i.e. $g = f \circ h$.

   Given a closed set $A \subseteq X$, means that A is covering compact as X is, hence $h(A) \subseteq Z$ is covering compact, too, because h is continuous. Since Z is Hausdorff, $h(A)$ is closed and so is h.

   Now let $B \subseteq Y$ be closed. Then $g^{-1}(B) = h^{-1}(f^{-1}(B))$, i.e. $f^{-1}(B) \overset{(*)}{=} h(h^{-1}(f^{-1}(B))) = h(g^{-1}(B))$. Since g is continuous and h closed, we have that $f^{-1}(B)$ is closed, which means f in continuous.

   $(*)$ h is surjective, so $h(h^{-1}(M)) = M$ for all sets $M \subseteq Z$.
   This property is equivalent to surjectivity.

   $(b) \Longrightarrow (c)$: If $f_1, f_2 : Z \longrightarrow Y$ with $f_1 h = g = f_2 h$, then $f_1 = f_2$ because surjectivity of h allows us a right cancellation.
   This property is equivalent to surjectivity.

   $(c) \Longrightarrow (a)$: Be $x, x' \in X$ with $h(x) = h(x')$ then $g(x) = f(h(x)) = f(h(x')) = g(x')$.

10. Given $y''' = y'' + y' - y$. Determine a fundamental system, and solve the initial value problem $y(0) = 1$, $y'(0) = 0$, $y''(0) = 3$.

    Reason: Differential Equation.

    Solution: The characteristic polynomial $p\left(\frac{d}{dx}\right)(y) = 0$ is given by $p(x) = x^3 - x^2 - x + 1 = (x-1)^2(x+1)$ with a double root at $c_1 = 1$ and a simple root at $c_2 = -1$. With $D = \frac{d}{dx}$ we verify

$$\begin{aligned}(D-1)(D-1)(e^x) &= (D-1)(e^x - e^x) = D(0) - 0 = 0\\ (D-1)(D-1)(xe^x) &= (D-1)(xe^x + e^x) = (e^x + xe^x) - (xe^x + e^x) = 0\\ (D+1)(e^{-x}) &= -e^{-x} + e^{-x} = 0\end{aligned}$$

    so we get three linear independent solutions and a basis by the fundamental system

$$\{\,\varphi_1 = e^x\,,\ \varphi_2 = xe^x\,,\ \varphi_3 = e^{-x}\,\}$$

    With the initial values for $y = \sum_{k=1}^3 a_k\varphi_k(x) = a_1e^x + a_2xe^x + a_3e^{-x}$

$$\begin{aligned}y(0) = 1 &\Longrightarrow a_1 + a_3 = 0\\ y'(0) = 0 &\Longrightarrow a_1 + a_2 - a_3 = 0\\ y''(0) = 3 &\Longrightarrow a_1 + 2a_2 + a_3 = 3\end{aligned}$$

    which results in $(a_1, a_2, a_3) = (0, 1, 1)$ and $y(x) = xe^x + e^{-x}$.

11. (HS-1) Assume we have put a Cartesian coordinate system on France and got the following positions: Paris $(0,0)$, Lyon $(3,-8)$ and Marseille $(4,-12)$. Look up the definitions and calculate the distance between Lyon and Marseille according to

    (a) the Euclidean metric.
    (b) the maximum metric.
    (c) the French railway metric.
    (d) the Manhattan metric.
    (e) the discrete metric.

    Reason: Internet search for the metrics.

    Solution: We have $P = (0,0), L = (3,-8), M = (4,-12)$.

    (a) the Euclidean metric.
    $|LM| = \sqrt{(4-3)^2 + (-12-(-8))^2} = \sqrt{17} \approx 4.123.$

(b) the maximum metric.

$|LM| = \max\{\,|4-3|\,,\,|-12-(-8)|\,\} = \max\{\,1,4\,\} = 4.$

(c) the French railway metric.

Paris isn't on the direct line between Lyon and Marseille, so $|LM| = |LP| + |PM| = \sqrt{3^2+(-8)^2} + \sqrt{4^2+(-12)^2} = \sqrt{73} + \sqrt{160} \approx 21.193.$

(d) the Manhattan metric.

$|LM| = |4-3| + |-12-(-8)| = 1+4 = 5.$

(e) the discrete metric.

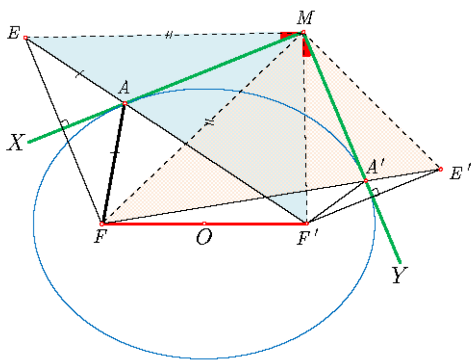
$|LM| = 1.$

12. (HS-2) Consider the ellipse in the first quarter of a Cartesian coordinate system

$$\frac{(x-2)^2}{4} + (y-1)^2 = 1$$

and rotate it such that the coordinate axes are always tangents to the ellipse. Which locus describes the center of the ellipse during a full rotation?

Reason: Geometry.



Solution:
Consider an Ellipse of foci F and F' and semi axis $a = 2, b = 1$. Let M be a point outside the ellipse. The tangents from M touch the Ellipse at A and A'. Let E be the symmetric of F with respect to MA and define E' similarly.

Step1. The points F', A and E are aligned. Indeed, by the optical property of the ellipse $\angle MAF' = \angle FAX = \angle XAE$. Similarly, the F, A' and F' are also aligned.

Step2. $\triangle FE'M$ and $\triangle F'EM$ are congruent. Because, $EF' = EA +$

$AF' = FA + AF' = 2a$ and similarly, $FE' = 2a$. Moreover, $ME = MF$ and $ME' = MF'$.

Step3. $\angle AMA' = \angle F'ME$. Indeed, from the previous step we conclude that

$$\angle XME = \frac{1}{2}\angle EMF = \frac{1}{2}\angle E'MF' = \angle YMF'.$$

Step4. It follows that $MA \perp MA'$ if and only if $\angle EMF' = \frac{\pi}{2}$, and (since $EM = FM$,) this equivalent to

$$FM^2 + F'M^2 = F'E^2 = 4a^2 \quad (*)$$

But using the parallelogram identity we know that

$$FM^2 + F'M^2 = 2OM^2 + 2OF^2 = 2OM^2 + 2e^2 = 2OM^2 + 2(a^2 - b^2)$$

Thus, $(*)$ is equivalent to $OM^2 = a^2 + b^2 = 5$, which is the desired conclusion, a segment of circle of radius $\sqrt{5}$ and center $(0,0)$ between $(2,1)$ and $(1,2)$.

13. (HS-3) Maximize $f(x,y,z) = 4x^2y^2 - (x^2+y^2-z^2)^2$ under the conditions $x+y+z = c$ and that $x,y,z > 0$.

Reason: Heron's Theorem.

Solution: In case x,y,z are the side lengths of a triangle, we have $f(x,y,z) = c(c-2x)(c-2y)(c-2z) > 0$ if we label the longest side z. Since the geometric mean is less or equal the arithmetic mean, we have

$$c(c-2x)(c-2y)(c-2z) \leq c\left((c-2x)+(c-2y)+(c-2z)\right)^3 = c^4$$

where equality holds if $c - 2x = c - 2y = c - 2z$, i.e. $x = y = z$.

The theorem of Heron says that $f(x,y,z) = 16\,F^2$ where F is the area of the triangle with side lengths x,y,z . The triangle with the maximal area by constant circumference is the equilateral triangle.

Now assume that $c > z \geq x+y > y \geq x > 0$. Then $f(x,y,z) \leq 0$ because $c, c-2x, c-2y > 0, c-2z = (x+y)-z \leq 0$. We can achieve the maximal value 0 by setting $z = c/2$ such that $f(x,y,c/2) = 0$. In order to match the restrictions on x,y, we could set $x = y = c/4$. The solution, however, isn't unique since e.g. $x = c/6, y = c/3$ match the requirements, too.

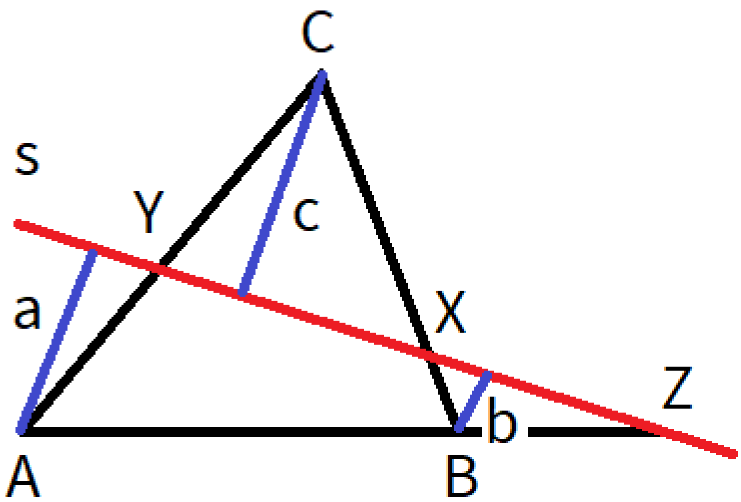
14. (HS-4) Let $\triangle ABC$ be a triangle and s a straight which intersects all three sides (or their prolongations), say $X \in BC$, $Y \in CA$, $Z \in AB$ are the intersection points. Prove

$$\overline{AZ} \cdot \overline{BX} \cdot \overline{CY} = \overline{AY} \cdot \overline{BZ} \cdot \overline{CX}$$

Reason: Menelaus's Theorem.

Solution: Let a, b, c be the perpendiculars in A, B, C resp. on s. From the intercept theorem we get

$$\overline{AZ} : \overline{BZ} = a : b\,,\ \overline{BX} : \overline{CX} = b : c\,,\ \overline{CY} : \overline{AY} = c : a$$



Multiplication yields

$$\frac{\overline{AZ}}{\overline{BZ}} \cdot \frac{\overline{BX}}{\overline{CX}} \cdot \frac{\overline{CY}}{\overline{AY}} = \frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a} = 1$$

15. (HS-5) Otto von Guericke, who invented the air pump, led an experiment in Berlin in 1654. Two groups of eight horses tried in vain to pull apart two bronze hemispheres between which a vacuum was created. Assume that the radius R of the hemispheres is so thin that we can neglect the difference between inner and outer radius.

Show that the force required to tear apart the hemispheres is $F =$

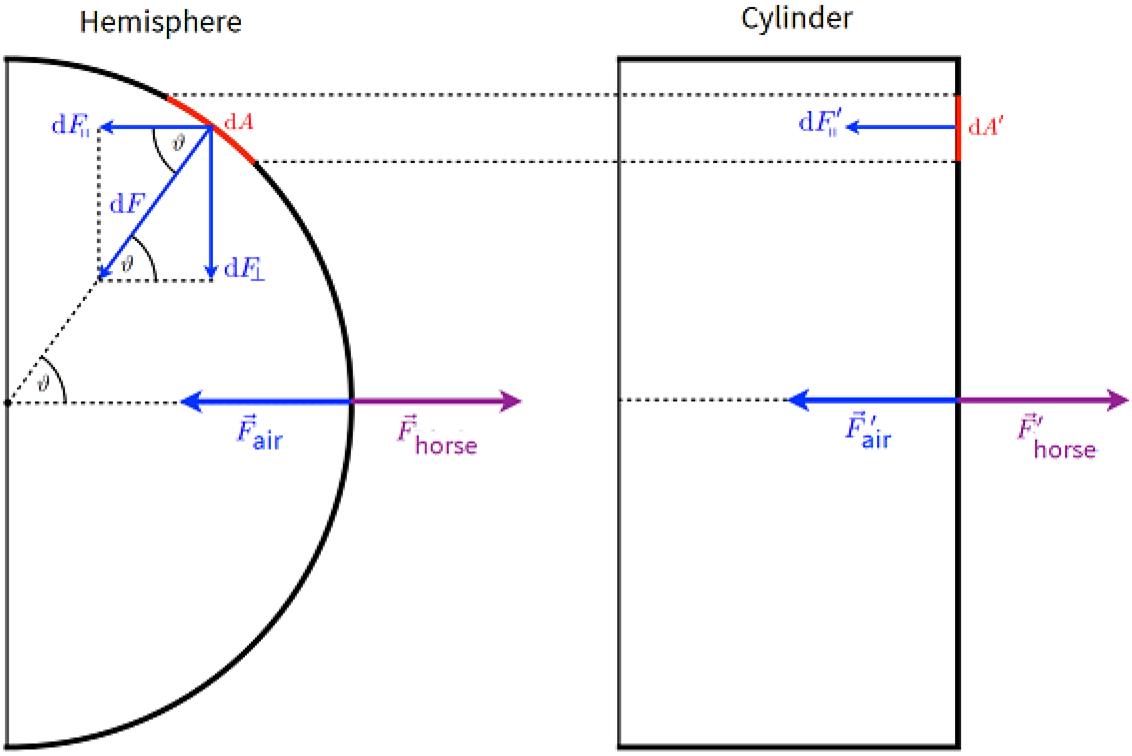
$\pi R^2 \cdot \Delta p$ where Δp is the difference of air pressure within and outside the hemispheres. Next assume $R = 30\,cm$, an inner pressure of $0.1\,bar$ and an outer pressure of $1.013\,bar$ Which force had each group of horses to apply in order to separate the hemispheres?

Reason: Magdeburg Hemispheres. Physics.

Solution: The evacuated Magdeburg hemispheres are affected by the difference of external and internal air pressure Δp which presses them together. To calculate the total force on one of the two hemispheres, we consider a surface element dA. The ambient air exerts a force $d\vec{F}$ on this area that is perpendicular to the surface element and of an amount $dF = \Delta p\, dA$. However, we are only interested in the horizontal part of this force $\vec{F}_{\|}$ which is parallel to the direction to which the horses pull, i.e. parallel to the horizontal symmetry axis of the hemisphere. The perpendicular components $\vec{F}_{\perp}$ cancel themselves out. If we denote the angle φ between the normal to the surface and the direction of pull, then the parallel component has an amount of

$$dF_{\|} = dF \cdot \cos\varphi = \Delta p\ dA \cdot \cos\varphi =: \Delta p\ dA' =: dF'_{\|}$$

The quantity $dA' = dA \cdot \cos\varphi$ can be viewed as parallel projection ot the surface area dA onto a cylinder (see the figure).



The parallel component $dF'_{\|}$ of the force which the air pressure exerts onto the projected surface area element dA' is thus of the same amount as the parallel component $dF_{\|}$ of the original force exerts on the original surface element dA .

The total amount of force exerted by the air pressure onto the hemisphere is the sum of all forces over the surface elements which compose the hemisphere. Since $dF_{\|} = dF'_{\|}$ we have for the total amount $F_{air} = F'_{air}$, the force onto the projection. So the two hemispheres are pressed together as two cylinders were, whose diameters correspond to the section of the hemispheres: R. The force of air pressure on a cylinder is easy to calculate. It's simply the product of pressure and circle area:

$$F_{air} = F'_{air} = \Delta p \; A_{\circ} = \pi R^2 \Delta p$$

The example then calculates to a force of

$$F_{air} = \pi \cdot (0.3)^2 (1.013 - 0.1) \; 10^5 \; N \approx 26 \; kN$$

which each group of horses has to come up with in order to separate the hemispheres. For comparison: One horsepower is approximately

735.5 Watt, so 30 horses would produce 22 kW. A horse pulls with approximately $10 - 12\,\%$ of its weight, i.e. with ca. 700 N or 21 kN for 30 horses.

7 June 2021

1. Let $\mathcal{D}_N := \left\{ x^n \dfrac{d}{dx}, \mid \mathbb{Z} \ni n \geq N \right\}$ be a set of linear operators on smooth real functions. For which values of $N \in \mathbb{Z} \cup \{\pm\infty\}$ do they generate a real Lie algebra, and are there isomorphic ones among them? Note that any linear combination of basis vectors has only finitely many nonzero coefficients.

   Reason: Infinite-Dimensional Lie Algebras.

   Solution: Let $\mathfrak{D}_N$ be the Lie algebra generated by $\mathcal{D}_N$. Then

$$\left[x^n \frac{d}{dx}, x^m \frac{d}{dx}\right] = x^n \frac{d}{dx} \circ x^m \frac{d}{dx} - x^m \frac{d}{dx} \circ x^n \frac{d}{dx} = (m-n) x^{n+m-1} \frac{d}{dx}$$

   defines a closed Lie structure for the values $N \in \{-\infty, 0, 1, \ldots, +\infty\}$. Now

$$\mathfrak{D}_N / [\mathfrak{D}_N, \mathfrak{D}_N] = \operatorname{span}\left\{ x^n \frac{d}{dx} \mid N \leq n \leq 2N-1 \right\}$$

   are of different dimensions N for different values of $N \geq 1$. Hence none of them are isomorphic. Since $[\mathfrak{D}_0, \mathfrak{D}_0] = \mathfrak{D}_0$ this is true for $N \geq 0$. Moreover $\mathfrak{D}_{+\infty} = \{0\}$ and $[\mathfrak{D}_{-\infty}, \mathfrak{D}_{-\infty}] = \mathfrak{D}_{-\infty}$.

   Since any linear combinations of the basis vectors contains only finitely many nonzero coefficients, $\mathfrak{D}_{-\infty} \not\cong \mathfrak{D}_0$:

   Let $D_n := x^n \dfrac{d}{dx}$. If $X = \sum_{k \geq 0} x_k D_k \in \mathfrak{D}_0$ with $m = \max\{k \mid x_k \neq 0\}$ then

$$\begin{aligned}
(\operatorname{ad} D_0)^{m+1}(X) &= \sum_{k=1}^{m} (\operatorname{ad} D_0)^m x_k k D_{k-1} \\
&= \sum_{k=2}^{m} (\operatorname{ad} D_0)^m x_k k(k-1) D_{k-2} \\
&\vdots \\
&= \sum_{k=m}^{m} (\operatorname{ad} D_0) x_k \frac{k!}{(k-m)!} D_{k-m} \\
&= x_m m! [D_0, D_0] = 0
\end{aligned}$$

   Assume there is an isomorphism $\varphi : \mathfrak{D}_0 \longrightarrow \mathfrak{D}_{-\infty}$. Then

$$(\operatorname{ad} \varphi(D_0))^k (\varphi(X)) = \varphi((\operatorname{ad} D_0)^k (X))$$

Hence there is an element $Y := \varphi(D_0) \in \mathfrak{D}_{-\infty}$ such that for any vector $\varphi(X) \in \mathfrak{D}_{-\infty}$ there is an $n \in \mathbb{N}$ such that $(\operatorname{ad} Y)^n(\varphi(X)) = 0$. Since φ is an isomorphism, this is especially true for all $\varphi(X) = D_n$ $(n \in \mathbb{Z})$. Let $V_m := \operatorname{span}\{\ldots, D_{m-1}, D_m\}$. Say

$$Y = D_m + R \text{ with } R \in V_{m-1}$$

$$\begin{aligned}
[Y, D_n] &= [D_m, D_n] + [R, D_n] \equiv (n-m)D_{n+m-1} \mod V_{n+m-2} \\
[Y,[Y,D_n]] &= (n-m)(n-1)D_{n+2m-2} \mod V_{n+2m-3} \\
[Y,[Y,[Y,D_n]]] &= (n-m)(n-1)(n+m-2)D_{n+3m-3} \mod V_{n+3m-4} \\
&\vdots
\end{aligned}$$

Thus $\operatorname{ad} Y$ acts nilpotent on D_n only if the leading coefficient becomes zero, i.e. if

$$\begin{aligned}
0 &= (n-m)(n-1)(n+m-2)(n+2m-3)(n+3m-4)\cdots \\
&= (n-m)(n-m+1\cdot(m-1)) + (n-m+2\cdot(m-1))\cdots \\
&\iff \\
0 &= (n-m) + k(m-1) = n-k+m(k-1) \text{ for some } k \in \mathbb{N}_0 \\
&\iff \\
k(m-1) &= m-n \text{ for some } k \in \mathbb{N}_0
\end{aligned}$$

If $m \notin \{0,1,2\}$ then we choose $n = 2$ so $k = \dfrac{m-2}{m-1} \notin \mathbb{N}_0$, and $(\operatorname{ad} Y)^k(D_2) \neq 0$ for all $k \in \mathbb{N}_0$. If $m = 2$ we choose $n = 3$, and again we get a $k = -1 \notin \mathbb{N}_0$. For $m = 1$ we can also choose $n = 2$ without a solution for $k \in \mathbb{N}_0$. It remains to consider the case $m = 0$ in which case we choose $n = -1$ so that there is no solution for $k \in \mathbb{N}_0$.

2. Let $c \in (0,1)$. Show that the function $f : [0,c] \longrightarrow \mathbb{R}$

$$f(x) = \begin{cases} -\dfrac{1}{\log x} & \text{if } 0 < x \leq c \\ 0 & \text{if } x = 0 \end{cases}$$

is uniformly continuous, but not Hölder continuous.

Reason: Hölder Continuity.

Solution: Set $\delta := e^{-1/\varepsilon}$ for an $\varepsilon > 0$. Then

$$|f(x) - f(0)| = |-\frac{1}{\log x} - 0| = -\frac{1}{\log x} < -\frac{1}{\log \delta} = \varepsilon$$

for all $|x| < \delta$ which shows that $f(x)$ is continuous at 0, hence continuous on $[0, c]$.

For the sake of completion we will show that every continuous function on a closed interval $[a, b]$ is automatically uniformly continuous (Theorem of Heine).

Assume $f(x)$ was not uniformly continuous. Then there is an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ exist points $y_n, z_n \in [a, b]$ with

$$|y_n - z_n| < \frac{1}{n} \text{ and } |f(y_n) - f(z_n)| \geq \varepsilon$$

We choose a convergent subsequence $(y_{n_k}) \subseteq (y_n)$ by the theorem of Bolzano-Weierstraß. Say $p := \lim_{n\to\infty} y_{n_k} \in [a, b]$. Since $|y_{n_k} - zn_k| < n_k^{-1}$ we have $p := \lim_{n\to\infty} z_{n_k}$, too. By continuity of $f(x)$ we thus get

$$\left| \lim_{n\to\infty} (f(y_{n_k}) - f(z_{n_k})) \right| = f(p) - f(p) = 0$$

which contradicts $|f(y_{n_k}) - f(z_{n_k})| \geq \varepsilon$ for all $k \in \mathbb{N}$.

It remains to show that $f(x)$ is not Hölder continuous. Assume it is and there are $C > 0$ and $\alpha \in (0, 1]$ such that

$$|f(x) - f(0)| = f(x) \leq Cx^\alpha \;\forall\; x \in (0, c]$$

This means by the rule of L'Hôpital

$$C \geq \lim_{x\downarrow 0} \frac{f(x)}{x^\alpha} = \lim_{x\downarrow 0} \frac{(-x^{-\alpha})'}{(\log x)'} = \lim_{x\downarrow 0} \alpha x^{-\alpha} = \lim_{x\downarrow 0} \frac{\alpha}{x^\alpha} = \infty$$

which is obviously a contradiction.

3. Consider the equation $pV - C(A - B\sqrt{p} + T) = 0$ where A, B, C are constant parameters, $p = p(T, V)$ vapor pressure, $V = V(T, p)$ molar volume, and $T = T(p, V)$ absolute temperature. Prove by three *different* methods that

$$\left(\frac{\partial V}{\partial T}\right)_p \cdot \left(\frac{\partial T}{\partial p}\right)_V \cdot \left(\frac{\partial p}{\partial V}\right)_T = -1$$

Reason: Antoine Equation.

Solution: Let $f(p, V, T) = pV - C(A - B\sqrt{p} + T) = 0$.

(a) Implicit Function Theorem.

We get from the implicit function theorem applied to resp.

$$f(T,V(T))=0 \Longrightarrow \frac{\partial f}{\partial T}+\frac{\partial f}{\partial V}\cdot\frac{\partial V}{\partial T}=0 \Longrightarrow \left(\frac{\partial V}{\partial T}\right)_p = \left(-\frac{\left(\frac{\partial f}{\partial T}\right)}{\left(\frac{\partial f}{\partial V}\right)}\right)$$

$$f(p,T(p))=0 \Longrightarrow \frac{\partial f}{\partial p}+\frac{\partial f}{\partial T}\cdot\frac{\partial T}{\partial p}=0 \Longrightarrow \left(\frac{\partial T}{\partial p}\right)_V = \left(-\frac{\left(\frac{\partial f}{\partial p}\right)}{\left(\frac{\partial f}{\partial T}\right)}\right)$$

$$f(V,p(V))=0 \Longrightarrow \frac{\partial f}{\partial V}+\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial V}=0 \Longrightarrow \left(\frac{\partial p}{\partial V}\right)_T = \left(-\frac{\left(\frac{\partial f}{\partial V}\right)}{\left(\frac{\partial f}{\partial p}\right)}\right)$$

Multiplication and reducing the quotients yield the result.

(b) Implicit Differentiation.

$$\begin{aligned}
\frac{\partial f}{\partial T} &= 0 = p\frac{\partial V}{\partial T}-C \Longrightarrow \left(\frac{\partial V}{\partial T}\right)_p = \frac{C}{p}\\
\frac{\partial f}{\partial p} &= 0 = V+\frac{CB}{2\sqrt{p}}-C\frac{\partial T}{\partial p} \Longrightarrow \left(\frac{\partial T}{\partial p}\right)_V = \frac{V}{C}+\frac{B}{2\sqrt{p}}\\
\frac{\partial f}{\partial V} &= 0 = V\cdot\frac{\partial p(V)}{\partial V}+p+BC\frac{\partial\sqrt{p(V)}}{\partial V}\\
&= V\left(\frac{\partial p}{\partial V}\right)_T+p+\frac{BC}{2\sqrt{p}}\cdot\left(\frac{\partial p}{\partial V}\right)_T\\
&\Longrightarrow \left(\frac{\partial p}{\partial V}\right)_T = -\frac{p}{V+\frac{BC}{2\sqrt{p}}}
\end{aligned}$$

Hence $\left(\dfrac{\partial V}{\partial T}\right)_p \cdot \left(\dfrac{\partial T}{\partial p}\right)_V \cdot \left(\dfrac{\partial p}{\partial V}\right)_T$ is equal to

$$\begin{aligned}
&\frac{C}{p}\cdot\left(\frac{V}{C}+\frac{B}{2\sqrt{p}}\right)\cdot\left(-\frac{2p\sqrt{p}}{2V\sqrt{p}+BC}\right)\\
&\quad = -\left(\frac{2V\sqrt{p}}{2p\sqrt{p}}+\frac{BC}{2p\sqrt{p}}\right)\cdot\left(-\frac{2p\sqrt{p}}{2V\sqrt{p}+BC}\right) = -1
\end{aligned}$$

(c) Solving for the Functions.

$$f(p,V,T)=0 \Longrightarrow V=\frac{C}{p}\cdot(A-B\sqrt{p}+T)\Longrightarrow \left(\frac{\partial V}{\partial T}\right)_p=\frac{C}{p}$$
$$f(p,V,T)=0 \Longrightarrow T=\frac{pV}{C}-A+B\sqrt{p}\Longrightarrow \left(\frac{\partial T}{\partial p}\right)_V=\frac{V}{C}+\frac{B}{2\sqrt{p}}$$

To solve $f(p,V,T)=0$ for p, we have to consider a quadratic equation in $\sqrt{p}>0$.

$$\begin{aligned}
0 &= p+\frac{CB}{V}\sqrt{p}-\frac{CA+CT}{V}\\
\sqrt{p} &= -\frac{CB}{2V}+\sqrt{\frac{C^2B^2}{4V^2}+\frac{CA+CT}{V}}\\
p &= \left(\frac{1}{2V}\left(-CB+\sqrt{C^2B^2+4VC(A+T)}\right)\right)^2
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial p}{\partial V}\right)_T &= \frac{\partial p}{\partial V} = \frac{\partial(\sqrt{p}^2)}{\partial V}) = 2\sqrt{p}\cdot\frac{\partial(\sqrt{p})}{\partial V}\\
&= 2\sqrt{p}\cdot\frac{\partial}{\partial V}\left\{\frac{1}{2V}\left(-CB+\sqrt{C^2B^2+4VC(A+T)}\right)\right\}\\
&= 2\sqrt{p}\cdot\left\{-\frac{1}{2V^2}\left(-CB+\sqrt{C^2B^2+4VC(A+T)}\right)\right\}\\
&+ 2\sqrt{p}\cdot\frac{1}{2V}\cdot\frac{1}{2}\cdot\frac{4C(A+T)}{\sqrt{C^2B^2+4VC(A+T)}}\\
&= -\frac{\sqrt{p}}{V^2}(-CB+CB+2V\sqrt{p}) + \frac{2\sqrt{p}C(A+T)}{V\cdot\left(CB+2V\sqrt{p}\right)}\\
&= -\frac{2p}{V} + \frac{2\sqrt{p}\left(pV+CB\sqrt{p}\right)}{V\cdot\left(CB+2V\sqrt{p}\right)}\\
&= \frac{-2pCB-4pV\sqrt{p}+2\sqrt{p}pV+2CBp}{V\cdot\left(CB+2V\sqrt{p}\right)}\\
&= -\frac{2p\sqrt{p}}{CB+2V\sqrt{p}} = \frac{-p}{V+\dfrac{CB}{2\sqrt{p}}}
\end{aligned}$$

and thus

$$\begin{aligned}
\left(\frac{\partial V}{\partial T}\right)_p\cdot\left(\frac{\partial T}{\partial p}\right)_V\cdot\left(\frac{\partial p}{\partial V}\right)_T &= \frac{C}{p}\cdot\left(\frac{V}{C}+\frac{B}{2\sqrt{p}}\right)\cdot\frac{-p}{V+\dfrac{CB}{2\sqrt{p}}}\\
&= \left(\frac{V}{p}+\frac{CB}{2p\sqrt{p}}\right)\cdot\frac{-2p\sqrt{p}}{2V\sqrt{p}+CB}\\
&= \frac{-2V\sqrt{p}}{2V\sqrt{p}-CB} - \frac{CB}{2V\sqrt{p}+CB} = -1
\end{aligned}$$

4. Calculate
$$\left(\frac{\partial V}{\partial T}\right)_p \text{ and } \left(\frac{\partial V}{\partial p}\right)_T$$
for $V = V(T,p)$ from the equation of state
$$\left(p+\frac{a}{V^2}\right)(V-b) = R\cdot T\,;\, a,b,R>0$$
Reason: Van der Waals Equation.

Solution:

(a) If we differentiate $\left(p+\dfrac{a}{V^2}\right)(V-b)=R\cdot T$ along T with constant p we get

$$\begin{aligned} R &= (V-b)\left(\frac{\partial}{\partial T}\left(p+\frac{a}{V^2}\right)\right)_p + \left(p+\frac{a}{V^2}\right)\left(\frac{\partial}{\partial T}(V-b)\right)_p \\ &= \frac{2(b-V)a}{V^3}\left(\frac{\partial V}{\partial T}\right)_p + \left(p+\frac{a}{V^2}\right)\left(\frac{\partial V}{\partial T}\right)_p \\ &= \frac{1}{V^3}\left(2ab-aV+pV^3\right)\left(\frac{\partial V}{\partial T}\right)_p \\ \left(\frac{\partial V}{\partial T}\right)_p &= \frac{RV^3}{pV^3-aV+2ab} \end{aligned}$$

(b) If we differentiate $\left(p+\dfrac{a}{V^2}\right)(V-b)=R\cdot T$ along p with constant T we get

$$\begin{aligned} 0 &= (V-b)\left(\frac{\partial}{\partial p}\left(p+\frac{a}{V^2}\right)\right)_T + \left(p+\frac{a}{V^2}\right)\left(\frac{\partial}{\partial p}(V-b)\right)_T \\ &= (V-b)\left(1-\frac{2a}{V^3}\left(\frac{\partial V}{\partial p}\right)_T\right) + \left(p+\frac{a}{V^2}\right)\left(\frac{\partial V}{\partial p}\right)_T \\ &= (V-b)+\left(\frac{2a(b-V)}{V^3}+\frac{pV^3}{V^3}+\frac{aV}{V^3}\right)\left(\frac{\partial V}{\partial p}\right)_T \\ \left(\frac{\partial V}{\partial p}\right)_T &= \frac{V^3(b-V)}{pV^3-aV+2ab} \end{aligned}$$

5. Let $\sigma\in \operatorname{Aut}(S_n)$ be an automorphism of the symmetric group S_n $(n\geq 4)$ such that σ sends transpositions to transpositions, then prove that σ is an inner automorphism. Determine the inner automorphism groups of the symmetric and the alternating groups for $n\geq 4$.

Reason: Inner Automorphisms of Permutation Groups.

Solution: Suppose that $\sigma(1,r)=(a_r,b_r)$ for each $r\in\{1,2,\ldots,n\,|\,n>3\}$. Then for $r\geq 3$

$$\sigma(1,r,2)=\sigma((1,2)(1,r))=\sigma(1,2)\sigma(1,r)=(a_2,b_2)(a_r,b_r)$$

is an element of order 3, hence either $a_r\in\{a_2,b_2\}$ or $b_r\in\{a_2,b_2\}$. By symmetry reasons we may assume that $a_r\in\{a_2,b_2\}$ for all $r\geq 3$.

We claim that either $a_r = a_2$ for all r or $a_r = b_2$ for all r. Assume that instead there are $r \neq s$ such that $a_r = a_2$ and $a_s = b_2$. Note that $(1, r, 2)(1, s, 2) = (1, s)(2, r)$ is of order 2. However,

$$\begin{aligned}\sigma((1,r,2)(1,s,2)) &= (a_2, b_2)(a_r, b_r)(a_2, b_2)(a_s, b_s)\\ &= (a_2, b_2)(a_2, b_r)(a_2, b_2)(b_2, b_s) = (b_2, b_s, b_r)\end{aligned}$$

is of order 3 if $2 \neq r \neq s \neq 2$. This is a contradiction. Thus we either have $a_2 = a_r$ or $b_2 = b_r$ for all $r \geq 3$. W.l.o.g. let $a_2 = a_r$, i.e. $\sigma(1, r) = (a_2, b_r)$ for all $r \geq 3$. Since σ is an isomorphism, we have $b_r \neq b_s$ if $r \neq s$ because $\sigma(1, r) \neq \sigma(1, s)$. Let π be a permutation for which $\pi(1) = a_2$ and $\pi(r) = b_r$ for all $r \geq 3$. This uniquely determines π, because we determined $n - 1$ values, and bijectivity determines the last value. Now

$$\sigma(1, r) = (a_r, b_r) = (a_2, b_r) = \pi \circ (1, r) \circ \pi^{-1}$$

and $\sigma = \operatorname{Inn}(\pi) \in \operatorname{Inn}(S_n)$.

Consider $G \xrightarrow{\pi} \operatorname{Inn}(G)$ defined by $g \longmapsto (x \mapsto g^{-1}xg)$. Then $\ker \pi = Z(G)$ and $\operatorname{Inn}(G) \cong G/Z(G)$. Since the alternating groups A_n are simple for $n > 4$, we have $\operatorname{Inn}(A_n) \cong A_n$. The symmetric groups S_n for $n > 4$ have only A_n as nontrivial normal subgroup, i.e. $S_n \cong A_n \rtimes \mathbb{Z}_2$. Furthermore the centers of S_n are trivial, i.e. $\operatorname{Inn}(S_n) \cong S_n$. In case of $n = 4$ we also have the Klein four-group

$$V_4 = [A_4, A_4] \triangleleft A_4 \triangleleft S_4.$$

Since V_4 is not abelian, it cannot be the center of either permutation group, i.e. $\operatorname{Inn}(A_4) \cong A_4$ and $\operatorname{Inn}(S_4) \cong S_4$.

6. Consider a code $C \subseteq \mathbb{F}_q^n$ with minimal Hamming distance $d > n \cdot \dfrac{q-1}{q}$. Prove that the number of possible code words is restricted by

$$c := \#C \leq \frac{d}{d - n \cdot \dfrac{q-1}{q}}$$

Reason: Plotkin Bound.

Solution: Let $s := \sum_{(x,y) \in C \times C} d(x, y)$ be the sum of all Hamming

distances in C. Since $d(x,y) \geq d$ for different code words x,y we immediately have
$$s \geq c \cdot (c-1) \cdot d.$$
We want to show that
$$s \leq c^2 \cdot n \cdot \left(1 - \frac{1}{q}\right) = c^2 \cdot n \cdot \frac{q-1}{q}$$
from which we get
$$c(c-1)d \leq c^2 n \frac{q-1}{q} \Longleftrightarrow cd - cn\frac{q-1}{q} \leq d \Longleftrightarrow c \leq \frac{d}{d - n\frac{q-1}{q}}$$
which we will have to show. Let $C = \{x^{(1)}, \ldots, x^{(c)}\}$. We define the number of code words that have an $a \in \mathbb{F}_q$ at $k-$th position by
$$t_k(a) := \#\{1 \leq j \leq c \,|\, x_k^{(j)} = a\}.$$
Obviously $\sum_{a\in\mathbb{F}_q} t_k(a) = c$. The number of pairs $(x,y) \in C \times C$ which are different at position k is $\sum_{a\in\mathbb{F}_q} t_k(a)(c - t_k(a))$. Therefore the sum of all Hamming distances equals
$$s = \sum_{k=1}^{n} \sum_{a\in\mathbb{F}_q} t_k(a)(c - t_k(a)) = \sum_{k=1}^{n} \left(c^2 - \sum_{a\in\mathbb{F}_q} t_k(a)^2\right)$$
According to the Cauchy-Schwarz inequality we have
$$\left(\sum_{a\in\mathbb{F}_q} t_k(a)\right)^2 \leq \left(\sum_{a\in\mathbb{F}_q} t_k(a)^2\right) \cdot \left(\sum_{a\in\mathbb{F}_q} 1^2\right) = q \cdot \sum_{a\in\mathbb{F}_q} t_k(a)^2$$
$$-\sum_{a\in\mathbb{F}_q} t_k(a)^2 \leq -\frac{1}{q}\left(\sum_{a\in\mathbb{F}_q} t_k(a)\right)^2$$
$$s = \sum_{k=1}^{n}\left(c^2 - \sum_{a\in\mathbb{F}_q} t_k(a)^2\right) \leq \sum_{k=1}^{n}\left(c^2 - \frac{1}{q}\left(\sum_{a\in\mathbb{F}_q} t_k(a)\right)^2\right)$$
$$= \sum_{k=1}^{n}\left(c^2 - \frac{1}{q}c^2\right) = nc^2\left(1 - \frac{1}{q}\right)$$

7. Prove that the Cantor dust on the real line contains uncountable infinitely many points, and that it is a fractal by calculating its Hausdorff-Besicovitch dimension.

   Reason: Cantor Set.

   Solution: Cantor dust is constructed by cutting out the interval $(1/3, 2/3)$ from $[0, 1]$, then cutting out the middle third from the remaining intervals and so on.

   We can ask at any step of the construction for any remaining point, whether a point is placed on the left or on the right from the nearest removed interval. Let's write a "0" for left and a "1" for right. The result is an infinite, binary sequence that determines the given point of the Cantor dust, and vice versa: each such sequence determines a point in the Cantor dust. The Cantor dust is therefore equivalent to $[0, 1]^{\mathbb{N}}$, which is equivalent to the real numbers. Since the real numbers are uncountable infinitely many, the Cantor dust is, too.

   To determine the fractal Hausdorff dimension D_H, we cover the object with the least number $N(\varepsilon)$ of circles of diameter ε and define

$$D_H = -\lim_{\varepsilon \to 0} \log_\varepsilon N(\varepsilon).$$

   This means in our case

ε	1	1/3	1/9	1/27	1/81	...
$N(\varepsilon)$	1	2	4	8	16	...

   i.e. $N(3^{-n}) = 2^n$ and $N(\varepsilon) = 2^{-\log_3 \varepsilon} = \varepsilon^{-\frac{\log 2}{\log 3}}$, i.e. $D_H = \dfrac{\log 2}{\log 3} \approx 0.631$. The Cantor dust is a fractal, since $D_H > D_T = 0$, the topological dimension D_T.

8. Define the harmonic number $H(p) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{p-1} = \dfrac{a}{b}$.
Show that $p^2 \,|\, a$ for primes $p > 3$.

Reason: Theorem of Wolstenholme.

Solution: Consider the polynomial

$$f(x) = \prod_{k=1}^{p-1}(x-k) = x^{p-1} - a_{p-2}(p)x^{p-2} \pm \ldots - a_1(p)x + a_0(p)$$

where $a_0(p) = (p-1)!$ and $a_1(p) = (p-1)!H(p)$. We want to show that $p^2 \,|\, a_1(p)$. We therefore pass to the finite field $\mathbb{F}_p$, so that $\mathbb{F}_p[x] \ni f(x) \stackrel{(*)}{=} x^{p-1} - 1$. Hence $p \,|\, a_k(p)$ for all $1 \leq k \leq p-2$ and $a_0(p) = (p-1)! \equiv -1 \mod p$. This is known as Wilson's theorem.

$(*)$ The polynomial $x^p - x = x(x^{p-1} - 1)$ has only simple zeros, because $(x^p - x)' = px^{p-1} - 1 \not\equiv 0 \mod p$. But there are at most p numbers in $\mathbb{F}_p$ for the p many zeros, hence $\mathbb{F}_p[x] \ni x(x-1)(x-2)\cdots(x-(p-1)) = x^p - x = x \cdot f(x)$.

$$\begin{aligned} f(p) = (p-1)! &= p^{p-1} - a_{p-2}(p)p^{p-2} \pm \ldots - a_1(p)p + a_0(p) \\ &= p^{p-1} - a_{p-2}(p)p^{p-2} \pm \ldots - a_1(p)p + (p-1)! \\ a_1(p) &= p^{p-2} - a_{p-2}(p)p^{p-3} \pm \ldots + a_2(p)p \end{aligned}$$

From $p > 3$ and $p \,|\, a_k$ follows that $p^2 \,|\, a_1(p) = (p-1)!H(p)$ and thus $p^2 \,|\, H(p)$ because p is prime.

9. An ideal coin is thrown three times in a row and then an ideal dice is thrown twice in a row. Each time you toss a coin you get one point if the coin shows "tails" and two points if the coin shows "heads". If you add the total of the two dice rolls to this number of points, you get the total number of points. Furthermore, let A be the event "the total number of points achieved is odd", B be the event "the total of the two dice rolls is divisible by 5", and C the event "the number of points achieved in the three coin tosses is at least 5". Investigate whether A, B, C are pairwise stochastically independent. Also investigate whether A, B, C are stochastically independent.

Reason: Stochastic.

Solution: The phase space $\Omega = \{1,2\}^3 \times \{1,2,\ldots,6\}^2$ and $p(\omega) := \dfrac{1}{|\Omega|} = \dfrac{1}{2^3 \cdot 6^2} = \dfrac{1}{288}$ for all $\omega \in \Omega$, such that (Ω, p) is a Laplace experiment. With $\vec{\omega} = (\omega_1, \ldots, \omega_5) \in \Omega$ let ω_i for $i \in \{1, 2, 3\}$ be the points achieved in the $i-$th coin flip, ω_4 the points of the first die roll, ω_5 the points of the second. Then

$$A = \left\{ (\omega_1, \ldots, \omega_5) \in \Omega \,|\, \sum_{i=1}^{5} \omega_i = 2n - 1 \text{ for some } n \in \mathbb{N} \right\},$$

$$B = \{ (\omega_1, \ldots, \omega_5) \in \Omega \,|\, \omega_4 + \omega_5 \in \{5, 10\} \},$$

$$C = \left\{ (\omega_1, \ldots, \omega_5) \in \Omega \,|\, \sum_{i=1}^{3} \omega_i = \in \{5, 6\} \right\}.$$

We can choose $\omega_1, \ldots, \omega_4$ arbitrarily and have 3 possibilities left to an odd total number of points. There are 4 possibilities to achieve 5 by rolling the dice, and 3 to get 10. To end up with 5, resp. 6 points in the coin flips, there are 3, resp. 1 chances. Hence

$$P(A) = \frac{2^3 \cdot 6 \cdot 3}{288} = \frac{1}{2}, P(B) = \frac{2^3 \cdot (4+3)}{288} = \frac{7}{36}, P(C) = \frac{(3+1) \cdot 6^2}{288} = \frac{1}{2}$$

Moreover

$$\vec{\omega} \in A \cap B \Longleftrightarrow \left(\sum_{k=1}^{3} \omega_k \in \{4, 6\} \wedge \omega_4 + \omega_5 = 5 \right) \vee \left(\sum_{k=1}^{3} \omega_k \in \{3, 5\} \wedge \omega_4 + \omega_5 = 10 \right)$$

$$P(A \cap B) = \frac{4 \cdot 4 + 4 \cdot 3}{288} = \frac{7}{72} = \frac{1}{2} \cdot \frac{7}{36} = P(A) \cdot P(B)$$

$$\vec{\omega} \in A \cap C \Longleftrightarrow \left(\sum_{k=1}^{3} \omega_k = 5 \wedge \omega_4 + \omega_5 \in \{2, 4, 6, 8, 10, 12\} \right) \vee \left(\sum_{k=1}^{3} \omega_k = 6 \wedge \omega_4 + \omega_5 \in \{3, 5, 7, 9, 11\} \right)$$

$$P(A \cap C) = \frac{3 \cdot 18 + 1 \cdot 18}{288} = \frac{72}{288} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(A) \cdot P(C)$$

$$\vec{\omega} \in B \cap C \Longleftrightarrow \sum_{k=1}^{3} \omega_k \in \{5, 6\} \wedge \omega_4 + \omega_5 \in \{5, 10\}$$

$$P(B \cap C) = \frac{4 \cdot 7}{288} = \frac{7}{72} = \frac{1}{2} \cdot \frac{7}{36} = P(B) \cdot P(C)$$

which proves that the events are pairwise independent. Finally we have

$$\begin{aligned}\vec{\omega} \in A \cap B \cap C \Longleftrightarrow & \left(\sum_{k=1}^{3} \omega_k = 6 \wedge \omega_4 + \omega_5 = 5\right) \\ & \vee \left(\sum_{k=1}^{3} \omega_k = 5 \wedge \omega_4 + \omega_5 = 10\right)\end{aligned}$$

$$P(A \cap B \cap C) = \frac{1 \cdot 4 + 3 \cdot 3}{288} = \frac{13}{288} \neq \frac{14}{288} = \frac{1}{2} \cdot \frac{7}{36} \cdot \frac{1}{2} = P(A) \cdot P(B) \cdot P(C)$$

which means that they are not independent as a whole.

10. Show

$$C_n := \binom{2n}{n} - \binom{2n}{n+1} = \prod_{k=1}^{n} \frac{4k-2}{k+1}$$

and determine all primes in $\{C_n\}$.

Reason: Catalan Numbers.

Solution:

$$\begin{aligned}C_n = \binom{2n}{n} - \binom{2n}{n+1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\ &= \frac{(2n)!(n+1) - (2n)!n}{n!(n+1)!} = \frac{(2n)!}{n!(n+1)!}\end{aligned}$$

$$\begin{aligned}\frac{C_{n+1}}{C_n} &= \frac{(2n+2)!}{(n+1)!(n+2)!} \cdot \frac{n!(n+1)!}{(2n)!} \\ &= \frac{(2n+2)(2n+1)}{(n+2)(n+1)} = \frac{2(2n+1)}{n+2} = \frac{4n+2}{n+2}\end{aligned}$$

Since $C_1 = 1$ we get

$$C_n = \frac{4n-2}{n+1} \cdot C_{n-1} = \frac{4n-2}{n+1} \cdot \frac{4n-6}{n} \cdot C_{n-2} = \ldots = \prod_{k=1}^{n} \frac{4k-2}{k+1}$$

$$\begin{aligned}C_0 = C_1 = 1\,,\; C_2 &= 2\,,\; C_3 = 5\,,\; C_4 = 14\,,\; C_5 = 42 \\ C_n = \frac{1}{n+1}\binom{2n}{n} &= \frac{2n(2n-1)\cdot \ldots \cdot (n+2)}{n!} \\ &= 2 \cdot \frac{2n-1}{n-1} \cdot \ldots \cdot \frac{n+2}{2} > 2^{n-1} \geq 2n \text{ for } n \geq 4\end{aligned}$$

All prime factors of

$$C_n = 2^n \cdot \frac{(2n-1)\cdot(2n-2)\cdot\ldots\cdot 1}{(n+1)\cdot n\cdot\ldots\cdot 1}$$

are less than $2n$. Thus it is impossible that C_n itself is prime if $n \geq 4$.

11. (HS-1) Check whether there is a natural number $n \in \mathbb{N}$ such that $\sqrt{n}+\sqrt{n+4} \in \mathbb{Q}$. Note that zero is no natural number.

    Reason: Irrational Numbers.

    Solution: Assume $\sqrt{n}+\sqrt{n+4} \in \mathbb{Q}$. Then

    $$\mathbb{Q} \ni \left(\sqrt{n}+\sqrt{n+4}\right)^2 = 2n+4+2\underbrace{\sqrt{n(n+4)}}_{\in\mathbb{Q}}$$

    If $\sqrt{n(n+4)} = \frac{a}{b}$, then $a^2 = b^2 \cdot n \cdot (n+4)$ and $n(n+4)$ is a square number (e.g. by the fundamental theorem of arithmetic).

    $$\begin{aligned} &n^2+2n+2 \leq n^2+4n \\ &\quad (n+1)^2 = n^2+2n+1 < n^2+4n < n^2+4n+4 = (n+2)^2 \\ &\quad n(n+4) = n^2+4n \in ((n+1)^2,(n+2)^2) \end{aligned}$$

    so $n(n+4)$ cannot be a square number contradicting our assumption.

12. (HS-2) Assume that $n \in \mathbb{N}$ is odd, and $\{a_1,a_2,\ldots a_n\} = \{1,2,\ldots,n\}$. Prove that

    $$(a_1-1)\cdot(a_2-2)\cdot\ldots\cdot(a_{n-1}-(n-1))\cdot(a_n-n)$$

    is always even.

    Reason: Pigeon Hole Principle.

    Solution: $n = 2m+1$ for some $m \in \mathbb{N}$. Among the numbers $\{a_1,\ldots,a_n\} = \{1,\ldots,n\}$ are therefore at most m numbers even, namely $\{2,4,\ldots,2m\}$. The set $\{a_1,a_3,\ldots,a_n\}$ of numbers with an odd index contains $m+1 = n-m$ many numbers, so at least one of them has to be odd. Thus at least one of the factors $(a_1-1),(a_3-3),\ldots,(a_n-n)$ is an even number, i.e. the product $\prod_{k=1}^n (a_k-k)$ is even, too.

13. (HS-3) Show that for every natural number $n \in \mathbb{N}$ there is a $c = c(n) \in \mathbb{R}$ such that for all real numbers $a > 0$

    $$a+a^2+a^3+\ldots+a^{2n-1}+a^{2n} \leq c(n)\cdot\left(1+a^{2n+1}\right).$$

Show that there is a smallest solution among all possible values $c(n)$ and determine it.

Reason: Calculus.

Solution: Let $a > 0$ be a real number. Then we have

$$0 \leq 1-(n+1)a^{2n+1} - a^{2n+2} + (n+1)a^{2n+3} \text{ for all } n \in \mathbb{N}_0$$

For $n = 0$ it is $1-a-a^2+a^3 = (1-a)(1-a^2) = (1-a)^2(1+a) \geq 0$ and for $n = 1$ we have $1-2a^3-a^4+2a^5 = (a-1)^2(a+1)(2a^2+a+1) \geq 0$.

$$\begin{aligned}
&1-(n+2)a^{2n+3} - a^{2n+4} + (n+2)a^{2n+5}\\
&= a^2(1-(n+1)a^{2n+1} - a^{2n+2} + (n+1)a^{2n+3}) + 1 - a^2 - a^{2n+3} + a^{2n+5}\\
&\geq 1-a^2-a^{2n+3}+a^{2n+5}\\
&= (1-a^2)(1-a^{2n+3}) = (1-a)^2(1+a) \cdot \frac{1-a^{2n+3}}{1-a}\\
&= (1-a)^2(1+a)(1+a+a^2+\ldots+a^{2n+2}) \geq 0
\end{aligned}$$

We now prove again by induction (case $n = 1$ see above) that

$$a + a^2 + a^3 + \ldots + a^{2n-1} + a^{2n} \leq n \cdot \left(1 + a^{2n+1}\right).$$

$$\begin{aligned}
&0 \leq -a - a^2 - a^3 + \ldots - a^{2n-1} - a^{2n} + n + na^{2n+1} \text{ and}\\
&0 \leq 1-(n+1)a^{2n+1} - a^{2n+2} + (n+1)a^{2n+3} \text{ hence}\\
&0 \leq 1-(n+1)a^{2n+1} - a^{2n+2} + (n+1)a^{2n+3} + n + na^{2n+1} - a - \ldots - a^{2n}\\
&0 \leq 1 + n - a - \ldots - a^{2n+1} - a^{2n+2} + (n+1)a^{2n+3}\\
&a + a^2 + a^3 + \ldots + a^{2n} + a^{2n+1} + a^{2n+2} \leq (n+1)(1+a^{2n+3})
\end{aligned}$$

Therefore $c = c(n) = n$ is a possible solution. It remains to show that it is already the minimal solution, i.e. for any real number $c < n$ we must find a real number $a > 0$ such that

$$a + a^2 + a^3 + \ldots + a^{2n-1} + a^{2n} > c(n) \cdot \left(1 + a^{2n+1}\right).$$

However, with $c < n$ we have $2c < 2n$ which is exactly our requirement if we choose $a = 1$.

14. (HS-4) Given an integer k, determine all pairs $(x, y) \in \mathbb{Z}^2$ such that

$$x^2 + k \cdot y^2 = 4 \text{ and } k \cdot x^2 - y^2 = 2$$

Reason: Conic Sections.

Solution: Assume (x, y) is an integer solution. Then

$$\begin{aligned} x^2 + k(kx^2 - 2) &= x^2(1 + k^2) - 2k = 4 \\ 0 < x^2(1 + k^2) &= 2(k + 2) \end{aligned}$$

We can therefore exclude all $k \leq -3$.

$$\begin{aligned}
k = -2 &: 0 < 2 = -2x^2 - y^2 \leq 0 \lightning \\
k = -1 &: 0 < 2 = -x^2 - y^2 \leq 0 \lightning \\
k = 0 &: 0 < 2 = -y^2 \leq 0 \lightning \\
k = 1 &: 2x^2 = 2 \cdot 3 = 6 \Longrightarrow x^2 = 3 \Longrightarrow x \notin \mathbb{Z} \lightning \\
k = 2 &: 5x^2 = 2 \cdot 4 = 8 \Longrightarrow x^2 = \frac{8}{5} \Longrightarrow x \notin \mathbb{Z} \lightning \\
k = 3 &: 10x^2 = 10 \Longrightarrow x = \pm 1 \Longrightarrow y = \pm 1 \\
k > 3 \wedge y^2 = 0 &\Longrightarrow 2 = k \cdot x^2 = 4 \cdot k > 12 \lightning \\
k > 3 \wedge x^2 = 0 &\Longrightarrow y^2 = -2 \lightning \\
k \geq 4 \wedge x^2 \geq 1 \wedge y^2 \geq 1 &\Longrightarrow 4 = x^2 + ky^2 \geq 1 + 4 \cdot 1 = 5 \lightning
\end{aligned}$$

Our equation system is thus not solvable, except $k = 3$, in which case all four pairs $\{(x, y) \in \mathbb{Z}^2 \,|\, x = \pm 1, y = \pm 1\}$ are the only solution. It is easy to check, that these pairs are indeed solutions.

15. (HS-5) Prove for every natural number $n \in \mathbb{N}$

$$\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n} < \frac{1}{\sqrt{2n + 1}}$$

Reason: Inequality.

Solution: For $k = 1, \ldots, n$

$$\begin{aligned}
(2k - 1) \cdot (2k + 1) = 4k^2 - 1 < 4k^2 &\Longrightarrow \frac{2k - 1}{2k} < \frac{2k}{2k + 1} \\
&\Longrightarrow \prod_{k=1}^{n} \frac{2k - 1}{2k} < \prod_{k=1}^{n} \frac{2k}{2k + 1} \\
&\Longrightarrow \left(\prod_{k=1}^{n} \frac{2k - 1}{2k} \right)^2 < \left(\prod_{k=1}^{n} \frac{2k - 1}{2k} \right) \cdot \left(\prod_{k=1}^{n} \frac{2k}{2k + 1} \right) = \frac{1}{2n + 1} \\
&\Longrightarrow \prod_{k=1}^{n} \frac{2k - 1}{2k} < \frac{1}{\sqrt{2n + 1}}
\end{aligned}$$

8 May 2021

1. Integrate
$$\int_0^\infty\int_0^\infty e^{-\left(x+y+\frac{\lambda^3}{xy}\right)}x^{-\frac{2}{3}}y^{-\frac{1}{3}}\,dx\,dy$$
Reason: Liouville's Formula.

Solution: Set $R(\lambda)=\int_0^\infty\int_0^\infty e^{-\left(x+y+\frac{\lambda^3}{xy}\right)}x^{-\frac{2}{3}}y^{-\frac{1}{3}}\,dx\,dy$ and $z:=\dfrac{\lambda^3}{xy}$. Then
$$\frac{dz}{dx}=-\frac{\lambda^3}{x^2y}=-\frac{\lambda^3}{y}\left(\frac{yz}{\lambda^3}\right)^2=-yz^2\Rightarrow dx=-\frac{\lambda^3}{yz^2}\,dz$$
$$\begin{aligned}R'(\lambda)&=-\int_0^\infty\int_0^\infty\frac{3\lambda^2}{xy}\cdot e^{-\left(x+y+\frac{\lambda^3}{xy}\right)}\cdot x^{-\frac{2}{3}}\cdot y^{-\frac{1}{3}}\,dx\,dy\\&=3\int_0^\infty\int_\infty^0\frac{z}{\lambda}\cdot e^{-\left(\frac{\lambda^3}{yz}+y+z\right)}\frac{\lambda^3}{yz^2}\cdot\frac{1}{\lambda^2}\cdot y^{\frac{2}{3}}\cdot z^{\frac{2}{3}}\cdot y^{-\frac{1}{3}}\,dz\,dy\\&=-3\int_0^\infty\int_0^\infty e^{-\left(\frac{\lambda^3}{yz}+y+z\right)}\cdot y^{-\frac{2}{3}}\cdot z^{-\frac{1}{3}}\,dz\,dy\\&=-3R(\lambda)\end{aligned}$$
Hence $R(\lambda)=R(0)e^{-3\lambda}$ and
$$\begin{aligned}R(0)&=\int_0^\infty\int_0^\infty e^{-x-y}\cdot x^{-\frac{2}{3}}\cdot y^{-\frac{1}{3}}\,dx\,dy=\int_0^\infty x^{\frac{1}{3}-1}e^{-x}\,dx\cdot\int_0^\infty y^{\frac{2}{3}-1}e^{-y}\,dy\\&=\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)=\frac{\pi}{\sin(\pi/3)}=\frac{2\pi}{\sqrt{3}}\text{ and }R(\lambda)=\frac{2\pi}{\sqrt{3}}\cdot e^{-3\lambda}\end{aligned}$$

2. Let F_n be the free group of rank n with generators $\{w_1,\ldots,w_n\}$. Then
$$\prod_{i=1}^m w_{a_i}^{b_i}\in[F_n,F_n]\Longleftrightarrow\forall_{k=1}^m\sum_{a_i=k}b_i=0$$
Reason: Abstract Algebra.

Solution: Denote the right-hand side property by (P). Then

(1) if two elements of F_n satisfy (P), then so does their product,
(2) each element of the form $[x,y]$ (i.e. each generator of $[F_n,F_n]$) satisfies (P).

which proves the direction from left to right.

Now assume (P). We proceed by the length of the word m. The cases $m=0$ and $m=1$ are obviously true.

$$w_k^{b_1}xw_k^{b_2}y=[w_k^{-b_1},x^{-1}]xw_k^{b_1+b_2}y$$

If $w_k^{b_1}xw_k^{b_2}y$ satisfies (P), so does $xw_k^{b_1+b_2}y$. But now the length of the latter is one less than the length of the former and the induction hypothesis applies.

Another way to see the statement is as follows: For an arbitrary group G, we have $g\in[G,G]$ if and only if $\bar{g}=g[G,G]=\bar{1}$ in the abelianization $G/[G,G]$. For the free group $G=F_n$ we have $F_n/[F_n,F_n]=\mathbb{Z}^n$ and the stated property follows immediately.

3. Calculate
$$\int_0^\pi\int_0^\pi\int_0^\pi \frac{1}{1-\cos x\,\cos y\,\cos z}\,dx\,dy\,dz$$

Reason: Watson Integral.

Solution: We start with the Weierstraß substitution $t=\tan(x/2)$.

$$\begin{aligned}\cos x&=\frac{1-t^2}{1+t^2}\\ \frac{dt}{dx}&=\frac{d}{dx}\tan\left(\frac{x}{2}\right)=\frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)=\frac{1+t^2}{2}\Longrightarrow dx=\frac{2\,dt}{1+t^2}\end{aligned}$$

$$\begin{aligned}x&=r\sin\theta\cos\varphi\,,\ dx=r\cos\theta\cos\varphi\,d\theta\\ y&=r\sin\theta\sin\varphi\,,\ dy=r\sin\theta\cos\varphi\,d\varphi\\ z&=r\cos\theta\,,\ dz=\cos\theta\,dr\end{aligned}$$

and rewrite

$$\begin{aligned}
I &:= \int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \frac{1}{1-\cos x\,\cos y\,\cos z}\,dx\,dy\,dz \\
&= 8\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{\dfrac{1}{1+x^2}\cdot\dfrac{1}{1+y^2}\cdot\dfrac{1}{1+z^2}}{1-\dfrac{1-x^2}{1+x^2}\cdot\dfrac{1-y^2}{1+y^2}\cdot\dfrac{1-z^2}{1+z^2}}\,dx\,dy\,dz \\
&= 8\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{dx\,dy\,dz}{(1+x^2)(1+y^2)(1+z^2)-(1-x^2)(1-y^2)(1-z^2)} \\
&= 4\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{dx\,dy\,dz}{x^2+y^2+z^2+x^2y^2z^2} \\
&= 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\infty} \frac{r^2\sin\theta\,dr\,d\theta\,d\varphi}{r^2+r^2\sin^2\theta\cos^2\varphi\; r^2\sin^2\theta\sin^2\varphi\; r^2\cos^2\theta} \\
&= 4\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\infty} \frac{dr}{1+\underbrace{\left(r\sin\theta\sqrt{\cos\theta}\sqrt{\sin\varphi\cos\varphi}\right)}_{=:s}^4}\sin\theta\,d\theta\,d\varphi \\
&= 4\int_0^{\infty}\frac{ds}{1+s^4}\cdot\int_0^{\pi/2}\frac{d\theta}{\sqrt{\cos\theta}}\cdot\int_0^{\pi/2}\frac{d\varphi}{\sqrt{\sin\varphi\cos\varphi}} \\
&= 4\cdot\frac{\pi}{2\sqrt{2}}\cdot\frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{2\pi}}\cdot\frac{\Gamma\left(\frac{1}{4}\right)^2}{2\sqrt{\pi}} \\
&= \frac{1}{4}\Gamma\left(\frac{1}{4}\right)^4 = 2\pi\varpi^2 = 2G^2\pi^3 \approx 43.198
\end{aligned}$$

with the Gauß constant $G = \dfrac{2}{\pi}\displaystyle\int_0^1 \frac{ds}{\sqrt{1-s^4}}$.

4. Let G be a finite group, $\mathbb{K}$ a field such that $\operatorname{char}(\mathbb{K}) \nmid |G|$, and (ρ, V) and (τ, W) linear representations of G over $\mathbb{K}$. The $\mathbb{K}-$linear mapping

$$\begin{aligned}
\operatorname{Sym} : \operatorname{Hom}_{\mathbb{K}}(V,W) &\longrightarrow \operatorname{Hom}_{\mathbb{K}}(V,W) \\
\varphi &\longmapsto \operatorname{Sym}(\varphi) = \frac{1}{|G|}\sum_{g\in G}\tau(g)\circ\varphi\circ\rho(g^{-1})
\end{aligned}$$

is a projection onto the subspace

$$\operatorname{Hom}_{\mathbb{K}}((\rho,V),(\tau,W)) = \{\vartheta : V \longrightarrow W \mid \forall_{g\in G} : \tau(g)\circ\vartheta\circ\rho(g^{-1}) = \vartheta\}$$

of $\operatorname{Hom}_{\mathbb{K}}(V,W)$. Prove (mention) all five claims.

Reason: Group Representations.

Solution:

(a) **Well-definition.** The element $|G| = 1 + \ldots + 1 \in \mathbb{K}$ has a multiplicative inverse since $\operatorname{char}(\mathbb{K}) \nmid |G|$, hence Sym is well-defined.

(b) **Homomorphism of representations.** Let $h \in G, \varphi \in \operatorname{Hom}_{\mathbb{K}}(V,W)$.

$$\begin{aligned}\tau(h) \circ \operatorname{Sym}(\varphi) &= \frac{1}{|G|}\sum_{g\in G}\tau(h)\circ\tau(g)\circ\varphi\circ\rho(g^{-1})\\ &= \frac{1}{|G|}\sum_{g\in G}\tau(hg)\circ\varphi\circ\rho(g^{-1}h^{-1}h)\\ &\stackrel{u=hg}{=} \frac{1}{|G|}\sum_{u\in G}\tau(u)\circ\varphi\circ\rho(u^{-1})\circ\rho(h)\\ &= \operatorname{Sym}(\varphi)\circ\rho(h)\end{aligned}$$

(c) **Linearity.**

$$\operatorname{Sym}(\alpha\varphi+\beta\vartheta) = \alpha\operatorname{Sym}(\varphi)+\beta\operatorname{Sym}(\vartheta)$$

is a direct consequence of the definition of Sym .

(d) **Image is a subspace of** $\operatorname{Hom}_{\mathbb{K}}(V,W)$. We just have proven that Sym is a $\mathbb{K}-$linear homomorphism of representations, i.e. especially spans a subspace of all $\mathbb{K}-$linear homomorphisms $V \longrightarrow W$.

(e) Sym **is a projection onto** $\operatorname{Hom}_{\mathbb{K}}((\rho,V),(\tau,W))$. Let ϑ be a homomorphism of representations, i.e. $\tau(g)\circ\vartheta\circ\rho(g^{-1}) = \vartheta$. Thus

$$\operatorname{Sym}(\vartheta) = \frac{1}{|G|}\sum_{g\in G}\tau(g)\circ\vartheta\circ\rho(g^{-1}) = \frac{1}{|G|}\sum_{g\in G}\vartheta = \frac{1}{|G|}\cdot|G|\cdot\vartheta = \vartheta$$

and especially $\operatorname{Sym}\circ\operatorname{Sym} = \operatorname{Sym}$ and

$$\operatorname{Sym}(\operatorname{Hom}_{\mathbb{K}}(V,W)) \subseteq \operatorname{Hom}_{\mathbb{K}}((\rho,V),(\tau,W))$$

5. Let $f(x) = x^3 - \frac{49}{6}x^2 + \frac{39}{2}x - \frac{31}{3}$. Prove that there are at least one a, b such that $f^2(a) = a\,,\ f(a) \neq a$ and $f^4(b) = b\,,\ f^k(b) \neq b\ (k < 4)$ where $f^n := f \circ f^{n-1}, f^1 = f$.

Is this true for every even power?

Reason: Theorem of Sharkovskii.

Solution: We observe that $f(1) = 2, f(2) = 4, f(4) = 1$, which means $f^3(1) = f(f(f(1))) = 1$. This means that $x = 1$ is a periodic point of order 3 of the continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$. A periodic point p of order m is a point such that $f^m(p) = p$ and $f^k(p) \neq p$ for some $m \in \mathbb{N}$ and all $0 < k < m$. The claim now follows from the theorem of Sharkovskii:

Consider the total (Sharkovskii) order "$\preccurlyeq_S$"

$$3, 5, 7, 9, \ldots, 2\cdot 3, 2\cdot 5, 2\cdot 7, \ldots, 2^2\cdot 3, 2^2\cdot 5, \ldots, 2^3\cdot 3, \ldots, \ldots, 2^4, 2^3, 2^2, 2, 1$$

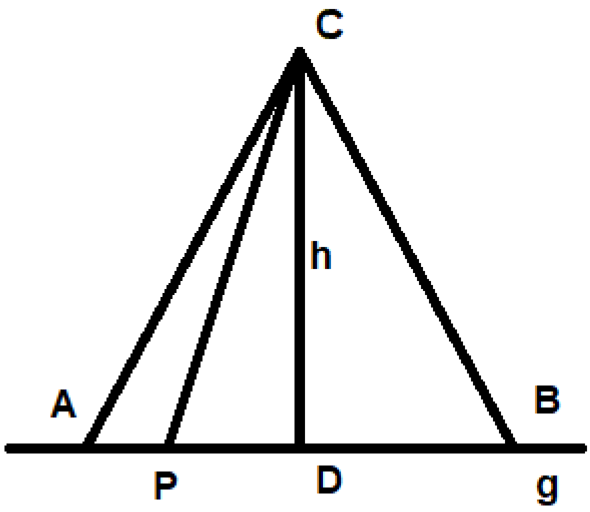
of the natural numbers. If the continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ has a periodic point of order m, and $m \preccurlyeq_S n$, then there is at least one periodic point of order n.

Since we have a periodic point of order 3, we have at least one periodic point of any order.

6. Prove the equivalence of the theorem of Pythagoras with the following transversal theorem about isosceles triangles:

   Given an isosceles triangle $\triangle ABC$ with baseline $\overline{AB} \subseteq g$, peak C, i.e. $|AC| = |BC|$, and g the straight along the baseline. Moreover let $P \in g$ be an arbitrary point. Then

$$|CP|^2 = |CA|^2 + |PA| \cdot |PB| \text{ if } P \notin \overline{AB}$$
$$|CP|^2 = |CA|^2 - |PA| \cdot |PB| \text{ if } P \in \overline{AB}$$



Reason: Geometry.

Solution: The height h divides the baseline $\overline{AB}$ into equal halves and intersects g at point D. We thus get two right triangles $\triangle PDC$ and $\triangle BDC$ on which we can apply the theorem of Pythagoras.

Case I: $P \notin \overline{AB}$

$$\begin{aligned}
|CP|^2 &= |CD|^2 + \left(\frac{|AB|}{2} + |PA|\right)^2 \\
|CB|^2 &= |CD|^2 + \left(\frac{|AB|}{2}\right)^2 \\
|CP|^2 &= |CB|^2 - \left(\frac{|AB|}{2}\right)^2 + \left(\frac{|AB|}{2} + |PA|\right)^2 \\
&= |CB|^2 + |PA|^2 + 2 \cdot |PA| \cdot |AB| \\
&= |CB|^2 + |PA| \cdot |PB| = |CA|^2 + |PA| \cdot |PB|
\end{aligned}$$

Case II: $P \in \overline{AB}$

$$\begin{aligned}
|CP|^2 &= |CD|^2 + \left(\frac{|AB|}{2} - |PA|\right)^2 \\
|CB|^2 &= |CD|^2 + \left(\frac{|AB|}{2}\right)^2 \\
|CP|^2 &= |CB|^2 - \left(\frac{|AB|}{2}\right)^2 + \left(\frac{|AB|}{2} - |PA|\right)^2 \\
&= |CB|^2 + |PA|^2 - 2 \cdot |PA| \cdot |AB| \\
&= |CB|^2 + |PA| \cdot |PB| = |CA|^2 - |PA| \cdot |PB|
\end{aligned}$$

Conversely we consider the special case $P = D$. Then

$$|CD|^2 = |CP|^2 = |CA|^2 - |PA| \cdot |PB| = |CA|^2 - |DA|^2$$

is the transversal theorem for the isosceles triangle $\triangle ABC$, which is the theorem of Pythagoras for the right triangle $\triangle ADC$. However, any given right triangle can be mirrored at one of its legs to make it an isosceles triangle for which the theorem of Pythagoras is a consequence of the transversal theorem.

7. Let α be an algebraic number of degree $n \geq 1$. Then there is a real number $c > 0$ such that for all $\mathbb{Q} \ni \dfrac{p}{q} \neq \alpha$

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c}{q^n}$$

Reason: Liouville's Approximation Theorem.

Solution: Let $f(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{Z}[x]$ be the minimal polynomial of α with $a_n \neq 0$. This means we can factorize $f(x) = (x - \alpha) \cdot g(x)$ in $\mathbb{C}[x]$. The function $\mathbb{R} \longrightarrow \mathbb{C}\,,\; x \longmapsto g(x)$ is continuous, i.e. there are real numbers $c_1, c_2 > 0$ such that $|g(x)| \leq c_1$ whenever $|\alpha - x| < c_2$. Since f has only finitely many zeros, we may assume w.l.o.g. that no other zero lies in the neighborhood of α, i.e. that $f(x) \neq 0$ for all $|\alpha - x| < c_2$ and $x \neq \alpha$. Set $c := \min\{c_2, c_1^{-1}\}$.

Assume that there are $p, q \in \mathbb{Z}, q \geq 1$ such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{c}{q^n} < c \leq c_2$$

hence $|g(p/q)| \leq c_1$ and

$$\left|f\left(\frac{p}{q}\right)\right| = \left|\frac{p}{q}-\alpha\right| \cdot g\left(\frac{p}{q}\right) < \frac{c}{q^n}\cdot c_1 \leq \frac{1}{q^n} \Longrightarrow \left|q^n f\left(\frac{p}{q}\right)\right| < 1$$

But $q^n f\left(\frac{p}{q}\right) = a_np^n + qa_{n-1}p^{n-1} + \ldots + q^{n-1}a_1p + q^na_0 \in \mathbb{Z}$ which implies that $f\left(\frac{p}{q}\right) = 0$ which can only happen if $\frac{p}{q} = \alpha$ in the chosen neighborhood of α.

Another way to prove Liouville's approximation theorem is the following. If $\alpha = a + ib \notin \mathbb{R}$ then

$$\frac{|b|}{q^n} \leq |b| \leq \sqrt{\left(a-\frac{p}{q}\right)^2 + b^2} = \left|\alpha - \frac{p}{q}\right|$$

and $c := |\mathfrak{Im}(\alpha)|$ proves the statement of the theorem. We may thus assume that $\alpha \in \mathbb{R}$. Let $r > 0$ and $M_r := \max\{f'(x) : |x-\alpha| \leq r\}$. Now we choose $c := \min\{r, M_r^{-1}\}$ and assume

$$\left|\alpha - \frac{p}{q}\right| \leq r.$$

There is a $\xi \in \left[\alpha, \frac{p}{q}\right]$ i.e. especially $|\xi - \alpha| \leq r$, such that

$$f\left(\frac{p}{q}\right) = f\left(\frac{p}{q}\right) - f(\alpha) = \left(\frac{p}{q} - \alpha\right) \cdot f'(\xi)$$

by the mean value theorem of differential calculus and $|f'(\xi)| \leq M_r$. Again we have $q^n f\left(\frac{p}{q}\right) \in \mathbb{Z}$. The polynomial f is irreducible over $\mathbb{Z}$ by its minimality and so irreducible over $\mathbb{Q}$ by Gauß's lemma for polynomials. Then either $\alpha = \frac{p}{q}$ or $f\left(\frac{p}{q}\right) \neq 0$. The former is impossible, so the latter must hold. Then

$$\begin{aligned}\left|q^n f\left(\frac{p}{q}\right)\right| \geq 1 &\Longrightarrow \frac{1}{q^n} \leq \left|f\left(\frac{p}{q}\right)\right| = \left|\frac{p}{q}-\alpha\right| \cdot |f'(\xi)| \\ &\Longrightarrow \frac{1}{q^n} \leq \left|\frac{p}{q} - \alpha\right| \cdot M_r \\ &\Longrightarrow \left|\alpha - \frac{p}{q}\right| \geq \frac{1}{M_r q^n} \geq \frac{c}{q^n}\end{aligned}$$

8. Let $a_{n+1} = 2 + \sqrt{4 + a_n}\,,\ a_0 \geq -4\,,$ be a sequence of real numbers. Determine - if existent - its limit in dependence of the initial value a_0, and show that $a_n \in [2, 5]$ in cases where $a_0 \in [-4, 5]$, and $a_n \geq 5$ in cases where $a_0 \geq 5$ ($n \in \mathbb{N}$).

   Reason: Recursion.

   Solution: By the monotony of the root function we get for $a_0 \in [-4, 5]$ that $a_1 \in [2, 5]$ and by induction $a_n \in [2+\sqrt{6}, 5] \subseteq [2, 5]$. In case $a_0 \geq 5$ we have again by monotony and induction $a_n \geq 5$.

$$\begin{aligned} a_{n+1} - a_n &= 2 + \sqrt{4 + a_n} - a_n = \frac{4 + a_n - (a_n - 2)^2}{\sqrt{4 + a_n} + a_n - 2} \\ &= \frac{a_n(5 - a_n)}{\sqrt{4 + a_n} + a_n - 2} = \begin{cases} \geq 0 & \text{if } a_n \in [2, 5] \\ \leq 0 & \text{if } a_n \geq 5 \end{cases} \end{aligned}$$

   The sequence is thus monotone increasing for $a_0 \in [2, 5]$ and monotone decreasing for $a_0 \geq 5$. This implies convergence in both cases (a_1 is an upper bound in the latter case). Now consider the fixed point equation

$$a = 2 + \sqrt{4 + a} \Longrightarrow (a - 2)^2 = 4 + a \Longrightarrow a(a - 5) = 0$$

   Testing both solutions gives us $a = 5$ as unique possible fixed point, i.e. $\lim_{n\to\infty} a_n = 5$ for any initial value $a_0 \geq -4$.

9. Calculate center, foci, semi-axis, and area of the maximal inscribed ellipse of the triangle $(1, 1), (5, 2), (3, 6)$.

   Reason: Geometry.

   Solution: Set

$$\begin{aligned} \mathbb{C}[x] \ni p(z) &= (z - 1 - i)(z - 5 - 2i)(z - 3 - 6i) \\ &= z^3 - (9 + 9i)z^2 + (3 + 52i)z + (33 - 39i) \end{aligned}$$

   The maximal inscribed ellipse of the triangle of zeros of $p(x)$ is thus the **Steiner inellipse**, where the sides of the triangle are tangents at their midpoints. The foci are the zeros of $p'(z)$ and the center the zero of $p''(x)$ by Marden's theorem.

$$\begin{aligned} 0 &= p'(z) = 3z^2 - (18 + 18i)z + (3 + 52i) \\ &= 3\left(z - (3 + 3i) - \sqrt{-1 + \frac{2i}{3}}\right)\left(z - (3 + 3i) + \sqrt{-1 + \frac{2i}{3}}\right) \\ &\approx 3\,(z - 3.32 - 4.05i)\,(z - 2.68 - 1.95i) \\ 0 &= p''(x) = 6\,(z - (3 + 3i)) \end{aligned}$$

Hence the center of the ellipse is $(3,3)$ and the foci $(2.68, 1.95), (3.32, 4.05)$. The area of an ellipse is given as $A = \pi ab$, so we have to compute the semi-axis of a Steiner inellipse within $\triangle ABC$ with center S.

$$A = (1,1)\,,\; B = (5,2)\,,\; C = (3,6)\,,\; S = (3,3)\,,$$
$$M := \frac{1}{4}\left(\overline{SC}^2 + \frac{1}{3}\overline{AB}^2\right)\,,\; N := \frac{1}{4\sqrt{3}} \cdot \left|\det\left(\vec{SC}, \vec{AB}\right)\right|$$
$$a = \frac{1}{2}\left(\sqrt{M+2N} + \sqrt{M-2N}\right)\,,\; b = \frac{1}{2}\left(\sqrt{M+2N} - \sqrt{M-2N}\right)$$

$$\begin{aligned}
M &= \frac{1}{4}\left(9 + \frac{1}{3}\cdot 17\right) = \frac{11}{3} \\
N &= \frac{1}{4\sqrt{3}}\left|\det\left(\begin{bmatrix}0 & 4\\ 3 & 1\end{bmatrix}\right)\right| = \sqrt{3} \\
a &= \frac{1}{2\sqrt{3}}\left(\sqrt{11+6\sqrt{3}} + \sqrt{11-6\sqrt{3}}\right) \\
b &= \frac{1}{2\sqrt{3}}\left(\sqrt{11+6\sqrt{3}} - \sqrt{11-6\sqrt{3}}\right) \\
ab &= \frac{1}{12}\left(\left(11+6\sqrt{3}\right) - \left(11-6\sqrt{3}\right)\right) = \sqrt{3} \\
A &= \pi\sqrt{3} \approx 5.44 \quad = \frac{\pi}{3\sqrt{3}}\cdot A_{\triangle}
\end{aligned}$$

10. Let $A, B \in \mathbb{M}(n, \mathbb{F})$ be two square $n \times n$ matrices over a field $\mathbb{F}$. Show that the minimal polynomials of AB and BA are the same in case A is regular. Is it true as well, if A is singular?

Reason: Matrices and Minimal Polynomials.

Solution: For any polynomial $p(x) \in \mathbb{F}[x]$ we have

$$p(AB)A = Ap(BA) \text{ i.e. } 0 = p(AB) = Ap(BA)A^{-1} \Longleftrightarrow p(BA) = 0$$

This means that the minimal polynomials of AB and BA divide each other, and are therefore equal. Now set

$$A := \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\,,\; B := \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$
$$AB = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\,,\; BA = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

so $m_{AB}(x) = x^2 \neq x = m_{BA}(x)$.

11. (HS-1) For which positive real numbers $\mathbb{R} \ni a, b > 0$ does

$$f(a,b) = \frac{a^4}{b^4} + \frac{b^4}{a^4} - \frac{a^2}{b^2} - \frac{b^2}{a^2} + \frac{a}{b} + \frac{b}{a}$$

assume a minimal value, and which one?

Reason: Real Numbers.

Solution:

$$\begin{aligned} f(a,b) &= \left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right)^2 - 2 - \left(\frac{a^2}{b^2} + \frac{b^2}{a^2}\right) + \left(\frac{a}{b} + \frac{b}{a}\right) \\ &= \left(\frac{a^2}{b^2} + \frac{b^2}{a^2} - \frac{1}{2}\right)^2 - \frac{9}{4} + \left(\frac{a}{b} + \frac{b}{a}\right) \\ &= \left(\left(\frac{a}{b} - \frac{b}{a}\right)^2 + \frac{3}{2}\right)^2 - \frac{9}{4} + \left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + 2 \\ &\geq \frac{9}{4} - \frac{9}{4} + 2 = 2 \end{aligned}$$

where equality holds if and only if $\frac{a}{b} = \frac{b}{a}$ which is equivalent to $a = b$ since both are positive. Hence $f(a,b)$ assumes its minimal value if and only if $a = b$ in which case $f(a,a) = 2$.

12. (HS-2) Find all pairs (x, y) of integers such that

$$y^2 = x \cdot (x+1) \cdot (x+7) \cdot (x+8)$$

Reason: Integers.

Solution: Assume (x, y) is a solution, $u := x + 4$ and $t = u^2 - (25/2)$. Then

$$y^2 = x(x+1)(x+7)(x+8) = (u-4)(u-3)(u+3)(u+4) = (u^2-9)(u^2-16)$$

$$y^2 = \left(t + \frac{7}{2}\right)\left(t - \frac{7}{2}\right) = t^2 - \frac{49}{4}$$

$$49 = 4t^2 - 4y^2 = (2t - 2y)(2t + 2y) \text{ with } 2t \in \mathbb{Z}$$

Therefore $\{2t-2y, 2t+2y\} = \{2x^2+16x+7 \pm 2y\} = \{\pm 1, \pm 7, \pm 49\}$.

$2t+2y$	$2t-2y$	t	y	u	x	$x(x+1)(x+7)(x+8)$
49	1	25/2	12	5	1	144
				−5	−9	144
				0	−4	144
7	7	7/2	0	3	−1	0
				−3	−7	0
				4	0	0
				−4	−8	0
1	49	25/2	−12	5	1	144
				−5	−9	144
				0	−4	144
−1	−49	−25/2	12	5	1	144
				−5	−9	144
				0	−4	144
−7	−7	−7/2	0	3	−1	0
				−3	−7	0
				4	0	0
				−4	−8	0
−49	−1	−25/2	−12	5	1	144
				−5	−9	144
				0	−4	144

Possible pairs (x,y) are thus $(-9,-12), (-9,12), (-8,0), (-7,0)$, $(-4,-12), (-4,12), (-1,0), (0,0), (1,-12), (1,12)$.

13. (HS-3) Show that

$$\underbrace{\left| x - \frac{\sin(x)(14+\cos(x))}{9+6\cos(x)} \right|}_{=:f(x)} \leq 10^{-4} \text{ for } x \in \left[0, \frac{\pi}{4}\right]$$

You may use $\pi = 3.14159 + \delta$, $\sqrt{2} = 1.41421 + \varepsilon$ with $\delta, \varepsilon \in (0, 10^{-5})$.

Reason: Error Margins.

Solution: To shorten notation we set $c := \cos(x)$ and $s := \sin(x)$.

Then

$$\begin{aligned} f'(x) &= 1 - \frac{(14c + c^2 - s^2)(9+6c) - (14s + cs)(-6s)}{(9+6c)^2} \\ &= \frac{1}{(9+6c)^2}\left[(9+6c)^2 - (9+6c)(14c + c^2 - s^2) - 6s(14s+cs)\right] \\ &= \frac{1}{(9+6c)^2}\left[81 - 18c - 57c^2 - 6c^3 - 75s^2\right] \\ &= \frac{1}{(9+6c)^2}\left(24 - 18c - 6c^3 - 18(1-c^2)\right) \\ &= \frac{6 - 18c + 18c^2 - 6c^3}{(9+6c)^2} = \frac{2(1-c)^3}{3(3+2c)^2} \end{aligned}$$

So $f'(x) > 0$ for $x \in (0, \pi/4]$ and $f'(0) = 0$. This means that $f(x)$ is strictly monotone increasing on the interval $[0, \pi/4]$. Hence

$$0 \leq f(x) \leq f(\pi/4) = \frac{\pi}{4} - \frac{1}{\sqrt{2}} \cdot \frac{14 + \frac{1}{\sqrt{2}}}{9 + 6\frac{1}{\sqrt{2}}} = \frac{\pi}{4} - \frac{41\sqrt{2} - 25}{42}$$

Now $\pi/4 = 0.7853975 + \frac{\delta}{4} < 0.7854$ and $\frac{41\sqrt{2}-25}{42} = \frac{41 \cdot 1.41421 - 25 + 41\varepsilon}{42} > \frac{32.98261}{42} > 0.7853$, i.e. $0 \leq f(x) < 0.7854 - 0.7853 = 10^{-4}$.

14. (HS-4) If $f(x) = a_n x^n + \ldots + a_1 x + a_0 \in \mathbb{R}[x]$ is a real polynomial of degree n which doesn't have real zeros, and $h \in \mathbb{R}$ a real number, then

$$F(x) := f(x) + h \cdot f'(x) + h^2 \cdot f''(x) + \ldots + h^n \cdot f^{(n)}(x)$$

doesn't have real zeros either.

Reason: Polynomial Zeros.

Solution: $F(x)$ is a real polynomial of degree n, too, and n has to be even, so we may assume w.l.o.g. that $f(x) > 0$ for all $x \in \mathbb{R}$, since we could otherwise work with $-f(x)$ instead. Since $F(x)$ and $f(x)$ are of the same degree with the same leading coefficient a_n, $F(x)$ has a global minimum, because $f(x)$ has, and their limits at $x \to \pm$ are the same. This means there is a point x_0 such that $F(x) \geq F(x_0)$ for all $x \in \mathbb{R}$

and it is sufficient to show that $F(x_0) > 0$.

$$F(x) = \sum_{k=0}^{n} h^k \frac{d^k}{dx^k} f(x) = \sum_{k=0}^{\infty} h^k \frac{d^k}{dx^k} f(x) = \left(1 - h\frac{d}{dx}\right)^{-1} f(x)$$
$$f(x) = \left(1 - h\frac{d}{dx}\right) F(x) = F(x) - hF'(x)$$
$$F(x) = f(x) + hF'(x)$$
$$F(x_0) = f(x_0) + hF'(x_0) = f(x_0) > 0$$

15. (HS-5) Solve the following real equations system:

$$x + y = az$$
$$x - y = bz$$
$$x^2 + y^2 = cz$$

Reason: Calculus.

Solution: The first two equations can be rewritten as

$$x = \frac{a+b}{2}\, z\,,\ y = \frac{a-b}{a}\, z$$

so

$$0 = \left(\frac{a+b}{2}\right)^2 z^2 + \left(\frac{a-b}{2}\right)^2 z^2 - cz = z \cdot \left(\frac{a^2+b^2}{2}\, z - c\right)$$

If $z = 0$ then $x = y = 0$ which is a solution for any choice of $a, b, c \in \mathbb{R}$.
If $a = b = 0$ then $x = y = 0$ and $cz = 0$.

Now let $a^2 + b^2 \neq 0$ and $z = \frac{2c}{a^2+b^2}$, $x = \frac{c(a+b)}{a^2+b^2}$, $y = \frac{c(a-b)}{a^2+b^2}$. These are necessary and sufficient conditions to solve the equations. We have therefore the following solutions:

a	b	c	x	y	z
0	0	0	0	0	z
a	b	c	0	0	0
a	b	c	$\frac{a+b}{a^2+b^2}\, c$	$\frac{a-b}{a^2+b^2}\, c$	$\frac{2}{a^2+b^2}\, c$

9 April 2021

1. Let T be a planet's orbital period, a the length of the semi-major axis of its orbit. Then
$$T'(a) = \gamma \sqrt[3]{T(a)}\,,\,T(0)=0$$
with a constant proportional factor $\gamma > 0$. Solve this equation for all $a \in \mathbb{R}$ and determine whether the solution is unique and why.

Reason: Uniqueness in the theorem of Picard-Lindelöf.

Solution: Division by $\gamma\sqrt[3]{T(a)}$ yields
$$\begin{aligned}\gamma^{-1}\int \frac{\frac{dT(a)}{da}}{\sqrt[3]{T(a)}}\,da &= \int 1\,da\\ \gamma^{-1}\int \frac{1}{\sqrt[3]{T(a)}}\,dT(a) &= a-C\\ \gamma^{-1}\frac{3}{2}\sqrt[3]{T(a)^2} &= a-C\\ T(a) &= \left(\frac{2\gamma}{3}(a-C)\right)^{3/2} = \gamma'(a-C)^{3/2}\end{aligned}$$
The global solutions are thus
$$T(a) = \begin{cases} 0 & ,\, a\leq C \\ \gamma'(a-C)^{3/2} & ,\, a>C\end{cases}$$
which are infinitely many, for any $C \in \mathbb{R}_0^+$ and $\gamma' = (2\gamma/3)^{3/2}$.

All conditions for the theorem of Picard-Lindelöf hold, except the Lipschitz continuity of $f(a,T) = \gamma\sqrt[3]{T(a)}$ at $T(0)=0$. The function $x \longmapsto \sqrt[3]{x}$ isn't Lipschitz continuous in any neighborhood of 0. This example shows that Lipschitz continuity is crucial for the uniqueness part in the (local and global version) of the theorem of Picard-Lindelöf.

2. Show that the Hadamard (elementwise) product of two positive definite complex matrices is again positive definite.

Reason: Schur product theorem.

Solution: A complex matrix A can be written as the sum of a Hermitian and a skew-Hermitian matrix:
$$A = \frac{1}{2}\left(A+A^\dagger\right) + \frac{1}{2}\left(A-A^\dagger\right)$$

Therefore, if A is positive definite, we have for $x \neq 0$

$$\begin{aligned} 0 < 2\langle Ax, x\rangle &= \langle \left(A + A^\dagger\right) x, x\rangle + \langle \left(A - A^\dagger\right) x, x\rangle \\ &= \langle x, \left(A + A^\dagger\right)^\dagger x\rangle + \langle x, \left(A - A^\dagger\right)^\dagger x\rangle \\ &= \langle x, \left(A + A^\dagger\right) x\rangle - \langle x, \left(A - A^\dagger\right) x\rangle \\ &= 2\langle x, A^\dagger x\rangle \end{aligned}$$

and A is Hermitian. As such it can be unitary diagonalized, i.e. $A = UDU^\dagger$ for a real diagonal matrix D and a unitary matrix U. As all eigenvalues of the positive definite matrix A are all positive, we can draw the square root of $D = R^2$. If we define

$$\sqrt{A} := URU^\dagger$$

then $\sqrt{A} \cdot \sqrt{A} = (URU^\dagger)(URU^\dagger) = URU^{-1}URU^\dagger = UDU^\dagger = A$ and the square root is Hermitian again: $\sqrt{A}^\dagger = \sqrt{A}$.

The Hadamard product of any two matrices $A \odot B$ is defined by elementwise multiplication, i.e.

$$\begin{aligned} \operatorname{tr}\left(A^\tau \operatorname{diag}(\bar{v}) B \operatorname{diag}(w)\right) &= \sum_{k=1}^{n} \left[\left(a_{ji}\bar{v}_j\right)\left(b_{ij}w_i\right)\right]_{kk} \\ &= \sum_{k=1}^{n} \left[\left(\sum_{l=1}^{n} a_{li}\bar{v}_l b_{lj} w_j\right)_{ij}\right]_{kk} \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} \left(a_{lk}\bar{v}_l b_{lk} w_k\right) \\ &= v^\dagger \left(A \odot B\right) w \end{aligned}$$

Now let's assume that A, B are positive definite. Then

$$\begin{aligned} \langle v, (A \odot B)v\rangle &= v^\dagger (A \odot B) v = \operatorname{tr}\left(A^\tau \operatorname{diag}(\bar{v}) B \operatorname{diag}(v)\right) \\ &= \operatorname{tr}\left(\sqrt{A}\sqrt{A}\operatorname{diag}(\bar{v})\sqrt{B}\sqrt{B}\operatorname{diag}(v)\right) \\ &= \operatorname{tr}\left(\left(\sqrt{A}\operatorname{diag}(\bar{v})\sqrt{B}\right)\left(\sqrt{B}\operatorname{diag}(v)\sqrt{A}\right)\right) \\ &= \operatorname{tr}\left(\underbrace{\left(\sqrt{B}\operatorname{diag}(v)\sqrt{A}\right)^\dagger}_{=:C^\dagger}\underbrace{\left(\sqrt{B}\operatorname{diag}(v)\sqrt{A}\right)}_{=:C}\right) \\ &= \operatorname{tr}\left(C^\dagger C\right) = \sum_{i,j} \bar{c}_{ij} c_{ij} > 0 \end{aligned}$$

for $v \neq 0$ and equal to zero if and only if $v = 0$.

3. A function $f : S^k \longrightarrow X$ is called antipodal if it is continuous and $f(-x) = -f(x)$ for all $x \in S^k$ and any topological space $X \subseteq \mathbb{R}^m$.

   Show that the following statements are equivalent:

   (a) For every antipodal map $f : S^n \longrightarrow \mathbb{R}^n$ there is a point $x \in S^n$ satisfying $f(x) = 0$.

   (b) There is no antipodal map $f : S^n \longrightarrow S^{n-1}$.

   (c) There is no continuous mapping $f : B^n \longrightarrow S^{n-1}$ that is antipodal on the boundary.

   Assume the conditions hold. Prove Brouwer's fixed point theorem:

   Any continuous map $f : B^n \longrightarrow B^n$ has a fixed point.

   Reason: Borsuk-Ulam Theorem.

   Solution: $(a) \Longrightarrow (b)$ Assume there is a antipodal map $f : S^n \longrightarrow S^{n-1}$. If we compose it with the inclusion $\iota : S^{n-1} \hookrightarrow \mathbb{R}^n$ then it remains antipodal and according to (a) there is a point $\iota(f(x)) = 0$ which can only occur if $f(x) = 0$. However, $f(S^n) \subseteq S^{n-1} \not\ni 0$, a contradiction.

   $(b) \Longrightarrow (a)$ Assume there is a antipodal map $f : S^n \longrightarrow \mathbb{R}^n$ such that $f(x) \neq 0$ for all $x \in S^n$. Then we define $g : S^n \longrightarrow S^{n-1}$ by $g(x) := \dfrac{f(x)}{\|x\|}$ which is antipodal, too, contradicting (b).

   $(c) \Longrightarrow (b)$ The map $\pi(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$ is a homeomorphism from the upper hemisphere of S^n to B^n. If we had an antipodal map $f : S^n \longrightarrow S^{n-1}$ then we would have an antipodal map $g : B^n \longrightarrow S^{n-1}$ by $g(x) := f(\pi^{-1}(x))$ which is antipodal on the boundary, contradicting (c).

   $(b) \Longrightarrow (c)$ Assume $g : B^n \longrightarrow S^{n-1}$ is continuous and antipodal on the boundary. Then we define $f : S^n \longrightarrow S^{n-1}$ by $f(x) := g(\pi(x))$ for x in the upper hemisphere and $f(-x) := -g(\pi(x))$. Thus f is antipodal, contradicting (b).

   That all these conditions actually hold, is the theorem of Borsuk-Ulam, which can be proven by topological algebra. E.g. see Theorem 2.2. in https://web.northeastern.edu/suciu/slides/Borsuk-Ulam-tapas05.pdf

   Suppose there exists a continuous function $f : B^n \longrightarrow B^n$ without fixed point. We define $g : B^n \longrightarrow S^{n-1}$ such that $g(x)$ is the point on S^{n-1} which intersects with the ray from $f(x)$ and x. This is a well-defined retraction, as there are no fixed points at which the function would be ill-defined. It is a retraction since a ray from anywhere in B^n to a point on the boundary $x \in \partial B^n = S^{n-1}$ intersects the boundary at x by construction. Now $g(-x) = -g(x)$ for $x \in S^{n-1}$ which is not possible according to (c).

4. Let Y be an affine, complex variety. Prove that Y is irreducible if and only if $I(Y)$ is a prime ideal.

   Reason: Hilbert's Nullstellensatz.

   Solution: Assume Y is irreducible and $f, g \in \mathcal{O}_{\mathbb{C}}(\mathbb{A}^n)$ such that $f \cdot g \in I(Y)$. Then $V(fg) = V(f) \cup V(g)$ and

$$Y = (V(f) \cap Y) \cup (V(g) \cap Y).$$

   Since Y is irreducible, we have w.l.o.g. $Y = V(f) \cap Y$, i.e. $Y \subseteq V(f)$ and $f \in I(Y)$.

   Now let $J := I(Y)$ be a prime ideal. Let $V(J) = Y_1 \cup Y_2$. Thus $J = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$. Assume $J \neq I(Y_1), I(Y_2)$. Then there are $f_i \in I(Y_i) - J$. Since

$$f_1 f_2 \in I(Y_1) \cdot \mathcal{O}_{\mathbb{C}}(\mathbb{A}^n) \cap \mathcal{O}_{\mathbb{C}}(\mathbb{A}^n) \cdot I(Y_2) = I(Y_1) \cap I(Y_2) = J$$

   is a prime ideal, we have that either $f_1 \in J$ or $f_2 \in J$ contradicting the choice of f_1, f_2. Hence w.o.l.g. we may assume $I(Y) = J = \sqrt{J} = I(Y_1)$. This implies $Y = V(I(Y)) = V(I(Y_1)) = Y_1$ and $V(J)$ is irreducible.

5. Let $p > 5$ be a prime number. Show that

$$\left(\frac{6}{p}\right) = 1 \Longleftrightarrow p \equiv k\,(24) \text{ with } k \in \{1, 5, 19, 23\}.$$

   The parentheses are the Legendre symbol.

   Reason: Quadratic Reciprocity Law.

   Solution:

$$\left(\frac{6}{p}\right) = \left(\frac{2}{p}\right) \cdot \left(\frac{3}{p}\right)$$

   which equals 1 if and only if both factors are $= 1$ or both are $= -1$.

By the quadratic reciprocity law we have

$$\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right) = \begin{cases} 1 & \text{if } p \equiv 1\,(4)\,,\, p \equiv 1\,(3)\,, \text{ i.e. } p \equiv 1\,(12) \\ -1 & \text{if } p \equiv 1\,(4)\,,\, p \equiv -1\,(3)\,, \text{ i.e. } p \equiv 5\,(12) \\ -1 & \text{if } p \equiv -1\,(4)\,,\, p \equiv 1\,(3)\,, \text{ i.e. } p \equiv -5\,(12) \\ 1 & \text{if } p \equiv -1\,(4)\,,\, p \equiv -1\,(3)\,, \text{ i.e. } p \equiv -1\,(12) \end{cases}$$

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}} = \begin{cases} 1 & \text{if } p \equiv \pm 1\,(8) \\ -1 & \text{if } p \equiv \pm 5\,(8) \end{cases}$$

If $p \equiv a\,(n)\,,\, p \equiv b\,(m)$ and $d = (n,m)$, then d can be written as $d = \mu m + \nu n$ with $(d,\mu) = (d,\nu) = 1$ by Bézout's Lemma. This means

$$\begin{aligned} \mu(a-b) &= \mu c_1 m - \mu c_2 n = c_1 \mu m - c_2 \mu n = c_1 d - c_1 \nu n - \mu c_2 n \\ &\Longrightarrow d \,|\, \mu(a-b) \Longrightarrow d \,|\, (a-b) \end{aligned}$$

For $n = 8, m = 12$ we get $d = 4$.

$$\begin{aligned} p \equiv 1\,(8) \wedge p \equiv -1\,(12) &\Longrightarrow 4 \,|\, (1-(-1)) = 2 \;\lightning \\ p \equiv -1\,(8) \wedge p \equiv 1\,(12) &\Longrightarrow 4 \,|\, (-1-1) = -2 \;\lightning \\ p \equiv 5\,(8) \wedge p \equiv -5\,(12) &\Longrightarrow 4 \,|\, (5-(-5)) = 10 \;\lightning \\ p \equiv -5\,(8) \wedge p \equiv 5\,(12) &\Longrightarrow 4 \,|\, (-5-5) = -10 \;\lightning \end{aligned}$$

Thus we have $\left(\frac{2}{p}\right) = \left(\frac{3}{p}\right) = \pm 1$ only if p has the same sign modulo 8 and 12

$$\begin{aligned} p = 8m \pm 1 = 12n + pm1 \Rightarrow 2m = 3n \Rightarrow 2|n, 3|m \Rightarrow p \equiv \pm 1\,(24) \\ p = 8m \pm 5 = 12n + pm5 \Rightarrow 2m = 3n \Rightarrow 2|n, 3|m \Rightarrow p \equiv \pm 5\,(24) \end{aligned}$$

Thus $p \mod 24 \in \{\pm 1, \pm 5\} = \{1, 5, 19, 23\}$.

6. Let $f \in L^2(\mathbb{R})$ and $g : \mathbb{R} \longrightarrow \overline{\mathbb{R}}$ be given as

$$g(t) := t \int_{\mathbb{R}} \chi_{[t,\infty)}(|x|) \exp(-t^2(|x|+1)) f(x)\, dx$$

Show that $g \in L^1(\mathbb{R})$.

Reason: Functional Analysis.

Solution: We observe $\chi_{[t,\infty)}(|x|) = \chi_{(-\infty,|x|]}(t)$ and define $u(x) = \int_{\mathbb{R}} \chi_{[t,\infty)}(|x|)|t|\exp(-t^2(|x|+1))\,dt$. Then

$$\begin{aligned}
u(x) &= \int_{-\infty}^{|x|} |t|\exp(-t^2(|x|+1))\,dt \\
&= -\int_{-\infty}^{0} t\exp(-t^2(|x|+1))\,dt + \int_{0}^{|x|} t\exp(-t^2(|x|+1))\,dt \\
&= \left[\frac{\exp(-t^2(|x|+1))}{-2(|x|+1)}\right]_0^{|x|} - \left[\frac{\exp(-t^2(|x|+1))}{-2(|x|+1)}\right]_{-\infty}^{0} \\
&= \frac{\exp(-|x|^2(|x|+1)) - 2}{-2(|x|+1)}
\end{aligned}$$

and thus

$$\|u\|_2^2 = \int_{\mathbb{R}} \frac{\left|\exp(-|x|^2(|x|+1)) - 2\right|^2}{4(|x|+1)^2}\,dx \leq \int_{\mathbb{R}} \frac{4}{4(|x|+1)^2}\,dx = 2$$

Now the integral of $|g|$ yields

$$\|g\|_1 = \int_{\mathbb{R}} |g(t)|\,dt \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |t|\chi_{[t,\infty)}(|x|)\exp(-t^2(|x|+1))|f(x)|\,dx\,dt$$

The conditions of the theorem of Tonelli hold, because the function under the integral is positive and as a product of continuous and measurable functions, itself measurable. Thus

$$\begin{aligned}
\|g\|_1 &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |t|\chi_{[t,\infty)}(|x|)\exp(-t^2(|x|+1))|f(x)|\,dt\,dx \\
&= \int_{\mathbb{R}} |u(x)f(x)|\,dx = \|uf\|_1 \leq \|u\|_2\|f\|_2 < \infty
\end{aligned}$$

by Hölder's inequality and the fact that $u(x), f(x) \in L^2(\mathbb{R})$.

7. (a) Let V be the pyramid with vertices $(0,0,1), (0,1,0), (1,0,0)$ and $(0,0,0)$. Calculate

$$\int_V \exp(x+y+z)\,dV$$

(b) Let $A \in \operatorname{GL}(d,\mathbb{R})$. Calculate

$$\int_{\mathbb{R}^d} \exp(-\|Ax\|_2^2)\,dx$$

Reason: Integration.

Solution:

(a) $V = \{(x,y,z) \in \mathbb{R}^3 | 0 \leq z \leq 1, 0 \leq x \leq 1-z, 0 \leq y \leq 1-z-x\}$ is compact, $g : V \longrightarrow \mathbb{R}$ with $g(x,y,z) = z\exp(x+y+z)$ is continuous, hence integrable, so by Fubini's theorem

$$\begin{aligned}\int_V \exp(x+y+z)\,dV &= \int_V \exp(x+y+z)\,d(x,y,z) \\ &= \int_0^1 \int_0^{1-z} \int_0^{1-z-x} \exp(x+y+z)\,dy\,dx\,dz \\ &= \int_0^1 \int_0^{1-z} (e - \exp(x+z))\,dx\,dz \\ &= \int_0^1 (e(1-z) - e + \exp(z))\,dz \\ &= e - \frac{1}{2}e - e + (e-1) = \frac{e}{2} - 1\end{aligned}$$

(b) $\varphi(x) = Ax$ is a C^1-diffeomorphism with $D\varphi = A$ and the transformation theorem can be applied:

$$\begin{aligned}\int_{\mathbb{R}^d} \exp(-\|Ax\|_2^2)\,dx &= \frac{1}{|\det(D\varphi)|} \int_{\mathbb{R}^d} \exp(-\|x\|_2^2)\,dx \\ &= \frac{1}{|\det(A)|} \int_{\mathbb{R}^d} \prod_{k=1}^{d} \exp(-x_k^2)\,dx_k \\ &\overset{\text{Tonelli}}{=} \frac{1}{|\det(A)|} \prod_{k=1}^{d} \int_{\mathbb{R}} \exp(-x_k^2)\,dx_k \\ &= \frac{\sqrt{\pi}^{\,d}}{|\det(A)|}\end{aligned}$$

8. Consider the Hilbert space $L^2([0,1])$ and its subspace $K := \operatorname{span}_{\mathbb{C}}\{x, 1\}$. Let $\pi^{\perp} : H \longrightarrow K$ be the orthogonal projection. Give an explicit formula for $\pi^{\perp}$ and calculate $\pi^{\perp}(e^x)$.

Reason: Hilbert Spaces.

Solution: If $\{u, v\}$ is a orthonormal basis of K, then the orthogonal projection is given by $\pi^{\perp}(f) = \langle f, u\rangle u + \langle f, v\rangle v$ and it is sufficient to determine a orthogonal basis. We set $u = 1$ and apply the Gram-Schmidt algorithm to get

$$\bar{v} = x - \langle x, 1\rangle 1 = x - \int_0^1 t\,dt = x - \frac{1}{2}$$

$$\|\bar{v}\|^2 = \int_0^1 \left(t-\frac{1}{2}\right)^2 dt = \frac{1}{3}-\frac{1}{2}+\frac{1}{4}=\frac{1}{12}$$

such that $\{u,v\} = \left\{u, \frac{\bar{v}}{\|\bar{v}\|}\right\} = \left\{1, \sqrt{12}\left(x-\frac{1}{2}\right)\right\}$ and

$$\pi^{\perp}(f) = \langle f,1\rangle 1 + 12\,\langle f, x-\frac{1}{2}\rangle \left(x-\frac{1}{2}\right)$$

$$\begin{aligned}\pi^{\perp}(e^x) &= \langle e^x,1\rangle 1 + 12\,\langle e^x, x-\frac{1}{2}\rangle \left(x-\frac{1}{2}\right)\\ &= \int_0^1 e^x\,dx + 3(2x-1)\int_0^1 e^x\,(2x-1)\,dx\\ &= (e-1)+3(2x-1)(3-e) = 6x(3-e)+4e-10\end{aligned}$$

9. Prove $\pi_1(S^n;x)=\{e\}$ for $n\geq 2$.

Reason: Theorem of Seifert-van Kampen.

Solution: Assume there is a point $a\in S^n$ such that $\pi_1(S^n;a)\ni g\neq e$. Set

$$U := S^n \cap \left\{x_n > -\frac{1}{2}\right\}\,,\; V := S^n \cap \left\{x_n < \frac{1}{2}\right\}$$

where we may assume without loss of generality that $a\in U\cap V$ by an appropriate choice of coordinates. Then U,V are homeomorphic to an open ball, i.e. $\pi_1(U;x)=\pi_1(V;x)=\{0\}$ because balls are simply connected, $S^n = U\cup V$, and $U\cap V \cong S^{n-1}\times(0,1)$, which is path connected for $n\geq 2$. Let

$$\begin{aligned}\iota_U &: \pi_1(U\cap V;a)\longrightarrow \pi_1(U;a)\\ \iota_V &: \pi_1(U\cap V;a)\longrightarrow \pi_1(V;a)\\ \kappa_U &: \pi_1(U;a)\longrightarrow \pi_1(S^n;a)\\ \kappa_V &: \pi_1(V;a)\longrightarrow \pi_1(S^n;a)\end{aligned}$$

be the embeddings of the according fundamental groups. We can now apply the theorem of Seifert-van Kampen which states, that for every pair $\varphi_U : \pi_1(U;a)\longrightarrow G$, $\varphi_V : \pi_1(V;a)\longrightarrow G$ of group homomorphisms, such that $\varphi_U\circ\iota_U = \varphi_V\circ\iota_V$, there is a unique group homomorphism $\varphi:\pi_1(S^n;a)\longrightarrow G$ with $\varphi_U=\varphi\circ\kappa_U$ and $\varphi_V=\varphi\circ\kappa_V$. We get, however, with $g \in G = \pi_1(S^n; a)$ two different homomorphisms $\varphi_1 = \mathrm{id}_G, \varphi_2 \equiv e$ in case $g \neq e$. These satisfy the conditions $k = 1, 2$

$$\varphi_k(\kappa_U([\gamma])) = \varphi_k(e) = e = \varphi_U([\gamma])$$
$$\varphi_k(\kappa_V([\gamma])) = \varphi_k(e) = e = \varphi_V([\gamma])$$

since both, U, V are simply connected, in contradiction to the theorem of Seifert-van Kampen. Hence $g = e$ and $\pi_1(S^n; a) = \{e\}$.

The same proof works in the more general case:

If $X = U \cup V$ with open sets U, V and $U \cap V$ is path connected, then $\pi_1(U; x) = \pi_1(V; x) = \{e\}$ implies $\pi_1(X; x) = \{e\}$.

10. Let $U \subseteq \mathbb{R}^{2n}$ be an open set and $f \in C^2(U, \mathbb{R})$ a twice continuously differentiable function at a point $\vec{a} \in U$. Prove that if f has a critical point in $\vec{a}$ and the Hessian matrix $Hf(\vec{a})$ has a negative determinant, then f has neither a local maximum nor a local minimum in $\vec{a}$.

    Reason: Taylor Series with Integral Remainder.

    Solution: The Hessian matrix is symmetric, and thus diagonalizable with real eigenvalues. Hence its determinant is negative, if there is at least one positive and one negative eigenvalue, say $Hf.\vec{v}_+ = \lambda_+\vec{v}_+$, $Hf.\vec{v}_- = \lambda_+\vec{v}_-$. Since a is a critical point, we have $Df(\vec{a}) = 0$, so the Taylor series with integral remainder is

$$f(\vec{a} + \vec{h}) = f(\vec{a}) + \int_0^1 (1 - t)\langle \vec{h}, Hf(\vec{a} + t\vec{h})\vec{h}\rangle\, dt$$

    $\langle \vec{h}, Hf(\vec{a}\vec{h})\rangle = \lambda_+\|\vec{h}\|^2 > 0$ for $\vec{h} \sim \vec{v}_+$ and this product remains positive in a neighborhood of $Hf(\vec{a})$ by continuity. So for sufficiently small $\vec{h}$ we have

$$\langle \vec{h}, Hf(a \vec{+} t\vec{h})\vec{h}\rangle > 0 \text{ for all } t \in [0, 1]$$

    which results in $f(\vec{a} + \vec{h}) > f(\vec{a})$. The same argument leads to the inequality $f(\vec{a} + \vec{h}) < f(\vec{a})$ for sufficiently small $\vec{h} \sim \vec{v}_-$.

11. (HS-1) Show that every non-negative real polynomial $p(x)$ can be written as $p(x) = a(x)^2 + b(x)^2$ with $a(x), b(x) \in \mathbb{R}[x]$.

    Reason: Fundamental Theorem of Algebra.

    Solution: Every real polynomial can be written as a product with its complex zeros $z_1, \ldots, z_n$

$$p(x)=\prod_{i=1}^{n}(x-z_i)^{r_i}=\underbrace{\prod_{i\leq m}\underbrace{(x-z_i)^{r_i}}_{z_i\in\mathbb{R}}}_{=:a_0\in\mathbb{R}[x]}\cdot\underbrace{\prod_{i>m}\underbrace{(x-z_i)^{r_i}}_{z_i\in\mathbb{C}}}_{=:b_0\in\mathbb{R}[x]}$$

All zeros of b_0 occur as pairs of conjugate complex numbers such that $z_i = u_i + iv_i$ can be are paired as

$$(x-z_i)^{r_i}(x-\bar{z}_i)^{r_i}=(\underbrace{(x-u_i)^2}_{=:b_i(x)^2\in\mathbb{R}[x]}+v_i^2)^{r_i}>0\;\forall\,x\in\mathbb{R}$$

Assume that some power r_j in the first factor a_0 is odd. Then $p(z_j-\varepsilon)$ and $p(z_j+\varepsilon)$ would be real numbers of different sign, because $b_0(z_j\pm\varepsilon)>0$ and the real roots z_j are discrete. As we excluded this possibility, $a_0(x) =: a_1(x)^2$ is already a square polynomial. Hence with possible repetitions

$$\begin{aligned}p(x)&=a_1(x)^2\cdot\prod_{i=1}^{k}(b_i(x)^2+v_i^2)=a_1(x)^2\cdot\prod_{i=1}^{k}\det\begin{pmatrix}b_i(x)&-v_i(x)\\v_i(x)&b_i(x)\end{pmatrix}\\&=a_1(x)^2\cdot\det\begin{pmatrix}B(x)&-V(x)\\V(x)&B(x)\end{pmatrix}=a_1(x)^2\cdot(B(x)^2+V(x)^2)\\&=(\underbrace{a_1(x)B(x)}_{=:a(x)})^2+(\underbrace{a_1(x)V(x)}_{=:b(x)})^2\end{aligned}$$

since matrix multiplication preserves this form:

$$\begin{pmatrix}b_i(x)&-v_i(x)\\v_i(x)&b_i(x)\end{pmatrix}\begin{pmatrix}b_j(x)&-v_j(x)\\v_j(x)&b_j(x)\end{pmatrix}$$
$$=\begin{pmatrix}b_i(x)b_j(x)-v_i(x)v_j(x)&-(b_i(x)v_j(x)+b_j(x)v_i(x))\\b_j(x)v_i(x)+b_i(x)v_j(x)&b_i(x)b_j(x)-v_i(x)v_j(x)\end{pmatrix}$$

12. (HS-2) Show that all Pythagorean triples $x^2+y^2=z^2$ can be found by

$$(x,y,z)=d\cdot(u^2-v^2,2uv,u^2+v^2)\text{ with }d,u,v\in\mathbb{N}\,,\;u>v$$

and which are primitive (no common divisor of x,y,z) if and only if u,v are coprime and one is odd and the other one even.

Reason: Pythagorean Triples.

Solution: All such triples form a Pythagorean triple, since

$(u^2-v^2)^2+(2uv)^2 = u^4-2u^2v^2+v^4+4u^2v^2 = u^4+2u^2v^2 = (u^2+v^2)^2$

If we have a Pythagorean triple $x^2+y^2=z^2$ then

$$\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2=1\ ;\ a:=\frac{x}{z}\ ,\ b:=\frac{y}{z}$$

and the straight from $(-1,0)$ to (a,b) on the unit circle intersects the $y-$axis at $(0,t)$ where $t=\dfrac{b}{a+1}=\dfrac{v}{u}$ with coprime natural numbers u,v is the rational slope of that straight.

$$\begin{aligned}
1&=a^2+b^2=a^2+t^2(a+1)^2\\
0&=(a+1)(a-1)+t^2(a+1)^2\\
0&=(a-1)+t^2(a+1)=a(t^2+1)+t^2-1\\
a&=\frac{1-t^2}{1+t^2}\ ,\ b=t\cdot(a+1)=\frac{2t}{1+t^2}
\end{aligned}$$

Hence

$$\left(\frac{x}{z},\frac{y}{z}\right)=\left(\frac{1-\dfrac{v^2}{u^2}}{1+\dfrac{v^2}{u^2}}\,,\,\frac{2\dfrac{v}{u}}{1+\dfrac{v^2}{u^2}}\right)=\left(\frac{u^2-v^2}{u^2+v^2}\,,\,\frac{2uv}{u^2+v^2}\right)$$

We can now write $(x,y,z)=d\cdot(u^2-v^2,2uv,u^2+v^2)$ and have the desired form. In case $u<v$ we can simply exchange u,v.

If u,v are both even, or both odd, then 2 divides all three numbers x,y,z and the triple isn't primitive. If u,v are not coprime, say $d|u$ and $d|v$, then $d^2|x,y,z$ and the triple isn't primitive either.

Let us now assume that the conditions hold, i.e. $u>v$ are coprime and one is even and one is odd. Then we have to show that (x,y,z) is primitive. Assume therefore that

$$d|x=u^2-v^2\ ,\ d|y=2uv\ ,\ d|z=u^2+v^2.$$

If d divides two of them, then it automatically divides the third one, too. Let $p|d$ an odd prime divisor. Then

$$\begin{aligned}
p\,|\,(u^2-v^2)\wedge p\,|\,(u^2+v^2)&\Longrightarrow p\,|\,2u^2\wedge p\,|\,2v^2\\
&\Longrightarrow p\,|\,u\wedge p\,|\,v
\end{aligned}$$

which cannot be as we assumed that u, v are coprime. So 2 is the only possible divisor of d and $u^2 - v^2 = (u-v)(u+v)$ is even, which can only be, if u, v are either both odd or both even, which we assumed is not true. Finally only $d = 1$ is possible if those conditions on u, v hold, i.e. (x, y, z) is primitive.

13. (HS-3) Write

$$\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \ldots}}}}$$

as $\dfrac{a + b\sqrt{c}}{d}$.

Reason: Continued Fraction.

Solution: Let's first consider the continued fraction. We define $L_0 = 2207\,,\ L_{n+1} = 2207 - 1/L_n$. It is a strictly decreasing sequence within the bounds $2206 < L_n < 2207$ for $n > 0$.

$2206 < L_1 = 2207 - \dfrac{1}{2207} < 2207 = L_0$ and by induction

$$0 < 1 < L_n \Longrightarrow 0 < \frac{1}{L_n} < 1 \Longrightarrow 2207 > 2207 - \frac{1}{L_n} = L_{n+1} > 2206$$

Set $L_0 = 2207 > 987\sqrt{5} =: C$ and assume n is minimal such that $L_0 > L_1 > \ldots > L_{n-1} > L_n > (1/2)(L_0 + C) \geq L_{n+1} = L_0 - 1/L_n$.

$$\begin{aligned} L_n > \frac{1}{2}(L_0 + C) &\Longrightarrow \frac{1}{L_n} < \frac{2}{L_0 + C} \\ &\Longrightarrow -\frac{1}{L_n} > -\frac{2}{L_0 + C} \\ &\Longrightarrow L_0 - \frac{1}{L_n} > L_0 - \frac{2}{L_0 + C} = \frac{L_0^2 + L_0 C - 2}{L_0 + C} \end{aligned}$$

Therfore $\dfrac{1}{2}(L_0 + C)^2 = \dfrac{1}{2}L_0^2 + \dfrac{1}{2}C^2 > L_0^2 - 2$ or $C^2 > L_0^2 - 4$ which cannot be true, since $C^2 = L_0^2 - 4$. Hence $(1/2)(L_0 + C)$ is a strict lower bound for all L_n. But this implies $L_n > L_{n+1}$:

$$L_n > L_{n+1} \Leftrightarrow L_n^2 - L_0 L_n + 1 > 0 \Leftrightarrow L_n > \frac{1}{2}\left(L_0 + \sqrt{L_0^2 - 4}\right) = \frac{1}{2}(L_0 + C)$$

This means we have a strictly decreasing sequence of real numbers which are bounded from below, i.e. its limit exists. Thus

$$L := \lim_{n\to\infty} L_n = \lim_{n\to\infty}\left(L_0 - \frac{1}{L_n}\right) = L_0 - \frac{1}{\lim_{n\to\infty} L_n} = L_0 - \frac{1}{L}$$
$$0 = L^2 - L_0L + 1 = L^2 - 2207L + 1 \Longleftrightarrow L = \frac{1}{2}\left(L_0 + C\right)$$

because $2206 < L \leq L_1 < 2207 = L_0$. This means we have to calculate $\sqrt[8]{\frac{1}{2}(L_0 + C)} = \sqrt{\sqrt{\sqrt{\frac{1}{2}(L_0 + \sqrt{L_0^2 - 4})}}}$.

If $0 = x^2 - ax + 1$ then

$$0 = (x^2 - ax + 1)(x^2 + ax + 1) = (x^2 + 1)^2 - a^2x^2 = x^4 - (a^2 - 2)x^2 + 1$$

so the positive square root of $y^2 - by + 1$ satisfies $x^2 - \sqrt{b+2}x + 1 = 0$.

$$y^2 - 2207y + 1 = 0 \Longrightarrow x^2 - \sqrt{2207 + 2}x + 1 = x^2 - 47x + 1 = 0$$
$$y^2 - 47y + 1 = 0 \Longrightarrow x^2 - \sqrt{47 + 2}x + 1 = x^2 - 7x + 1$$
$$y^2 - 7y + 1 = 0 \Longrightarrow x^2 - \sqrt{7 + 2}x + 1 = x^2 - 3x + 1$$
$$\Longrightarrow L = \frac{1}{2}\left(3 + \sqrt{5}\right)$$

14. (HS-4) To each positive integer with n^2 decimal digits, we associate the determinant of the matrix obtained by writing the digits in order across the rows. For example, for $n = 2$, to the integer 8617 we associate $\det\left(\begin{bmatrix} 8 & 6 \\ 1 & 7 \end{bmatrix}\right) = 50$. Find, as a function of n, the sum of all the determinants associated with n^2-digit integers. Leading digits are assumed to be nonzero; for example, for $n = 2$, there are 9000 determinants: $f(2) = \sum_{1000 \leq N \leq 9999} \det(N)$.

    Reason: Determinants.

    Solution: If $n > 2$ then all determinants appear as positive and equal negative value in the sum: Fix all but the last two columns (c_{n-1}, c_n). Then (c_n, c_{n-1}) is in the sum, too, but of opposite sign. $f(1) = \sum_{k=1}^{9} \det(k) = 1 + \ldots + 9 = 45$. So it remains to determine $f(2)$.

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + \det\left(\begin{bmatrix} a & b \\ e & f \end{bmatrix}\right) = ad - bc + af - be = a(d+f) - b(c+e)$$

By the multilinearity of the determinant, the answer is the determinant of the matrix whose first (resp. second) row is the sum of all possible first (resp. second) rows. There are 90 first rows whose sum is the vector $(450, 405)$, and 100 second rows whose sum is $(450, 450)$. Thus the answer is

$$450 \cdot 450 - 450 \cdot 405 = 450 \cdot 45 = 20250$$

15. (HS-5) All squares on a chessboard are labeled from 1 to 64 in reading order (from left to right, row by row top-down). Then someone places 8 rooks on the board such that none threatens any other. Let S be the sum of all squares which carry a rook. List all possible values of S.

    Reason: Chessboard.

    Solution: The squares on a chessboard are usually labeled by numbers $1, \ldots, 8$ for the ranks and letters $A, \ldots, H$ for the files. Any order of rooks require that each number and each letter is assumed exactly once, so

$$\begin{aligned} S &= 1+2+3+4+5+6+7+8+A+B+C+D+E+F+G+H \\ &= 1+2+3+4+5+6+7+8+0+8+16+24+32+40+48+56 \\ &= 1+2+3+4+5+6+7+8+8\cdot(1+2+3+4+5+6+7) \\ &= 9 \cdot \binom{8}{2} + 8 = 9 \cdot 28 + 8 = 260 \end{aligned}$$

10 March 2021

1. Prove that all derivations $D := \operatorname{Der}(L)$ of a semisimple Lie algebra L are inner derivations $M := \operatorname{ad}(L)$.

   Reason: Semisimple Lie Algebras.

   Solution: Since the center $Z(L) = \ker \operatorname{ad}(L) = \{0\}$ of L is trivial, we have an isomorphism $M \cong L$. Furthermore

$$\begin{aligned}[\delta, \operatorname{ad}(X)](Y) &= \delta(\operatorname{ad}(X)(Y)) - \operatorname{ad}(X)(\delta(Y)) \\ &= [\delta(X), Y] + [X, \delta(Y)] - [X, \delta(Y)] \\ &= \operatorname{ad}(\delta(X))(Y)\end{aligned}$$

   i.e. $M \trianglelefteq D$ is an ideal, and the Killing-form K_M of M is the restriction of the Killing-form K_D of D. Let

$$M^{\perp} = \{\delta \in D \mid K_D(\delta, \operatorname{ad}(X)) = 0 \,\forall\, X \in L\}$$

   As the Killing-form of L is non-degenerated, so is K_M, hence $M^{\perp} \cap M = \{0\}$. Since both, $M^{\perp}$ and M are ideals in D, we obtain $[M, M^{\perp}] = \{0\}$. This means $D = M \oplus M^{\perp}$ because $\dim M + \dim M^{\perp} = \dim D$. Let $\delta \in M^{\perp}$. Then by the above equation

$$\{0\} = [M^{\perp}, M] \ni [\delta, \operatorname{ad}(X)] = \operatorname{ad}(\delta(\mathrm{X}))$$

   we get $\delta(X) \in Z(L) = \{0\}$, i.e. $M^{\perp} = \{0\}$ and $D = M$.

2. Give four possible non-isomorphic meanings for $\mathbb{Z}_p$.

   Reason: Localizations, p-adics and Factorization.

   Solution: The most common meaning is probably the factor ring

$$\mathbb{Z}_p = \mathbb{Z}/p \cdot \mathbb{Z} = \mathbb{Z}/\sim_p = \{0, 1, \ldots, p-1 \mid p \equiv 0\} = \mathbb{F}_p$$

   with the equivalence relation $a \sim_p b \Longleftrightarrow p \mid (a - b)$.

   A subset S of a commutative ring R with 1 is called multiplicative closed, if $1 \in S$ and $a, b \in S$ implies $a \cdot b \in S$. Now $(s, a) \sim_S (t, b) \Longleftrightarrow u \cdot (sb - at) = 0$ for some $u \in S$ defines an equivalence relation on $S \times R$. Its factor ring $S \times R/\sim_S =: S^{-1}R$ is called localization of R according to S. There are two natural multiplicative sets:

   The set $\{1, p, p^2, p^3, \ldots\}$ is obviously multiplicative closed for any given $p \in \mathbb{Z} - \{0\}$. In this case we can write

$$\{p^n \mid n \in \mathbb{N}_0\}^{-1}\mathbb{Z} = \mathbb{Z}_p = \left(p^{\mathbb{N}_0}\right)^{-1}\mathbb{Z} = p^{\mathbb{N}_0} \times \mathbb{Z}/\sim_{p^{\mathbb{N}_0}}$$

where $(p^n,a) \sim_{p^{\mathbb{N}_0}} (p^m,b) \Longleftrightarrow p^k(bp^n-ap^m)=0$ for some $k\in\mathbb{N}_0$. Hence
$$\mathbb{Z}_p=\left\{\frac{a}{p^n}\,\middle|\, a\in\mathbb{Z}\wedge n\in\mathbb{N}_0\right\}\subseteq\mathbb{Q}$$
is the localization of $\mathbb{Z}$ at the element p.

Another naturally multiplicative closed set is $S=R-P$ where P is a prime ideal of R, since this is exactly the definition of a prime ideal. We can thus localize R at the prime ideal P and write it R_P. This means in our case for a prime ideal $(p)=p\cdot\mathbb{Z}$ in $\mathbb{Z}$ that
$$\mathbb{Z}_p=(p\mathbb{Z})^{-1}\mathbb{Z}=(p)\times\mathbb{Z}/\sim_{(p)}=\left\{\frac{a}{b}\,\middle|\, a,b\in\mathbb{Z}\wedge p\nmid b\right\}\subseteq\mathbb{Q}$$

Finally we look at the p-adic numbers $\mathbb{Q}_p$ for a given prime p. They are an extension field of the rationals and can be written as $\sum_{i=-\infty}^{\infty}a_ip^i$ with coefficients $a_i\in\{0,1,\ldots,p-1\}$. Then we have the subring of integers of $p-$adic numbers
$$\mathbb{Q}_p\supseteq\mathbb{Z}_p=\left\{\sum_{i=-\infty}^{\infty}a_i\cdot p^i\,\middle|\, a_i=0\,\forall\, i<0\right\}=\varprojlim_{n\in\mathbb{N}}\mathbb{Z}/p^n\mathbb{Z}\supseteq\mathbb{Z}$$

3. Let $T\subseteq(\mathbb{Z}_+^n,\preccurlyeq)$ with the partial natural ordering. Then there is a finite subset $S\subseteq T$ such that for every $t\in T$ exists a $s\in S$ with $s\preccurlyeq t$.
$$\alpha\preccurlyeq\beta\Longleftrightarrow\alpha_i\leq\beta_i \text{ for all } i=1,\ldots,n$$

Reason: Gordan-Dickson Lemma.

Solution: There are no infinite descending chains under the natural ordering in $(\mathbb{Z}_+^n,\preccurlyeq)$, i.e. the set T_{min} of minimal elements of T is the smallest subset with the required property, and we must show that $S=T_{min}$ is finite.

We proceed by induction along n. The case $n=1$ is obvious, since $|T_{min}|=1$ in this case. Now assume $n>1$ and that the statement is true for all $k<n$. For $k\geq 0$ we define
$$U_k=\{t'\in\mathbb{Z}_+^{n-1}\,|\,(t',k)\in T\}\ \wedge\ U:=\bigcup_{k\geq 0}U_k$$

The sets $(U_k)_{min}$ and U_{min} are finite by induction hypothesis. Therefore exists an $m\geq 0$ such that $U_{min}\subseteq U_0\cup\ldots\cup U_m$. Set
$$S:=\bigcup_{k=0}^{m}((U_k)_{min}\times\{k\})\subseteq T.$$

which is thus finite. Let $t = (t', k) \in T$ with $t' \in \mathbb{Z}_+^{n-1}$ and $k \geq 0$. If $k \leq m$, then there is a $u \in (U_k)_{min}$ with $u \preccurlyeq t'$. Therefore $(u, k) \in S$ and $(u, k) \preccurlyeq (t', k)$. If $k > m$, then there is by the choice of m a $l \leq m$ and a $u \in (U_l)_{min}$ with $u \preccurlyeq t'$, i.e. $(u, l) \preccurlyeq (t', k)$. Since $(t', k) \in T$ was an arbitrary element of T, we have proven that S has the required property.

Another proof is possible by using Hilbert's basis theorem. The monomial ideal $\langle x^\alpha \,|\, \alpha \in T\rangle$ is generated by a finite set $\{x^\alpha \,|\, \alpha \in T\}$ by Hilbert's basis theorem. This set is necessarily of the form $\{x^\alpha \,|\, \alpha \in S\}$ for a finite subset $S \subseteq T$. As a generating set of the ideal, S has the required property. The Lemma of Gordan-Dickson is therefore a corollary of Hilbert's basis theorem.

4. (a) Solve the following linear differential equation system:

$$\dot{y}_1(t) = 11y_1(t) - 80y_2(t) \;\wedge\; \dot{y}_2(t) = y_1(t) - 5y_2(t)$$
$$y_1(0) = 0 \;\wedge\; y_2(0) = 0$$

(b) Which solutions do $y_1(0) = \pm\varepsilon \;\wedge\; y_2(0) = \pm\varepsilon$ have?

(c) How does the trajectory for $y_1(0) = 0.001 \;\wedge\; y_2(0) = 0.001$ behave for $t \to \infty$?

(d) What will change if we substitute the coefficient -80 by -60?

(e) Calculate (approximately) the radius of the osculating circle at $t = \pi/12$ for both trajectories with initial condition $\mathbf{y}(0) = (-1, 1)$.

Reason: Unstable Vortex and Repeller.

Solution:

(a) The system can be written as $\dot{\mathbf{y}}(t) = A\mathbf{y}(t)$ with $\mathbf{y}(t) = (y_1(t), y_2(t))$ and $A = \begin{bmatrix} 11 & -80 \\ 1 & -5 \end{bmatrix}$. Obviously solves $\mathbf{y}(t) \equiv 0$ the problem, and is a stable solution.

(b) To determine a general solution, we set $\mathbf{y}(t) = e^{\lambda t}\mathbf{u}$ and find

$$\dot{\mathbf{y}}(t) = \lambda e^{\lambda t}\mathbf{u} = A\mathbf{y}(t) = e^{\lambda t}A\mathbf{u}$$

so we have to solve $(A - \lambda)\mathbf{u} = 0$ which yields the eigenvalues $\lambda_{1,2} = 3 \pm 4i$ and eigenvectors

$$\mathbf{u} \in \{(1 - 2i, -i/4)^\tau,\, (-20i, 1 - 2i)^\tau\}$$

The general solution is thus the linear combination

$$\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{u_1} + c_2 e^{\lambda_2 t}\mathbf{u_2}$$

$$\dot{\mathbf{y}}(t) = c_1\lambda_1 e^{\lambda_1 t}\mathbf{u_1} + c_2\lambda_2 e^{\lambda_2 t}\mathbf{u_2}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = c_1 e^{(3+4i)t}\begin{bmatrix} 1-2i \\ -i/4 \end{bmatrix} + c_2 e^{(3-4i)t}\begin{bmatrix} -20i \\ 1-2i \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} &= c_1(3+4i)e^{(3+4i)t}\begin{bmatrix} 1-2i \\ -i/4 \end{bmatrix} + c_2(3-4i)e^{(3-4i)t}\begin{bmatrix} -20i \\ 1-2i \end{bmatrix} \\
&= c_1 e^{(3+4i)t}\begin{bmatrix} 11-2i \\ 1-3i/4 \end{bmatrix} + c_2 e^{(3-4i)t}\begin{bmatrix} -80-60i \\ -5-10i \end{bmatrix} \\
&= c_1 e^{(3+4i)t}\begin{bmatrix} 11(1-2i)-80(-i/4) \\ (1-2i)-5(-i/4) \end{bmatrix} \\
&\quad + c_2 e^{(3-4i)t}\begin{bmatrix} 11(-20i)-80(1-2i) \\ (-20i)-5(1-2i) \end{bmatrix} \\
&= c_1 e^{\lambda_1 t}A\mathbf{u_1} + c_2 e^{\lambda_2 t}A\mathbf{u_2} \\
&= c_1\dot{\mathbf{y}}_\mathbf{1}(t) + c_2\dot{\mathbf{y}}_\mathbf{2}(t)
\end{aligned}$$

$$\mathbf{y_1}(t) = e^{3t}\cdot e^{4it}\cdot\begin{bmatrix} 1-2i \\ -i/4 \end{bmatrix} = e^{3t}\cdot(\cos(4t)+i\sin(4t))\cdot\begin{bmatrix} 1-2i \\ -i/4 \end{bmatrix}$$

$$\frac{\mathbf{y_1}(t)}{e^{3t}} = \begin{bmatrix} \cos(4t)+2\sin(4t) \\ (1/4)\sin(4t) \end{bmatrix} + i\begin{bmatrix} \sin(4t)-2\cos(4t) \\ -(1/4)\cos(4t) \end{bmatrix}$$

$$\mathbf{y_2}(t) = e^{3t}\cdot e^{-4it}\cdot\begin{bmatrix} -20i \\ 1-2i \end{bmatrix} = e^{3t}\cdot(\cos(4t)-i\sin(4t))\cdot\begin{bmatrix} -20i \\ 1-2i \end{bmatrix}$$

$$\frac{\mathbf{y_2}(t)}{e^{3t}} = \begin{bmatrix} -20\sin(4t) \\ \cos(4t)-2\sin(4t) \end{bmatrix} + i\begin{bmatrix} -20\cos(4t) \\ -2\cos(4t)-\sin(4t) \end{bmatrix}$$

Hence the real solutions are given by

$$\mathbf{y}(t) = \alpha e^{3t}\begin{bmatrix} \cos(4t)+2\sin(4t) \\ (1/4)\sin(4t) \end{bmatrix} + \beta e^{3t}\begin{bmatrix} -20\sin(4t) \\ \cos(4t)-2\sin(4t) \end{bmatrix}$$

and

$$\begin{bmatrix} y_1(\pm\varepsilon) \\ y_2(\pm\varepsilon) \end{bmatrix} = \alpha e^{\pm 3\varepsilon}\begin{bmatrix} \cos(4\varepsilon)\pm 2\sin(4\varepsilon) \\ \pm(1/4)\sin(4\varepsilon) \end{bmatrix} + \beta e^{\pm 3\varepsilon}\begin{bmatrix} \mp 20\sin(4\varepsilon) \\ \cos(4\varepsilon)\mp 2\sin(4\varepsilon) \end{bmatrix}$$

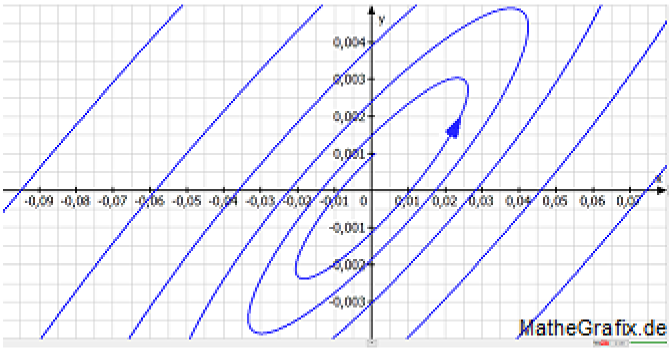
and

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} \pm\varepsilon \\ \pm\varepsilon \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \pm\varepsilon\begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm \varepsilon\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This means that the solutions next to the fixed point $(0,0)$ drift away from it, and that any disturbance of the fixed point's initial condition repels from it. The origin is an unstable vortex.

(c) The trajectory from the initial condition $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1/1000 \\ 1/1000 \end{bmatrix}$ is

$$\mathbf{y}(t) = \frac{1}{1000} e^{3t} \begin{bmatrix} \cos(4t) - 18\sin(4t) \\ \cos(4t) - (7/4)\sin(4t) \end{bmatrix} \overset{t\to\infty}{\longrightarrow} \begin{bmatrix} \text{oscillates} & \pm\infty \\ \text{oscillates} & \pm\infty \end{bmatrix}$$



(Image produced with MatheGrafix.de - not scaled $e^{3t} \to e^{0.3t}$)

(d) The system now writes $\dot{\mathbf{y}}(t) = B\mathbf{y}(t)$ with $\mathbf{y}(t) = (y_1(t), y_2(t))$ and $B = \begin{bmatrix} 11 & -60 \\ 1 & -5 \end{bmatrix}$. Again $\mathbf{y}(t) \equiv 0$ solves the linear problem, and is a stable solution. However we now have the characteristic polynomial

$$t^2 - 6t + 5 = (t-1)(t-5)$$

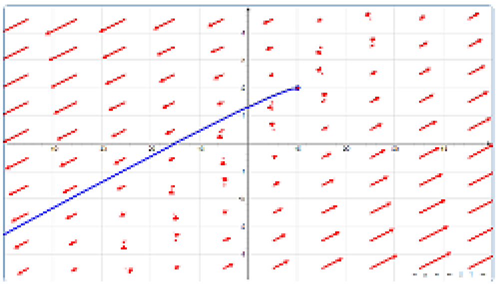
and real eigenvalues $\mu_{1,2}$ with eigenvectors

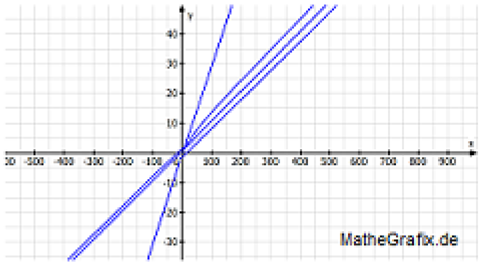
$$\mathbf{u} \in \{(6,1)^\tau, (10,1)^\tau\}$$

and general solution

$$\mathbf{y}(t) = \alpha e^t \begin{bmatrix} 6 \\ 1 \end{bmatrix} + \beta e^{5t} \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

Thus we not only have two real and positive eigenvalues now, we also don't have a vortex anymore at $(0,0)$. The vector field has now a repeller at the origin.





(e) The radius of the osculating circle is the reciprocal curvature and defined by the formula

$$R(\mathbf{y}(t)) = \left|\frac{(\dot{\mathbf{y}}_1^2(t) + \dot{\mathbf{y}}_2^2(t))^{3/2}}{\dot{\mathbf{y}}_1(t)\cdot\ddot{\mathbf{y}}_2(t) - \ddot{\mathbf{y}}_1(t)\cdot\dot{\mathbf{y}}_2(t)}\right|$$

The initial condition $\mathbf{y}(0) = (-1, 1)$ result in the two trajectories

$$\mathbf{y}_A(t) = e^{3t}\begin{bmatrix} -22\sin(4t) - \cos(4t) \\ -(9/4)\sin(4t) + \cos(4t) \end{bmatrix}$$

$$\dot{\mathbf{y}}_A(t) = e^{3t}\begin{bmatrix} -62\sin(4t) - 91\cos(4t) \\ -(43/4)\sin(4t) - 6\cos(4t) \end{bmatrix}$$

$$\ddot{\mathbf{y}}_A(t) = e^{3t}\begin{bmatrix} 550\sin(4t) - 521\cos(4t) \\ -(33/4)\sin(4t) - 61\cos(4t) \end{bmatrix}$$

$$\mathbf{y}_B(t) = e^{t}\begin{bmatrix} 33/2 \\ 11/4 \end{bmatrix} - e^{5t}\begin{bmatrix} 35/2 \\ 7/4 \end{bmatrix}$$

$$\dot{\mathbf{y}}_B(t) = e^{t}\begin{bmatrix} 33/2 \\ 11/4 \end{bmatrix} - e^{5t}\begin{bmatrix} 175/2 \\ 35/4 \end{bmatrix}$$

$$\ddot{\mathbf{y}}_B(t) = e^{t}\begin{bmatrix} 33/2 \\ 11/4 \end{bmatrix} - e^{5t}\begin{bmatrix} 875/2 \\ 175/4 \end{bmatrix}$$

$$\dot{\mathbf{y}}_A(\pi/12) = e^{\pi/4}\begin{bmatrix} -31\sqrt{3} - 91/2 \\ -(43/8)\sqrt{3} - 3 \end{bmatrix} \approx \begin{bmatrix} -217.56 \\ -27 \end{bmatrix}$$

$$\ddot{\mathbf{y}}_A(\pi/12) = e^{\pi/4}\begin{bmatrix} 275\sqrt{3} - 521/2 \\ -(33/8)\sqrt{3} - (61/2) \end{bmatrix} \approx \begin{bmatrix} 473.34 \\ -82.57 \end{bmatrix}$$

$$R(\mathbf{y}_A(t)) \approx 342.71$$

$$\dot{\mathbf{y}}_B(\pi/12) = e^{\pi/12}\begin{bmatrix} 33/2 \\ 11/4 \end{bmatrix} - e^{5\pi/12}\begin{bmatrix} 175/2 \\ 35/4 \end{bmatrix} \approx \begin{bmatrix} -302.53 \\ -28.82 \end{bmatrix}$$

$$\ddot{\mathbf{y}}_B(\pi/12) = e^{\pi/12}\begin{bmatrix} 33/2 \\ 11/4 \end{bmatrix} - e^{5\pi/12}\begin{bmatrix} 875/2 \\ 175/4 \end{bmatrix} \approx \begin{bmatrix} -1598.39 \\ -158.41 \end{bmatrix}$$

$$R(\mathbf{y}_B(t)) \approx 15,104.4$$

This calculation shows, that both solutions are very different even for small paths ($t \approx 0.2618$). The unstable vortex (A) has a low curvature, but it is still large compared to the curvature of the repeller (B). And all due to a reduction of one single coefficient about 25%. It also shows the importance of stability considerations for the solutions of even simple differential equations.

5. Consider the ideal $I = \langle x^2y + xy, xy^2 + 1 \rangle \subseteq \mathbb{R}[x, y]$ and compute a reduced Gröbner basis to determine the number of irreducible components of the algebraic variety $V(I)$.

   Reason: Gröbner Basis.

   Solution: $\mathbb{R}[x, y]$ is partially ordered by $x \prec y$ according to which we define $LT(f)$ as the leading term of the polynomial $f \in \mathbb{R}[x, y]$ and $LC(f)$ as the leading coefficient of f. A **Gröbner basis** of I is a generating system $G = (g_1, \ldots, g_n)$ of polynomials, such that for all $f \in I - \{0\}$ there is a $g \in G$ whose leading term divides the one of $f : LT(g) \,|\, LT(g)$. A Gröbner basis is called **minimal**, if for all $g \in G$

   $$LT(g) \notin \langle LT(G - \{g\}) \rangle \,\wedge\, LC(g) = 1.$$

   and **reduced** if no monomial of its elements $g \in G$ is an element of $\langle LT(G - \{g\}) \rangle$ and $LC(g) = 1$. Reduced Gröbner basis are automatically minimal. They are also unique whereas the minimal ones do not need to be.

   Gröbner bases can be found by the Buchberger algorithm. We define for two polynomials $p, q \in I - \{0\}$ the division

   $$S(p, q) := \frac{lcm(LT(p), LT(q))}{LT(p)} \cdot p - \frac{lcm(LT(p), LT(q))}{LT(q)} \cdot q$$

Then Buchberger's algorithm can be written as

$$\begin{aligned}
\text{INPUT: } & \{I\} = \{f_1, \ldots, f_n\} \\
\text{OUTPUT: } & \text{Gröbner basis } G = (g_1, \ldots, g_m) \\
\text{INIT: } & G := \{I\}
\end{aligned}$$

1. DO
2. $\quad G' := G$
3. $\quad$ FOREACH $p, q \in G'$, $p \neq q$
4. $\qquad s = remainder(S(p, q), G)$
5. $\qquad$ IF $s \neq 0$ THEN $G := G \cup \{s\}$
6. $\quad$ NEXT
7. UNTIL $G = G'$

We start with $f_1(x, y) = x^2y + xy$, $f_2(x, y) = xy^2 + 1$ and compute

$$\begin{aligned}
S(f_1, f_2) &= \frac{lcm(x^2y, xy^2)}{x^2y} f_1 - \frac{lcm(x^2y, xy^2)}{xy^2} f_2 \\
&= yf_1 - xf_2 = xy^2 - x = 1 \cdot f_2 - x - 1 \\
G' &= G \cup \{f_3 := -x - 1\} = \{f_1, f_2, f_3\} \\
S(f_1, f_3) &= \frac{lcm(x^2y, x)}{x^2y} f_1 - \frac{lcm(x^2y, x)}{-x} f_3 \\
&= f_1 + xyf_3 = x^2y + xy + xy(-x - 1) = 0 \\
S(f_2, f_3) &= \frac{lcm(xy^2, x)}{xy^2} f_2 - \frac{lcm(xy^2, x)}{-x} f_3 \\
&= f_2 + y^2 f_3 = xy^2 + 1 + y^2(-x - 1) = -y^2 + 1 \\
G' &= G \cup \{f_4 := -y^2 + 1\} = \{f_1, f_2, f_3, f_4\} \\
S(f_1, f_4) &= \frac{lcm(x^2y, y^2)}{x^2y} f_1 - \frac{lcm(x^2y, y^2)}{-y^2} f_4 = yf_1 + x^2 f_4 \\
&= x^2y^2 + xy^2 - x^2y^2 + x^2 = xy^2 + 1 + x^2 - 1 \\
&= f_2 - (x - 1)(-x - 1) = f_2 - xf_3 + f_3 \equiv 0 \mod G \\
S(f_2, f_4) &= \frac{lcm(xy^2, y^2)}{xy^2} f_2 - \frac{lcm(xy^2, y^2)}{-y^2} f_4 = f_2 + xf_4 \\
&= xy^2 + 1 - xy^2 + x = x + 1 = -f_3 \equiv 0 \mod G \\
S(f_3, f_4) &= \frac{lcm(x, y^2)}{-x} f_3 - \frac{lcm(x, y^2)}{-y^2} f_4 = -y^2 f_3 + xf_4 \\
&= y^2(x + 1) - xy^2 + x = y^2 + x \\
&= -f_2 - y^2 f_3 - f_3 \equiv 0 \mod G
\end{aligned}$$

Hence we get a Gröbner basis $\{x^2y+xy, xy^2+1, -x-1, -y^2+1\}$ of I.

$$LT(f_1) = x^2y = (-xy)\cdot(-x) = (-xy)\cdot LT(f_3)$$
$$LT(f_2) = xy^2 = (-x)\cdot(-y^2) = (-x)\cdot LT(f_4)$$

means that $\{x+1, y^2-1\}$ is a minimal Gröbner basis, which is already reduced, because we cannot omit another leading term and the leading coefficients are normed to 1. The vanishing variety are thus the points $\{(-1,-1),(-1,1)\}$ which are two separated points, i.e. two irreducible components.

6. Define the complex function

$$\Gamma(z) := \lim_{n\to\infty} \frac{n!\,n^z}{z(z+1)\cdot\ldots\cdot(z+n)}$$

and prove

(a) $\Gamma(z) = \displaystyle\int_0^\infty e^{-t}\,t^{z-1}\,dt\ ;\quad \Re(z) > 0$

(b) $\Gamma(z)^{-1} = e^{\gamma z} z \displaystyle\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}$

where $\gamma := \displaystyle\lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log(n)\right)$ is the Euler-Mascheroni constant.

Reason: Gamma Function.

Solution:

(a) From $e^{-t} = \displaystyle\lim_{n\to\infty}\left(1-\frac{t}{n}\right)^n$ we get

$$\begin{aligned}
\int_0^\infty e^{-t}\,t^{z-1}\,dt &= \lim_{n\to\infty}\int_0^n \left(1-\frac{t}{n}\right)^n t^{z-1}\,dt \\
&\stackrel{t=ns}{=} \lim_{n\to\infty}\int_0^1 (1-s)^n n^z s^{z-1}\,ds \\
&= \lim_{n\to\infty} n^z\left(\left[\frac{s^z}{z}(1-s)^n\right]_0^1 + \frac{n}{z}\int_0^1 (1-s)^{n-1}s^z\,ds\right) \\
&\vdots \\
&= \lim_{n\to\infty} n^z\left(\frac{n}{z}\cdot\frac{n-1}{z+1}\cdot\ldots\cdot\frac{1}{z+n-1}\int_0^1 s^{z+n-1}\,ds\right) \\
&= \Gamma(z)
\end{aligned}$$

(b) Weierstraß expression.

$$\begin{aligned}\frac{z(z+1)\ldots(z+n)}{n!n^z} &= \frac{1}{n^z}\cdot z\cdot\left(1+\frac{z}{1}\right)\cdot\ldots\cdot\left(1+\frac{z}{n}\right)\\ &= \frac{e^{\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)z}}{e^{(\log n)z}}\cdot z\cdot\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\\ &= e^{\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right)\cdot z}\cdot z\cdot\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\\ &\stackrel{n\to\infty}{\longrightarrow} e^{\gamma z}z\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}\end{aligned}$$

7. Let $u : [0,1]\times[a,b]\longrightarrow\mathbb{C}$ be a continuous function, such that the partial derivative in the first coordinate exists everywhere and is continuous. Define

$$U(\lambda):=\int_a^b u(\lambda,t)\,dt\ ,\ V(\lambda):=\int_a^b\frac{\partial u}{\partial\lambda}(\lambda,t)\,dt.$$

Show that U is continuously differentiable and $U'(\lambda)=V(\lambda)$ for all $0\leq\lambda\leq 1$.

Reason: Complex Integration.

Solution: Let $\varepsilon>0$. We have to show that there is a $\delta>0$ such that for all $\lambda,h\in\mathbb{R}$ with $0<|h|<\delta$, $0\leq\lambda\leq 1$, and $0\leq\lambda+h\leq 1$

$$|V(\lambda+h)-V(\lambda)|<\varepsilon\,,\ \left|\frac{U(\lambda+h)-U(\lambda)}{h}-V(\lambda)\right|<\varepsilon.$$

Every continuous function on a compact interval is uniformly continuous, hence there is a $\delta>0$ such that for all $(\lambda,t),(\lambda',t')\in[0,1]\times[a,b]$

$$|\lambda'-\lambda|+|t'-t|<\delta\Longrightarrow\left|\frac{\partial u}{\partial\lambda}(\lambda',t')-\frac{\partial u}{\partial\lambda}(\lambda,t)\right|<\frac{\varepsilon}{b-a}.$$

By definition of V we have

$$\begin{aligned}|V(\lambda+h)-V(\lambda)| &= \left|\int_a^b\left(\frac{\partial u}{\partial\lambda}(\lambda+h,t)-\frac{\partial u}{\partial\lambda}(\lambda,t)\right)\,dt\right|\\ &\leq\int_a^b\left|\frac{\partial u}{\partial\lambda}(\lambda+h,t)-\frac{\partial u}{\partial\lambda}(\lambda,t)\right|\,dt\\ &<\varepsilon\end{aligned}$$

since the integrand is continuous and takes its maximum in $[a,b]$.

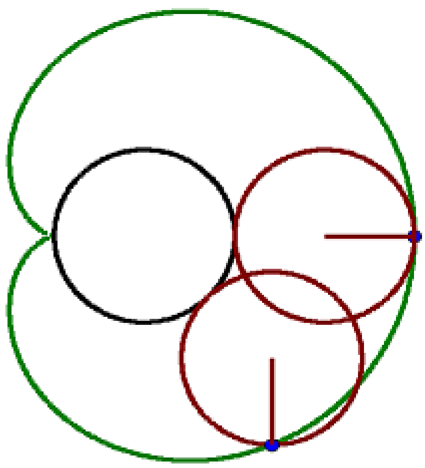
Now assume $0<h<\delta$ such that $0\leq\lambda<\lambda+h\leq 1$. (The case $h<0$ is proven accordingly.) Therefore

$$\begin{aligned}\left|\dfrac{u(\lambda+h,t)-u(\lambda,t)}{h}-\dfrac{\partial u}{\partial\lambda}(\lambda,t)\right| &= \left|\dfrac{1}{h}\int_\lambda^{\lambda+h}\left(\dfrac{\partial u}{\partial\lambda}(\lambda',t)-\dfrac{\partial u}{\partial\lambda}(\lambda,t)\right)d\lambda'\right|\\ &\leq \dfrac{1}{h}\int_\lambda^{\lambda+h}\left|\dfrac{\partial u}{\partial\lambda}(\lambda',t)-\dfrac{\partial u}{\partial\lambda}(\lambda,t)\right|d\lambda'\\ &< \dfrac{\varepsilon}{b-a}\end{aligned}$$

and so

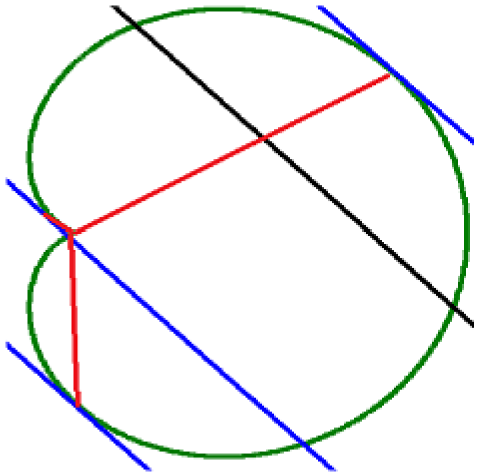
$$\begin{aligned}\left|\dfrac{U(\lambda+h)-U(\lambda)}{h}-V(\lambda)\right| &= \left|\int_a^b\left(\dfrac{u(\lambda+h,t)-u(\lambda,t)}{h}-\dfrac{\partial u}{\partial\lambda}(\lambda,t)\right)dt\right|\\ &\leq \int_a^b\left|\dfrac{u(\lambda+h,t)-u(\lambda,t)}{h}-\dfrac{\partial u}{\partial\lambda}(\lambda,t)\right|dt\\ &< \int_a^b\dfrac{\varepsilon}{b-a}\,dt=\varepsilon\end{aligned}$$

8. A cardiod is defined as the trace of a point on a circle that rolls around a fixed circle of the same size without slipping.



It can be described by $(x^2+y^2)^2+4x(x^2+y^2)-4y^2=0$ or in polar coordinates by $r(\varphi)=2(1-\cos\varphi)$. Show that:

(a) Given any line, there are exactly three tangents parallel to it. If we connect the points of tangency to the cusp, the three segments meet at equal angles of $2\pi/3$.



(b) The length of a chord through the cusp is 4.

(c) The midpoints of chords through the cusp lie on the perimeter of the fixed generator circle (black one in the first picture).

(d) Calculate length, area and curvature.

Reason: Cardioid.

Solution: The cartesian coordinates are given by

$$\begin{bmatrix} x(\varphi) \\ y(\varphi) \end{bmatrix} = r(\varphi) \cdot \begin{bmatrix} \cos(\varphi) \\ \sin(\varphi) \end{bmatrix} = 2 \cdot \begin{bmatrix} \cos(\varphi) - \cos^2(\varphi) \\ \sin(\varphi) - \cos(\varphi)\sin(\varphi) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}(\varphi) \\ \dot{y}(\varphi) \end{bmatrix} = 2 \cdot \begin{bmatrix} \sin(2\varphi) - \sin(\varphi) \\ \cos(\varphi) - \cos(2\varphi) \end{bmatrix}, \left\| \begin{bmatrix} \dot{x}(\varphi) \\ \dot{y}(\varphi) \end{bmatrix} \right\|^2 = 8 - 8\cos(\varphi)$$

$$(x+1)^2 + y^2 = 1 \text{ generating circle}$$

This yields three vertical tangents ($\dot{x}(\varphi) = 0$) at $\varphi = \pi/3, \varphi = \pi, \varphi = 5\pi/3$ and three horizontal tangents ($\dot{y}(\varphi) = 0$) at $\varphi = 0$ (degenerated), $\varphi = 2\pi/3, \varphi = 4\pi/3$ and proves the first statement for tangent slopes $\{0, \infty\}$. In general we have with $\alpha = 2\pi/3$

(a)

$$\begin{aligned}\frac{\cos(\varphi+\alpha)-\cos(2\varphi+2\alpha)}{\sin(2\varphi+2\alpha)-\sin(\varphi+\alpha)} &= \frac{-\sqrt{3}\sin(\varphi)-\cos(\varphi)+\cos(2\varphi)-\sqrt{3}\sin(2\varphi)}{-\sqrt{3}\cos(2\varphi)-\sin(2\varphi)-\sqrt{3}\cos(\varphi)+\sin(\varphi)}\\ &= \frac{\sqrt{3}(\sin(2\varphi)+\sin(\varphi))+(\cos(\varphi)-\cos(2\varphi))}{\sqrt{3}(\cos(\varphi)+\cos(2\varphi))+(\sin(2\varphi)-\sin(\varphi))}\\ &\overset{(*)}{=} \frac{\cos(\varphi)-\cos(2\varphi)}{\sin(2\varphi)-\sin(\varphi)}\end{aligned}$$

We want to prove $(*)$ which is equivalent to show

$$\begin{aligned}\frac{\sin(2\varphi)+\sin(\varphi)}{\cos(\varphi)+\cos(2\varphi)} &= \frac{(\sin(2\varphi)+\sin(\varphi))(\cos(\varphi)-\cos(2\varphi))}{\cos^2(\varphi)-\cos^2(2\varphi)}\\ &= \frac{(\sin(2\varphi)+\sin(\varphi))(\cos(\varphi)-\cos(2\varphi))}{\sin^2(2\varphi)-\sin^2(\varphi)}\\ &= \frac{\cos(\varphi)-\cos(2\varphi)}{\sin(2\varphi)-\sin(\varphi)}\end{aligned}$$

(b) A chord through the cusp (origin) intersects the cardiod in $P = 2(1-\cos\varphi)$ and $Q = 2(1-\cos(\varphi+\pi))$. Thus

$$\begin{aligned}|PQ| &= r(\varphi)+r(\varphi+\pi)\\ &= 4-2\left(\cos(\varphi)+\cos(\varphi)\cos(\pi)-\sin(\varphi)\sin(\pi)\right)\\ &= 4-2(\cos(\varphi)-\cos(\varphi)-0)=4\end{aligned}$$

(c)

$$\begin{aligned}(1/2)\overline{PQ} &= 1-\cos(\varphi)-(1-\cos(\varphi+\pi))\\ &= -\cos(\varphi)+\cos(\varphi)\cos(\pi)-\sin(\varphi)\sin(\pi)\\ &= -2\cos(\varphi)\\ (x+1)^2+y^2 &= (1-2\cos(\varphi)\cos(\varphi))^2+(-2\cos(\varphi)\sin(\varphi))^2\\ &= 1-4\cos^2(\varphi)+4\cos^4(\varphi)+4\cos^2(\varphi)\sin^2(\varphi)\\ &= 1-4\cos^2(\varphi)(1-\cos^2(\varphi)-\sin^2(\varphi)=1\end{aligned}$$

(d)

$$\begin{aligned}A &= 2\cdot\frac{1}{2}\int_0^\pi |r(\varphi)\times\dot{r}(\varphi)|\,d\varphi\\ &= 4\int_0^\pi (1-\cos(\varphi))^2\left|\begin{bmatrix}\cos(\varphi) & -\sin(\varphi)\\ \sin(\varphi) & \cos(\varphi)\end{bmatrix}\right|\,d\varphi\\ &= 4\cdot\left[\varphi-2\sin(\varphi)+(\varphi/2)+(1/2)\sin(\varphi)\cos(\varphi)\right]_0^\pi\\ &= 4(\pi+\pi/2)=6\pi\end{aligned}$$

$$\begin{aligned}
L &= 2\int_0^{\pi} \sqrt{r(\varphi)^2+\dot r(\varphi)^2}\,d\varphi \\
&= 4\int_0^{\pi} \sqrt{(1-\cos(\varphi))^2+(\sin(\varphi))^2}\,d\varphi \\
&= 4\int_0^{\pi} \sqrt{2-2\cos(\varphi)}\,d\varphi = 8\int_0^{\pi} \sin(\varphi/2)\,d\varphi \\
&= -16\,[\cos(\varphi/2)]_0^{\pi} = 16
\end{aligned}$$

$$\begin{aligned}
\kappa(\varphi) &= \frac{|r(\varphi)^2+2\dot r(\varphi)^2-r(\varphi)\ddot r(\varphi)|}{(r(\varphi)^2+\dot r(\varphi)^2)^{3/2}} \\
&= \frac{1}{2}\cdot\frac{|(1-\cos(\varphi))^2+2\sin^2(\varphi)-(1-\cos(\varphi))\cos(\varphi)|}{((1-\cos(\varphi))^2+(\sin(\varphi))^2)^{3/2}} \\
&= \frac{1}{2}\cdot\frac{3-3\cos(\varphi)}{(2-2\cos(\varphi))^{3/2}} = \frac{3}{4\sqrt{2}}\cdot\frac{1}{\sqrt{(1-\cos(\varphi))}} \\
&= \frac{3}{4\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{\sin(\varphi/2)} = \frac{3}{8}\cdot\frac{1}{\sin\left(\frac{\varphi}{2}\right)}
\end{aligned}$$

9. Let A be a complex Banach algebra with 1. Prove that the spectrum

$$\sigma(a) = \{\lambda\in\mathbb{C}\,|\,\lambda\cdot 1 - a \text{ is not invertible }\} \subseteq \{\lambda\in\mathbb{C}\,|\,|\lambda|\leq\|a\|\}$$

for any $a\in A$ is not empty, bounded and closed.

Reason: Theorem of Gelfand.

Solution: Let $G(A) := \{a\in A\,|\,a \text{ is invertible }\}$. We first show that in case $\|a\|<1$ we have

$$1-a\in G(A)\,,\;(1-a)^{-1}=\sum_{k=0}^{\infty} a^k$$

which is referred to as the Neumann series for $(1-a)^{-1}$.

The norm of A is submultiplicative so the series converges absolutely, and by completeness of A it converges in A, say to $s\in A$. Then

$$(1-a)s = (1-a)\lim_{n\to\infty}\sum_{k=0}^{n} a^k = \lim_{n\to\infty}(1-a^{k+1}) = 1$$

because $\|a^{k+1}\|\leq\|a\|^{k+1}\stackrel{k\to\infty}{\longrightarrow} 0$. By the same argument we get the left inverse $s(1-a)=1$, hence $s=(1-a)^{-1}$.

Next we show that $G(A) \subseteq A$ is open and inversion $f : G(A) \longrightarrow A\,,\, f(a) = a^{-1}$ is continuous.

Let $a \in G(A)$ and $b \in A$ such that $\|a - b\| < \|a^{-1}\|^{-1}$. Then

$$\|1 - ba^{-1}\| = \|(a - b)a^{-1}\| \leq \|a - b\| \cdot \|a^{-1}\| < 1$$

so $1 - (1 - ba^{-1}) = ba^{-1} \in G(A)$ by the previous statement, and thus $b = (ba^{-1})a \in G(A)$ which means that $G(A) \subseteq A$ is open, because $G(A)$ is closed under multiplication.

Let's consider left- and right multiplications $l_b(a) = ba\,,\, r_b(a) = ab$ by $b \in A$. Now we can write inversion as

$$f(a) = (l_{a^{-1}} \circ f \circ r_{a^{-1}})(a)$$

for all $a \in G(A)$. Since $r_{a^{-1}}(a) = 1, f(1) = 1, l_{a^{-1}}(1) = a^{-1}$, and the maps $l_{a^{-1}}, r_{a^{-1}}$ are continuous at 1 and a, respectively for all $a \in G(A)$, it is sufficient to show that f is continuous at 1.
Let $\|1 - b\| < \varepsilon < 1/2 < 1$. Then we get for $a := 1 - b$ the Neumann series $\sum_{k=0}^{\infty} a^k = (1 - a)^{-1} = b^{-1}$ and thus

$$\begin{aligned} \|f(b) - f(1)\| &= \|b^{-1} - 1\| \leq \|1 - b\| \cdot \|b^{-1}\| \\ &\leq \|a\| \sum_{k=0}^{\infty} \|a\|^k = \|a\|(1 - \|a\|)^{-1} \leq 2\varepsilon \end{aligned}$$

which shows that f is continuous at 1.

If $|\lambda| > \|a\|$, then $\|\lambda^{-1}a\| < 1$ and $1 - \lambda^{-1}a \in G(A)$. Hence $\lambda - a = \lambda(1 - \lambda^{-1}a) \in G(A)$ and $\lambda \notin \sigma(a)$ which proves the inclusion in the statement and that $\sigma(a)$ is bounded. Now define $g : \mathbb{C} \longrightarrow A$ by setting $g(\lambda) := \lambda \cdot 1 - a$. Then g is continuous, and thus $g^{-1}(G(A)) = \mathbb{C} - \sigma(a)$ is open as the preimage of the open set $G(A)$ which means that $\sigma(a)$ is closed.

It remains to show that $\sigma(a) \neq \emptyset$. Let $\lambda \in \mathbb{C} - \sigma(a)$, i.e. $\lambda - a \in G(A.)$ Since $G(A)$ is open, there is an $r > 0$ such that $\mu - a \in G(A)$ whenever $|\mu - \lambda| < r$.

$$\begin{aligned} (\mu - a)^{-1} - (\lambda - a)^{-1} &= (\mu - a)^{-1}((\lambda - a)(\lambda - a)^{-1}) \\ &\quad - ((\mu - a)^{-1}(\mu - a))(\lambda - a)^{-1} \\ &= (\mu - a)^{-1}((\lambda - a) - (\mu - a))(\lambda - a)^{-1} \\ &= (\lambda - \mu)(\mu - a)^{-1}(\lambda - a)^{-1} \end{aligned}$$

Hence for $\phi \in A^*$ we get by continuity of inversion and continuity of ϕ

$$\begin{aligned}\frac{\phi((\mu-a)^{-1})-\phi((\lambda-a)^{-1})}{\mu-\lambda} &= \frac{\phi((\mu-a)^{-1}-(\lambda-a)^{-1})}{\mu-\lambda}\\ &= \phi(-(\mu-a)^{-1}(\lambda-a)^{-1})\\ &\stackrel{\mu\to\lambda}{\longrightarrow} \phi(-(\lambda-a)^{-2})\end{aligned}$$

which means that $\lambda \longmapsto \phi((\lambda-a)^{-1})$ is analytic on $\mathbb{C}-\sigma(a)$. Using continuity of inversion again yields

$$\phi((\lambda-a)^{-1}) = \lambda^{-1}\phi((1-\lambda^{-1}a)^{-1}) \stackrel{|\lambda|\to\infty}{\longrightarrow} 0$$

Assume $\sigma(A)=\emptyset$. Then $\lambda \longmapsto \phi((\lambda-a)^{-1})$ is analytic on the whole plane $\mathbb{C}$, and bounded as we have seen above. By the theorem of Liouville the function is identically zero. Consequently $\phi(a^{-1})=0$ for all $\phi\in A^*$. By the theorem of Hahn-Banach this implies $a^{-1}=0$, which is a contradiction. Thus $\sigma(a)\neq\emptyset$.

10. (a) Determine all primes which occur as orders of an element from $G := \operatorname{SL}_3(\mathbb{Z})$.
    (b) Let $I \trianglelefteq R$ be a two-sided ideal in a unitary ring with group of unities U. Show by two different methods that

    $$M := \{u\in U \,|\, u-1\in I\} \trianglelefteq U$$

    is a normal subgroup.

Reason: Group Theory.

Solution:

(a) There are eight vectors in $\mathbb{Z}_2^3$. For a basis we can choose among $7=8-1$ of them as first basis vector, $6=8-2$ for the second, and $4=8-4$ for the last one. Hence there are $7\cdot 6\cdot 4 = 168 = 2^3\cdot 3\cdot 7$ possible ordered basis, which equals $|\operatorname{GL}(Z_2^3)| = |\operatorname{GL}_3(\mathbb{Z}_2)|$. Next we consider the induced homomorphism

$$\varphi\,:\, G=\operatorname{SL}_3(\mathbb{Z}) \rightarrowtail \operatorname{GL}_3(\mathbb{Z}) \twoheadrightarrow \operatorname{GL}_3(\mathbb{Z}_2)$$

For all elements $g\in G$ of order n holds $1=\varphi(1)=\varphi(g^n)=\varphi(g)^n\in\operatorname{GL}_3(\mathbb{Z}_2)$ and thus $n|168$ and possible prime orders are $\{2,3,7\}$. With

$$\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\;,\;\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

we have elements of order 2 and 3 in G.

Assume there is an element $a \in G$ of order 7. Then a is a zero of

$$\mathbb{Z}[x] \ni p(x) := X^7 - 1 = \prod_{k=1}^{7}(x-\zeta_k) = \prod_{k=1}^{7}\left(x - e^{\frac{2k\pi i}{7}}\right)$$

The Jordan normal form of a states that a is a conjugate of a complex upper triangular matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ on its diagonal. Those eigenvalues are all among the 7−th roots of unity, the zeros of $p(x)$, because the characteristic polynomial of a divides $p(x)$. We may assume that $\lambda_1 \neq 1$, since if $\lambda_1 = \lambda_2 = \lambda_3 = 1$ then a is a conjugate to a matrix of the Heisenberg group, and the Heisenberg group is torsion free. Thus λ_1 is a primitive 7−th root of unity, and by choosing a suitable power of a we may assume that

$$\lambda_1 = \zeta_1 = e^{\frac{2\pi i}{7}} = \cos\left(\frac{2\pi}{7}\right) + i \cdot \sin\left(\frac{2\pi}{7}\right)$$

Moreover we have $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = \det(a) = 1$ and $\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{tr}(a) \in \mathbb{Z}$. Thus $\lambda_3 = \zeta_1^{-1}\lambda_2^{-1}$ and so $\zeta_1 + \lambda_2 + \zeta_1^{-1}\lambda_2^{-1} \in \mathbb{Z}$. As λ_2 is also a 7−th root of unity, we have only seven possible values for $\zeta_1 + \lambda_2 + \zeta_1^{-1}\lambda_2^{-1}$. Those values are only real if $\lambda_2 = 1$ or $\lambda_2 = \zeta_1^{-1}$. In either case we have

$$\zeta_1 + 1 + \zeta^{-1} \approx 2.247 \notin \mathbb{Z}$$

which disproves our assumption that there is an element of order 7. The only possible (and occurring) prime orders are 2 and 3.

(b) Let $n, m \in M$. Obviously we have $0 \in I$ which means $1 \in M$. Moreover $nm - 1 = (n-1)m + (m-1) \in I$ so $nm \in M$. Thirdly $n^{-1} - 1 = n^{-1}(1-n) \in I$ so $n^{-1} \in M$ which shows that $M \leq U$ is a subgroup. With $u \in U$ we have $unu^{-1} - 1 = u(n-1)u^{-1} \in I$ hence $unu^{-1} \in M$.

Alternatively we can consider the projection $\pi : R \twoheadrightarrow R/I$. Since π is a ring homomorphism, we have $1 = \pi(1) = \pi(n \cdot n^{-1}) = \pi(n) \cdot \pi^{-1}(n)$. Thus π induces a group homomorphism on the according groups of units:

$$\overline{\pi} : R^* = U \longrightarrow (R/I)^*$$

i.e. $\ker \overline{\pi} \trianglelefteq U$ is a normal subgroup. But $\ker \overline{\pi} = \{u \in U \,|\, \overline{\pi}(u) = u + I = 1 + I\} = \{u \in U \,|\, u - 1 \in I\} = M$.

11. (HS-1) If a, b, c are real numbers such that $a+b+c = 2$ and $ab+ac+bc = 1$, show that $0 \leq a, b, c \leq \dfrac{4}{3}$.

    Reason: Discriminant.

    Solution: We get the quadratic equation

    $$ab+(a+b)(2-(a+b)) = -(b^2+(a-2)b+(a^2-2a)) = 1 \text{ or } 0 = b^2+(a-2)b+(a-1)^2$$

    from the given conditions, which must have a non-negative discriminant

    $$\begin{aligned}(2-a)^2 - 4(a-1)^2 = -3a^2 + 4a \geq 0 &\Longrightarrow 4a \geq 3a^2 \geq 0 \\ &\Longrightarrow \frac{4}{3} \geq a \geq 0\end{aligned}$$

    which holds for symmetry reasons for b, c, too.

12. (HS-2) Determine all pairs (m, n) of (positive) natural numbers such that $2022^m - 2021^n$ is a square.

    Reason: Modular Arithmetics.

    Solution: $(1, 1)$ is obviously a solution. If $m = 1$ then $n = 1$ for otherwise the sum would be negative and cannot be a square. Hence we may assume $m \geq 2$. As a consequence $4 \,|\, 2022^m$ and

    $$x^2 := 2022^m - 2021^n \equiv -1 \equiv 3 \mod 4$$

    If x is even, then $x^2 \equiv 0 \mod 4$, and if $x = 2k+1$ is odd, then $x^2 = 4k^2 + 4k + 1 \equiv 1 \mod 4$. The only possible solution is thus $(n, m) = (1, 1)$.

13. (HS-3)

    (a) Prove for any $n \in \mathbb{N}$, $n \geq 4$

    $$Q(n) := \frac{4^2-9}{4^2-4} \cdot \frac{5^2-9}{5^2-4} \cdot \ldots \cdot \frac{n^2-9}{n^2-4} > \frac{1}{6}.$$

    (b) Is the above statement still true, if we replace $1/6$ on the right hand side by 0.1667 ?

    Reason: Inequality.

    Solution:

(a)

$$\begin{aligned}
&\frac{4^2-9}{4^2-4}\cdot\frac{5^2-9}{5^2-4}\cdot\ldots\cdot\frac{n^2-9}{n^2-4}\\
&\quad=\frac{(4-3)(4+3)}{(4-2)(4+2)}\cdot\frac{(5-3)(5+3)}{(5-2)(5+2)}\cdot\ldots\cdot\frac{(n-3)(n+3)}{(n-2)(n+2)}\\
&\quad=\frac{1\cdot 2\cdots(n-3)}{2\cdot 3\cdots(n-2)}\cdot\frac{7\cdot 8\cdots(n+3)}{6\cdot 7\cdots(n+2)}\\
&\quad=\frac{1}{n-2}\cdot\frac{n+3}{6}>\frac{1}{n}\cdot\frac{n}{6}=\frac{1}{6}
\end{aligned}$$

(b) For any $n > 25,002$ we get

$$\begin{aligned}
0.0002n > 5.00040 \Longrightarrow 1.0002(n-2) > n+3 &\Longrightarrow \frac{n+3}{n-2} < 1.0002\\
&\Longrightarrow \frac{1}{n-2}\cdot\frac{n+3}{6} < 0.1667\\
&\Longrightarrow 0.1667 > Q(25,002) > \frac{1}{6}
\end{aligned}$$

Alternatively consider the sequence $Q(n) := \dfrac{n+3}{6(n-2)}$. This sequence converges $\lim_{n\to\infty} Q(n) = \dfrac{1}{6}$. Hence for any $\varepsilon > 0$, especially for $\varepsilon = 0.1667 - (1/6)$, there are only finitely many exceptions n to

$$\left|Q(n)-\frac{1}{6}\right| < \varepsilon \Longleftrightarrow \frac{1}{6}-\varepsilon < Q(n) < \frac{1}{6}+\varepsilon = 0.1667$$

The answer is therefore 'no'. The lower limit cannot be improved.

14. (HS-4) Determine all pairs $(x,y)\in\mathbb{R}^2$ such that

$$\begin{aligned}
5 &= \sqrt{1+x+y}+\sqrt{2+x-y}\\
2-x+y &= \sqrt{18+x-y}
\end{aligned}$$

Reason: Quadratic Equations.

Solution: Set $u := x+y, v := x-y$. Then the equations become

$$\begin{aligned}
5 &= \sqrt{1+u}+\sqrt{2+v}\\
2-v &= \sqrt{18+v}
\end{aligned}$$

From the second equation we get $2-v\geq 0$ and $v^2-5v-14=0$, i.e.

$$v_{1,2}=\frac{5}{2}\pm\frac{1}{2}\sqrt{25+56}\in\{-2,7\}$$

hence $v=-2$ since $2-7<0$. Thus $\sqrt{1+u}=5-\sqrt{2+(-2)}=5$ and $u=24$. This means $x=11, y=13$ which also satisfy the initial equation system.

15. (HS-5) Given a real, continuous function $f:\mathbb{R}\longrightarrow\mathbb{R}$ such that $f(f(f(x)))=x$. Prove that $f(x)=x$ for all $x\in\mathbb{R}$.

    Reason: Functions.

    Solution: $f(x)$ is injective (into, one-to-one), i.e.

    $$f(r)=f(s)\Longrightarrow r=f(f(f(r)))=f(f(f(s)))=s$$

    and monotone. Assume that given $r<sr<t$ we have either

    $$\begin{aligned}&f(r)<f(s)\,,\,f(s)>f(t)\quad\text{or}\\&f(r)>f(s)\,,\,f(s)<f(t)\end{aligned}$$

    then there would be a real number $y\in\mathbb{R}$ with

    $$\begin{aligned}&f(r)<y<f(s)\,\wedge\,f(s)>y>f(t)\quad\text{or}\\&f(r)>y>f(s)\,\wedge\,f(s)<y<f(t)\end{aligned}$$

    and therefore also real numbers $x,x'\in\mathbb{R}$ such that

    $$r<x<s<x'<t\,\wedge\,f(x)=f(x')=y$$

    by the meanvalue theorem for continuous functions. By the previous part, it follows $x=x'$, a contradiction. $f(x)$ is therefore strictly monotone. Assume $f(x)$ is strictly monotone decreasing, i.e. $f(x)\neq x$ for an $x\in\mathbb{R}$, as $f(x)=x$ would be strictly monotone increasing. W.l.o.g. let

    $$\begin{aligned}x<f(x)&\Longrightarrow f(x)>f(f(x))\\&\Longrightarrow f(f(x))<f(f(f(x)))=x\\&\Longrightarrow f(f(f(x)))=x>f(x)\end{aligned}$$

    which is impossible. Hence $f(x)$ is strictly monotone increasing and one-to-one. Finally assume that $f(x)\neq x$ for an $x\in\mathbb{R}$ and $x<f(x)$ or $x>f(x)$. Then we apply f again twice and get $x<f(x)<f(f(x))<f(f(f(x)))=x$ or $x>f(x)>f(f(x))>f(f(f(x)))=x$ which is impossible. Therefore $f(x)=x$ for all $x\in\mathbb{R}$.

11 February 2021

1. Let f be a real, differentiable function such that there is no $x \in \mathbb{R}$ with $f(x) = 0 = f'(x)$. Show that f has at most finitely many zeros in the interval $[0,1]$.

   Reason: Nice Proof.

   Solution: Set $S := \{x \in \mathbb{R} \,|\, f(x) = 0\} = [0,1] \cap f^{-1}(\{0\})$. Then S is a compact set. If S is infinite, then it has a limit point

   $$S \ni x = \lim_{n\to\infty} x_n$$

   with a sequence $(x_n) \subseteq S$ of distinct points. Therefore $f(x_n) = f(x) = 0$ for all $n \in \mathbb{N}$. Now

   $$f'(x) = \lim_{x_n \to x} \frac{f(x + (x_n - x)) - f(x)}{x_n - x} = \lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} = 0$$

   which contradicts our assumption.

2. Let (X, Ω, ω) be a measure space and f be a $\omega-$integrable function. Show that for every $\varepsilon > 0$ there is a set $W \in \Omega$ such that $\omega(W) < \infty$ and $\int_{X-W} |f| \, d\omega < \varepsilon$.

   Reason: Measure Theory.

   Solution: We define $A_n := \{x \in X \,|\, 1/n \leq |f(x)| < n\}$ for $n \in \mathbb{N}$ and $A_1 \subseteq A_2 \subseteq \ldots =: A = \cup_{n=1}^{\infty} A_n$. All $A_n = v^{-1}([1/n, n])$ with the continuous function $v(x) = |f(x)|$ are measurable. If we add $A_0 := \{x \in X \,|\, f(x) = 0\}$ and $A_\infty := \{x \in X \,|\, |f(x)| = \infty\}$ then $X = A_0 \cup A \cup A_\infty$ is a disjoint union and

   $$\int_X |f| \, d\omega = \int_{A_0} |f| \, d\omega + \int_A |f| \, d\omega + \int_{A_\infty} |f| \, d\omega = \int_A |f| \, d\omega$$

   so it is sufficient to find $W \subseteq A$.

   With $f_n := |f| \circ \chi_{A_n}$ we get a sequence of non-negative measurable functions, which converge pointwise to $|f| \circ \chi_A$. As $A_n \subseteq A_{n+1}$, we have $0 \leq f_1(x) \leq f_2(x) \leq \ldots$ and by the monotone convergence theorem

   $$\lim_{n\to\infty} \int_X f_n \, d\omega = \lim_{n\to\infty} \int_{A_n} |f| \, d\omega = \int_A |f| \, d\omega = \int_X |f| \, d\omega \, .$$

   Hence there is some $N > 0$ for which

   $$\int_{X - A_N} |f| \, d\omega < \varepsilon$$

and since $1/N \leq |f| < N$ on $W := A_N$

$$\omega(W) \leq N \int_W |f|\, d\omega \leq N \int_X |f|\, d\omega < \infty\,.$$

3. Prove or find a counterexample to:

   (a) L^2 convergence implies pointwise convergence.

   (b) $\lim\limits_{n\to\infty} \displaystyle\int_0^\infty \frac{\sin x^n}{x^n}\, dx = 1$

   (c) Let (f_n) be a sequence of measurable functions which converge uniformly to zero on $[0,\infty)$. Then

$$\lim_{n\to\infty} \int_{[0,\infty)} f_n(x)\, dx = \int_{[0,\infty)} \lim_{n\to\infty} f_n(x)\, dx\,.$$

Reason: Convergence.

Solution: False - True - False.

(a) For each $k \in \mathbb{N}$ and $1 \leq i \leq k$ we define with the characteristic function $\chi(.)$ the functions $f_{k,i} := \chi\left(\left[\frac{i-1}{k}, \frac{i}{k}\right)\right)$ and the sequence (g_n) defined as

$$\begin{aligned} &g_1 = f_{1,1}\,,\\ &g_2 = f_{2,1}\,,\; g_3 = f_{2,2}\,,\\ &g_4 = f_{3,1}\,,\; g_5 = f_{3,2}\,,\; g_6 = f_{3,3}\\ &g_7 = f_{4,1}\,,\;\ldots \end{aligned}$$

Then $\int |f_{k,i}|^2\, d\mu = 1/k$ for each $1 \leq i \leq k$, i.e.

$$\lim_{k\to\infty} \|f_{k,i}\|_2 = \lim_{k\to\infty} \frac{1}{k} = 0 \Longrightarrow \lim_{n\to\infty} \|g_n\|_2 = 0$$

But (g_n) does not converge pointwise:

For every $N \in \mathbb{N}$ and every $x \in [0,1]$ there is a pair (k,i) such that $g_n(x) = f_{k,i}(x) = 1$ for all $n \geq N$, and we can find a pair (k',i') such that $g_{n'}(x) = f_{k',i'}(x) = 0$ for all $n' \geq N$.

(b) $\sin\varphi = \varphi - \dfrac{\varphi^3}{3!} + \dfrac{\varphi^5}{5!} \mp \ldots$ so for $0 < x < 1$ we get $\lim\limits_{n\to\infty} \dfrac{\sin x^n}{x^n} = 1$, and for $\varphi \geq 0$

$$|\sin\varphi| \leq \int_0^\varphi |\cos x|\, dx \leq \int_0^\varphi 1\, dx = \varphi\,.$$

For $\varphi < 0$ we have $|\sin\varphi| = |\sin(-\varphi)| \leq -\varphi = |\varphi|$.

In particular we have $\left|\dfrac{\sin x^n}{x^n}\right| \leq 1$ on $(0,1)$ and by the dominant convergence theorem

$$\lim_{n\to\infty}\int_0^1 \frac{\sin x^n}{x^n}\,dx = \int_0^1 \lim_{n\to\infty}\frac{\sin x^n}{x^n}\,dx = \int_0^1 1\,dx = 1$$

Substitute $u = x^n$ and $du = nx^{n-1}dx = nu^{1-\frac{1}{n}}dx$ for

$$\int_1^N \frac{\sin x^n}{x^n}\,dx = \int_1^{N^n}\frac{\sin u}{u}\,\frac{du}{nu^{1-\frac{1}{n}}} = \frac{1}{n}\int_1^{N^n}\frac{\sin u}{u^{2-\frac{1}{n}}}\,du$$

Now for $n \geq 2$

$$\begin{aligned}\left|\int_1^\infty \frac{\sin x^n}{x^n}\,dx\right| &= \lim_{N\to\infty}\frac{1}{n}\left|\int_1^{N^n}\frac{\sin u}{u^{2-\frac{1}{n}}}\,du\right| \leq \lim_{N\to\infty}\frac{1}{n}\,u^{\frac{1}{n}-2}\,du \\ &= \lim_{N\to\infty}\frac{1}{n}\,\frac{u^{\frac{1}{n}-1}}{\frac{1}{n}-1}\Bigg|_1^{N^n} = \frac{1}{1-n}\,\left(N^{1-n}-1\right) = \frac{1}{n-1}\end{aligned}$$

which implies

$$\lim_{n\to\infty}\int_1^\infty \frac{\sin x^n}{x^n}\,dx = 0$$

and thus

$$\lim_{n\to\infty}\int_0^\infty \frac{\sin x^n}{x^n}\,dx = \lim_{n\to\infty}\int_0^1 \frac{\sin x^n}{x^n}\,dx + \lim_{n\to\infty}\int_1^\infty \frac{\sin x^n}{x^n}\,dx = 1+0 = 1$$

(c) The sequence $f_n = \frac{1}{n}\chi([0,n)$ converges uniformly to zero, i.e. $\int \lim f_n = 0$. But $\int f_n = 1$ for all $n \in \mathbb{N}$, i.e. $\lim \int f_n = 1 \neq 0$.

4. Let (a_n) be a sequence of positive real numbers such that the series $\displaystyle\sum_{n=1}^\infty a_n =: C < \infty$ converges. Show that $\displaystyle\sum_{n=1}^\infty \left(\prod_{k=1}^n a_k\right)^{1/n} \leq e\cdot C.$

Reason: Carleman's inequality.

Solution: We denote the geometric and arithmetic means by

$$\operatorname{GM}(a_1,\ldots,a_n) = \left(\prod_{k=1}^n a_k\right)^{1/n} \;<\; \operatorname{AM}(a_1,\ldots,a_n) = \frac{a_1+\ldots+a_n}{n}\,.$$

We first show $e \geq (n+1)(n!)^{-1/n}$. From Stirling's formula we get

$$\begin{aligned}\sqrt{2\pi}n^{n+1/2}e^{-n} &\leq n!\\ \sqrt[n]{2\pi}\cdot\frac{n}{e}\cdot\sqrt[2n]{n} &\leq \sqrt[n]{n!}\\ \frac{1}{\sqrt[n]{n!}} &\leq \frac{1}{\sqrt[n]{2\pi}\cdot\sqrt[2n]{n}}\cdot\frac{e}{n}\end{aligned}$$

and

$$\begin{aligned}\left(1+\frac{1}{n}\right)^{2n} &\leq e^2 \leq n\cdot 4\pi^2\\ 1+\frac{1}{n} &\leq \sqrt[n]{2\pi}\cdot\sqrt[2n]{n}\\ \frac{1}{n\cdot\sqrt[n]{2\pi}\cdot\sqrt[2n]{n}} &\leq \frac{1}{n+1}\end{aligned}$$

Hence combining both we get

$$\frac{1}{\sqrt[n]{n!}} \leq \frac{e}{n+1}\,.$$

With the notation above we have

$$\begin{aligned}\operatorname{GM}(a_1,\ldots,a_n) &= \operatorname{GM}(a_1,2a_2,\ldots,na_n)(n!)^{-1/n}\\ &\leq \operatorname{AM}(a_1,2a_2,\ldots,na_n)(n!)^{-1/n}\\ &\leq \frac{e}{n(n+1)}\sum_{k=1}^{n}ka_k\end{aligned}$$

Thus

$$\sum_{n=1}^{\infty}\operatorname{GM}(a_1,\ldots,a_n) \leq e\sum_{k=1}^{\infty}\underbrace{\left(\sum_{n=k}^{\infty}\frac{k}{n(n+1)}\right)}_{\stackrel{(*)}{=}1}a_k = e\sum_{k=1}^{\infty}a_k = e\cdot C$$

If the inequality wasn't a strict one, then

$$\operatorname{GM}(a_1,2a_2,\ldots,na_n) = \operatorname{AM}(a_1,2a_2,\ldots,na_n) \Longrightarrow a_k = \frac{a_1}{k}$$

but the harmonic series is divergent.

$$(*)\quad \begin{aligned}\sum_{n=k}^{m}\frac{k}{n(n+1)} &= \sum_{n=1}^{m}\frac{k}{n(n+1)} - \sum_{n=1}^{k-1}\frac{k}{n(n+1)}\\ &= k\cdot\left(\frac{m}{m+1}-\frac{k-1}{k}\right) = 1-\frac{k}{m+1} \stackrel{m\to\infty}{\longrightarrow} 1\end{aligned}$$

5. Let $(E, \mathcal{T})$ be a normal Hausdorff space, and $U_1, \ldots, U_n$ a finite open covering of E. Then there are continuous functions $g_1, \ldots, g_n : (E, \mathcal{T}) \longrightarrow [0,1]$ such that $g_1 + \ldots + g_n = 1$ on E and $g_j(E - U_j) = \{0\}$ for all $1 \leq j \leq n$.

   Reason: Important Topological Result: Partition of Unity.

   Solution: We first show that there are n closed subsets $F_1, \ldots, F_n \subseteq E$, such that $F_j \subseteq U_j$ for all $1 \leq j \leq n$ and $F_1 \cup \ldots \cup F_n = E$.

   The set $G_1 := E - (U_2 \cup \ldots \cup U_n) \subseteq U_1$ is closed, and we can find an open set V_1 such that $G_1 \subseteq V_1 \subseteq \overline{V}_1 \subseteq U_1$ where $V_1 \cup U_2 \ldots \cup U_n = E$ is still a finite open covering. Now we proceed by setting $G_2 := E - (V_1 \cup U_3 \ldots \cup U_n) \subseteq U_2$ which is closed, i.e. again we find an open set V_2 such that $G_2 \subseteq V_2 \subseteq \overline{V}_2 \subseteq U_2$ where $V_1 \cup V_2 \cup U_3 \ldots \cup U_n = E$ is still a finite open covering. Iteration up to n yields the closed sets $F_1 := \overline{V}_1, \ldots, F_n := \overline{V}_n$, such that $E = V_1 \cup \ldots \cup V_n$ is still a finite open covering with $F_j \subseteq U_j$ for all $1 \leq j \leq n$.

   Since F_j and $E - U_j$ are disjoint closed sets, we may apply Urysohn's lemma and find continuous functions $f_j : E \longrightarrow [0,1]$ such that $f_j = 1$ on F_j and $f_j = 0$ on $E - U_j$ for all $1 \leq j \leq n$. We finally define $g_1 := f_1\,,\, g_2 := f_2 \cdot (1 - f_1)\,, \ldots ,\, g_n := f_n \cdot (1 - f_{n-1}) \cdot \ldots \cdot (1 - f_1)$. With these functions we get $g_j(E - U_j) = \{0\}$ for all $1 \leq j \leq n$ and $1 - (g_1 + \ldots + g_j) = (1 - f_1) \cdot \ldots \cdot (1 - f_j)$. The case $j = n$ finishes the proof, since the A_j are a (closed) covering of E.

6. Let (X, Ω, ω) be a measure space and $1 \leq p < \infty$. Show that

   (a) $\tilde{L}^p := L^p(X, \Omega, \omega)$ is a Banach space with respect to $\|\cdot\|_p$.

   (b) The sequence $(\|f_n\|_p) \subseteq \mathbb{R}$ is bounded for every Cauchy sequence $(f_n) \subseteq L^p(X, \Omega, \omega)$.

   Reason: Functional Analysis.

   Solution:

   (a) Let $(f_n)_{n\in\mathbb{N}} \subseteq \tilde{L}^p$ be a Cauchy sequence, i.e. for every $\varepsilon > 0$ there is a N_ε such that $\|f_n - f_m\|_p < \varepsilon$ for all $n, m \geq N_\varepsilon$. Thus we have a subsequence $(f_{n_k})_{k\in\mathbb{N}} \subseteq (f_n)_{n\in\mathbb{N}}$ such that $\|f_{n_k} - f_m\|_p < 2^{-k}$ for all $m \geq n_k$. If we define $g_k := f_{n_k} - f_{n_{k+1}}$, then

$$\left\| \sum_{k=1}^{n} |g_k| \right\|_p \leq \sum_{k=1}^{n} \|g_k\|_p \leq \sum_{k=1}^{n} \frac{1}{2^k} < 1$$

for all $1 \leq n < \infty$, so the sequence of partial sums is convergent if it is bounded by the theorem of monotone convergence:

$$\left\| \sum_{k=1}^{\infty} |g_k| \right\|_p \leq \lim_{n\to\infty} \sum_{k=1}^{n} \|g_k\|_p \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

Hence the sequence $(\sum_{k=1}^{n} g_k)_{n\in\mathbb{N}}$ converges $\omega-$almost everywhere (a.e.) absolutely, and $f_{n_1} - f_{n_k} = \sum_{j=1}^{n_{k-1}} g_j$ converges $\omega-$a.e. for $k \to \infty$. So $f_{n_k} = f_{n_1} + \sum_{j=1}^{n_{k-1}} g_j$ converges $\omega-$a.e. for $k \to \infty$ to a function $f = f_{n_1} + \sum_{j=1}^{\infty} g_j$. We have thus found a convergent subsequence and it remains to show that $f \in \tilde{L}^p$, i.e. $\|f\|_p < \infty$, and $\lim_{n\to\infty} \|f_n - f\|_p = 0$.

For $\varepsilon > 0$ we choose $N_\varepsilon \in \mathbb{N}$ such that $\|f_{n_k} - f_m\|_p < \varepsilon$ for all $n_k, m \geq N_\varepsilon$ and apply Fatou's Lemma on the sequence $(|f_{n_k} - f_m|^p)_{k\in\mathbb{N}}$ and get for all $m \geq N_\varepsilon$

$$\int |f-f_m|^p d\omega = \int \liminf_{k\to\infty} |f_{n_k}-f_m|^p d\omega = \liminf_{k\to\infty} \int |f_{n_k}-f_m|^p d\omega \leq \varepsilon^p$$

hence $\lim_{m\to\infty} \|f_m - f\|_p = 0$ and $\|f\|_p \leq \|f - f_{n_1}\|_p + \|f_{n_1}\|_p < \infty$, i.e. $f \in \tilde{L}_p$.

(b) This follows immediately from the previous part, since there is a $N_0 \in \mathbb{N}$ such that $\|f_n - f\|_p \leq C < \infty$ for all $n \geq N_0$ and $\|f_n\|_p \leq \|f_n - f\|_p + \|f\|_p \leq C + \|f\|_p < \infty$.

7. We know that there are only two sets in $(\mathbb{R}, |\cdot|)$, which are open and closed, the empty set and the entire topological space. Prove it.

Reason: Open And Closed Sets.

Solution: Let $\emptyset \neq M \subsetneq \mathbb{R}$ be open and closed, and $x \in \mathbb{R} - M, y \in M$. W.l.o.g. we may assume that $x > y$. Then x is an upper bound of

$$N := \{z \in \mathbb{R} \,|\, [y, z] \subseteq M\}$$

and the real number $s := \sup N$ exists. Then $s - 1/n$ is no upper bound for all $n \in \mathbb{N}$ and we can find numbers $z_n \in N$ such that $s - 1/n < z_n \leq s$, i.e. $\lim_{n\to\infty} z_n = s$. Since $N \subseteq M$ we have $z_n \in M$, and since M is closed, $s \in M$. Now M is open as well, so there is a $r > 0$ such that

$$[s - r, s + r] \subseteq M.$$

From $z_n \in N$, i.e. $[y, z_n] \subseteq M$, and $\lim_{n\to\infty} z_n = s$ we get $[y, s+r] \subseteq M$, i.e. $s + r \in N$. However, s is an upper bound of N and there cannot be an element greater than s in N. This means that one of our assumptions was wrong and either $M=\emptyset$ or $\mathbb{R}-M=\emptyset$.

8. Let (V,α) and (W,β) be irreducible representations of an associative, complex algebra $\mathcal{A}$. Assume that V and W are complex and of countable dimension. Then

$$\dim \operatorname{Hom}_{\mathcal{A}}(V,W)=\begin{cases}1 & \text{if } (V,\alpha)\cong(W,\beta)\\ 0 & \text{otherwise}\end{cases}$$

Reason: Schur's Lemma.

Solution: Given a $\mathcal{A}-$homomorphism $\varphi : V \longrightarrow W$ kernel and range are invariant subspaces of V,W, resp. If $\varphi\neq 0$ then $\ker\varphi\neq V$ and $\operatorname{range}\varphi\neq\{0\}$. By irreducibility we get $\ker\varphi=\{0\}$ and $\operatorname{range}\varphi=V$, which means that φ is a linear isomorphism. Hence $\operatorname{Hom}_{\mathcal{A}}(V,W)\neq\{0\}$ if and only if $(V,\alpha)\cong(W,\beta)$.

Next we have to show that $\operatorname{Hom}_{\mathcal{A}}(V,W)$ is one dimensional in case the representations are equivalent, which means that the only possible homomorphism is a multiple of the identity operator. Let $\varphi,\psi\in \operatorname{Hom}_{\mathcal{A}}(V,W)-\{0\}$ and $\rho:=\psi^{-1}\varphi\in\operatorname{End}_{\mathcal{A}}(V)$. Assume further that $\rho\notin\mathbb{C}\cdot\operatorname{Id}_V$. This means that $\rho-\lambda I$ is non zero for any $\lambda\in\mathbb{C}$ and thus invertible. We will show that the set

$$\{(\rho-\lambda_k I)^{-1}(v)\,|\,1\leq k\leq m\}$$

is linear independent for any $v\in V-\{0\}$ and pairwise distinct complex numbers $\lambda_1,\ldots,\lambda_m$, which contradicts the countable dimensionality of V.

Let

$$\sum_{k=1}^m c_k(\rho-\lambda_k I)^{-1}(v)=0 \text{ and } f(x):=\sum_{k=1}^m c_k\prod_{l\neq k}(x-\lambda_l I)$$

Then

$$f(\rho)(v)=\sum_{k=1}^m c_k\prod_{l\neq k}(\rho-\lambda_l I)(v)=\prod_{l=1}^m(\rho-\lambda_l I)\cdot\sum_{k=1}^m c_k(\rho-\lambda_k)^{-1}(v)=0$$

Now $f(\lambda_j)=c_j\prod_{l\neq j}(\lambda_j-\lambda_l)$. If $c_j\neq 0$, then $f(x)$ is a nonzero polynomial and has a factorization $f(x)=c(x-z_1)\ldots(x-z_{m-1})$ with $c\neq 0$ and $z_i\in\mathbb{C}$. We know that all $\rho-z_iI$ are invertible and so is $f(\rho)$, in which case $f(\rho)(v)$ cannot be zero for $v\neq 0$. Hence $c_j=0$ for all $1\leq j\leq m$.

9. Let $U \subseteq \mathbb{C}$ be an open connected neighborhood and $f : U \longrightarrow \mathbb{C}$ a holomorphic function. If $|f|$ has a local maximum in $z_0 \in U$, i.e. there is an open neighborhood $z_0 \in U_0 \subseteq U$ with $|f(z_0)| \geq |f(z)|$ for all $z \in U_0$, then f is constant.

   Reason: Maximum Principle.

   Solution: Since f is holomorphic, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

   f as a power series in a neighborhood of z_0 (**Cauchy-Taylor**). Assume s is its radius of convergence and $0 < r < s$. Note that for $m, n \in \mathbb{N}_0$

$$\int_0^{2\pi} e^{it(n-m)}\,dt = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}$$

   so uniform convergence and $f(z_0 + re^{it}) = \sum_{n\in\mathbb{N}_0} a_n r^n e^{itn}$ yields

$$\int_0^{2\pi} f(z_0 + re^{it})e^{-itm}\,dt = \sum_{n\in\mathbb{N}_0} a_n r^n \int_0^{2\pi} e^{it(n-m)}\,dt = 2\pi a_m r^m$$

   From

$$|f(z_0 + re^{it})|^2 = f(z_0 + re^{it})\overline{f(z_0 + re^{it})} = \sum_{n\in\mathbb{N}_0} \overline{a}_n r^n f(z_0 + re^{it})e^{-itn}$$

   we get (again with uniform convergence) the **Gutzmersche formula**

$$\begin{aligned} \int_0^{2\pi} |f(z_0 + re^{it})|^2\,dt &= \sum_{n=0}^{\infty} \overline{a}_n r^n \int_0^{2\pi} f(z_0 + re^{it})e^{-itn}\,dt \\ &= 2\pi \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \\ &\leq 2\pi(\max\{|f(z)| \,:\, |z - z_0| = r\})^2 \end{aligned}$$

   From $|f(z_0)| \geq |f(z)|$ we get for small $r > 0$

$$\begin{aligned} (\max\{|f(z)| \,:\, |z - z_0| = r\})^2 + \sum_{n=1}^{\infty} |a_n|^2 r^{2n} &\leq |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \\ &\leq (\max\{|f(z)| \,:\, |z - z_0| = r\})^2 \\ &\Longrightarrow \end{aligned}$$

$$a_n = 0 \quad (n \geq 1) \quad \wedge \quad f(z) = a_0 \quad (|z - z_0| < r)$$

This means that the set $\{z \in U \,:\, f(z) = g(z)\}$ with $g(z) := a_0$ has a limit point in U. Since U is connected we can apply the **identity theorem** $(*)$ and conclude

$$f(z) \equiv g(z) \equiv a_0 \quad (z \in U)$$

Proof of the identity theorem $(*)$:

Let z_1 be a limit point of the coincidence set $V := \{z \in U \,:\, f(z) = g(z)\}$. For the sake of simplicity in notation we choose $z_1 = 0$. Now assume that there is a natural number $n \in \mathbb{N}_0$ such that $f^{(n)}(0) \neq g^{(n)}(0)$ and that n is the smallest among them. Then we have in a neighborhood of $z = 0$

$$f(z) - g(z) = z^n \underbrace{\sum_{k=0}^{\infty} \frac{f^{(n+k)}(0) - g^{(n+k)}(0)}{(n+k)!}\, z^k}_{=:h(z)}$$

and $V = \{z \,:\, h(z) = 0\}$ since h is continuous. In particular we have $0 = h(0) = \dfrac{f^{(n)} - g^{(n)}}{n!}$ contradicting minimality and choice of n. This means that $f^{(n)}(z_1) = g^{(n)}(z_1)$ for all $n \in \mathbb{N}_0$.

Because U is connected, it is sufficient to show that

$$A := \{z \in U \,:\, f^{(n)}(z) = g^{(n)}(z) \,\forall\, n \in \mathbb{N}_0\}$$

is nonempty, open and closed in U, in order to conclude $f = g$ on U. With $z_1 \in A$ we have $A \neq \emptyset$ and by

$$A_n := \{z \in U \,:\, f^{(n)}(z) = g^{(n)}(z)\} = (f^{(n)} - g^{(n)})^{-1}(\{0\})$$

we see that $A = \cap_{n\in\mathbb{N}_0} A_n$ is closed. If $z \in A$ then the holomorphic function $f - g$ is identical to its Taylor series in a neighborhood of z (**Cauchy-Taylor**), i.e. identical zero. However, this neighborhood is part of A, hence A is also open.

10. Show that all groups G of order pqr with pairwise distinct primes $p < q < r$ are solvable.

    Reason: Groups of Order pqr.

    Solution: The number of $r-$Sylow-subgroups of G is congruent 1 modulo r and a divisor of pq according to the Sylow theorems, i.e. it equals either 1 or pq. If there is only one $r-$Sylow-subgroup H of G, then H is a normal subgroup of prime order, thus cyclic and solvable. But G/H is a group of order pq, which are solvable as well (see problem no. 10 from Dec. 2018). Thus G is solvable.

Now we assume that there are pq many $r-$Sylow-subgroups of G. The number of elements in an intersection of two different $r-$Sylow-subgroups is a proper divisor of the prime r, hence 1. Every nontrivial element of a $r-$Sylow-subgroup is of order r, and all elements of order r in G are contained in one $r-$Sylow-subgroup. Thus G contains exactly $pq(r-1)$ elements of order r. The number m of $q-$Sylow-subgroups of G is congruent 1 modulo q and a divisor of pr. The only possibilities are $1, r, pr$, since $1 < p < q < r$. $(1 < p = m \cdot q + 1 < q \Longrightarrow m = 0$ ↯ $)$

If $m = 1$ then this $q-$Sylow-subgroup H is normal, of prime order q, hence cyclic and thus solvable, and as $|G/H| = pr$, G/H is solvable as well (see problem no. 10 from Dec. 2018), hence G is solvable. If $m \in \{r, pr\}$ then there are exactly $r(q-1)$, or $pr(q-1)$ resp. many elements of order q according to the same argument as above. Now we have $pq(r-1)$ elements of order r and at least $r(q-1)$ many elements of order q, i.e.

$$|G| = pqr \geq pq(r-1) + r(q-1) \geq pq(r-1) + rp = pqr + p(r-q) > pqr$$

which is not possible. So the cases $m \in \{r, pr\}$ do not exist and G is solvable.

11. (HS-1) Let z be a natural number with 1995 decimal digits and $1 \leq n \leq 1994$. Then we note the number, which we get by cutting off the first n digits and append them in the same order at the end of z by $z^{[n]}$. Show that if z is divisible by 27, then all $z^{[n]}$ are divisible by 27, too.

    Reason: Puzzle with 1995.

    Solution: It is sufficient to show the statement for $n = 1$, because if it is true for $n = 1$ we can repeat the process as long as we need for any $1 \leq n \leq 1994$. Let a be the first digit of z, i.e. $z = a \cdot 10^{1994} + b$ and $z^{[1]} = 10b + a$ for some $b \in \mathbb{N}$.

$$10z - z^{[1]} = a \cdot (10^{1995} - 1) = a \cdot (1000^{665} - 1)$$

    From $1000 = 37 \cdot 27 + 1$ we get $1000^{665} \equiv 1^{665} \equiv 1 \mod 27$. Hence $27 \,|\, (10z - z^{[1]})$ and if $27 \,|\, z$ then $27 \,|\, z^{[1]} = 10z - (10z - z^{[1]})$.

12. (HS-2) Let a, b, c, d be positive real numbers. Prove (in the logically correct order)
$$\frac{1}{\frac{1}{a}+\frac{1}{b}}+\frac{1}{\frac{1}{c}+\frac{1}{d}} \leq \frac{1}{\frac{1}{a+c}+\frac{1}{b+d}}$$
Reason: Inequality.

Solution:
$$\begin{aligned}
0 &\leq (ad-bc)^2 \\
2abcd &\leq a^2d^2+b^2c^2 \|_{+ab(c^2+d^2)+cd(a^2+2ab+b^2)} \\
ab(c+d)^2+cd(a+b)^2 &\leq (ad+bc)(ac+ad+bc+bd) \|_{+(ab+cd)(a+b)(c+d)} \\
ab(c+d)(a+b+c+d)+cd(a+b)(a+b+c+d) & \\
&\leq (a+c)(b+d)(a+b)(c+d) \|_{:[(a+b)(c+d)(a+b+c+d)]} \\
\frac{ab}{a+b}+\frac{cd}{c+d} &\leq \frac{(a+c)(b+d)}{a+b+c+d} \\
\frac{1}{\frac{1}{a}+\frac{1}{b}}+\frac{1}{\frac{1}{c}+\frac{1}{d}} &\leq \frac{1}{\frac{1}{a+c}+\frac{1}{b+d}}
\end{aligned}$$

13. (HS-3) Let $m \geq 2$ be a given natural number. We define a sequence $(x_0, x_1, x_2, \ldots)$ of numbers by $x_0 = 0, x_1 = 1$, and for $n \geq 0$ we set x_{n+2} to be the remainder of $x_{n+1} + x_n$ by division by m, chosen such that $0 \leq x_{n+2} < m$. Decide whether for every $m \geq 2$ there exists a natural number $k \geq 1$, such that $x_{k+2} = 1\,,\ x_{k+1} = 1\,,\ x_k = 0$.

Reason: Sequence.

Solution: There are at most m^3 possible triplets $(x_j, x_{j+1}, x_{j+2})_{j\in\mathbb{N}} \in \{0, 1, \ldots, m-1\}^3$ so there have to be repetitions. Hence there are $j \geq 0, k \geq 1$ with
$$x_j = x_{k+j}\,,\ x_{j+1} = x_{k+j+1}\,,\ x_{j+2} = x_{k+j+2}$$
Assume $j > 0$. Per construction we know that $x_{j+1} \equiv x_j + x_{j-1} \mod m$ and $x_{k+j+1} \equiv x_{k+j} + x_{k+j-1} \mod m$, hence subtraction yields
$$\begin{aligned}
&x_{j+1} - x_{k+j+1} = 0 = x_j + x_{j-1} - x_{k+j} - x_{k+j-1} \equiv x_{j-1} - x_{k+j-1} \mod m \\
&\qquad \Longrightarrow m \,|\, (x_{j-1} - x_{k+j-1})
\end{aligned}$$
which is only possible if $x_{j-1} = x_{k+j-1}$ for a number between 0 and $m-1$. We can repeat this argument until
$$0 = x_0 = x_k\ ,\ 1 = x_1 = x_{k+1}\ ,\ 1 = x_2 = x_{k+2}$$

14. (HS-4) We define real functions

$$f_n(x) := x^3 + (n+3)\cdot x^2 + 2n\cdot x - \frac{n}{n+1}$$

for every non-negative integer $n \geq 0$. Determine all values of n, such that all zeros of $f_n(x)$ are contained in an interval of length 3.

Reason: Polynomial.

Solution: For $n=0$ we have $f_0(x) = x^3 + 3x^2 = x^2(x-(-3))$ with the zeros $0, -3 \in [-3,0]$. Now assume $n > 0$. Here we have

$$\begin{aligned} f_n(-n-3) &= 2n(-n-3) - \frac{n}{n+1} = -2n^2 - 6n - \frac{n}{n+1} < 0 \\ f_n(-2) &= -8 + 4(n+3) - 4n - \frac{n}{+n+1} = 4 - \frac{n}{n+1} > 0 \\ f_n(0) &= -\frac{n}{n+1} < 0 \\ f_n(1) &= 1 + (n+3) + 2n - \frac{n}{n+1} = 3n + 4 - \frac{n}{n+1} > 0 \end{aligned}$$

hence all f_n have three pairwise distinct real zeros, say $a < b < c$. Vieta's formulas are thus

$$a+b+c = -(n+3)\,,\; ab+ac+bc = 2n\,,\; abc = \frac{n}{n+1}$$

Now

$$\begin{aligned} (c-a)^2 &= (a+b+c)^2 - 3(ab+ac+bc) + \underbrace{(c-b)(b-a)}_{>0} \\ &> (n+3)^2 - 6n = n^2 + 9 > 9 \Longrightarrow c - a > 3 \end{aligned}$$

Thus we have only for $n=0$ that all zeros of f_0 are within a distance of three, whereas they are further apart for all other f_n $(n \geq 1)$.

15. (HS-5)

   (a) Determine the number of all pairs of integers $(x,y) \in \mathbb{N}_0^2$ with $\sqrt{x} + \sqrt{y} = 1993$.

   (b) Determine for every $n \in \mathbb{N}$ the greatest power of 2 which divides $[(4+\sqrt{18})^n]$.

Reason: Calculus.

Solution:

(a) From $\sqrt{x}+\sqrt{y}=1993$ we get

$$\begin{aligned} y &= (1993-\sqrt{x})^2 = 1993^2 - 3986\sqrt{x} + x \\ \sqrt{x} &= \frac{1993^2 + x - y}{3986} \in \mathbb{Q} \\ x &= \left(\frac{u}{v}\right)^2 \text{ for some } u,v \in \mathbb{N}_0 \\ u^2 &= v^2 \cdot x \end{aligned}$$

Because u^2, v^2 have an even number of primes, so does x, i.e. $x = a^2$ for some $a \in \mathbb{N}_0$. For the same reason is $y = b^2$ for some $b \in \mathbb{N}_0$, hence combined: $a + b = 1993$.

On the other hand are any two integers $a, b \geq 0$ for which $a+b = 1993$ holds, a solution to $\sqrt{x}+\sqrt{y}=1993$ with $x = a^2, y = b^2$.

Thus we have shown that there are as many integer solutions as there are pairs (a, b), which are the following

$$(0,1993)\,,\,(1,1992)\,,\,\ldots\,,\,(1993,0)$$

1994 possible pairs.

(b) For the sequences $(a_n)_{n\in\mathbb{N}_0}, (b_n)_{n\in\mathbb{N}_0}, (c_n)_{n\in\mathbb{N}_0}$ defined by

$$a_n := (4+\sqrt{18})^n\,,\; b_n := (4-\sqrt{18})^n\,,\; c_n := a_n + b_n \in \mathbb{Z}$$

we have the following recursions

$$\begin{aligned} a_{n+2} &= a_n(4+\sqrt{18})^2 = a_n(34+8\sqrt{18}) \\ &= 2a_n(4(4+\sqrt{18})+1) = 2\cdot(4a_{n+1}+a_n) \\ b_{n+2} &= b_n(4-\sqrt{18})^2 = b_n(34-8\sqrt{18}) \\ &= 2b_n(4(4-\sqrt{18})+1) = 2\cdot(4b_{n+1}+b_n) \\ c_{n+2} &= 2\cdot(4c_{n+1}+c_n) \in \mathbb{Z} \end{aligned}$$

with $c_0 = 2\,,\, c_1 = 8$. Next we prove that for all integers c_n holds:

$$S(k) := \begin{cases} 2^{k+1} \,|\, c_{2k} & \wedge \quad 2^{k+2} \nmid c_{2k} \\ 2^{k+3} \,|\, c_{2k+1} & \wedge \quad 2^{k+4} \nmid c_{2k+1} \end{cases}$$

We know already that $S(0)$ is true. Now assume $S(k)$ is also true. Then there are odd integers s, t such that $c_{2k} = s \cdot 2^{k+1}\,,\, c_{2k+1} =$

$t \cdot 2^{k+3}$. Thus

$$\begin{aligned} c_{2k+2} &= 2 \cdot (4c_{2k+1} + c_{2k}) = 2 \cdot (4(t \cdot 2^{k+3}) + (s \cdot 2^{k+1})) \\ &= t \cdot 2^{k+6} + s \cdot 2^{k+2} \equiv \begin{cases} 0 & \mod 2^{(k+1)+1} \\ 1 & \mod 2^{(k+1)+2} \end{cases} \\ c_{2k+3} &= 2 \cdot (4c_{2k+2} + c_{2k+1}) \\ &= 2 \cdot (4(t \cdot 2^{k+6} + s \cdot 2^{k+2}) + t \cdot 2^{k+3}) \\ &= t \cdot (2^{k+9} + 2^{k+5}) + s \cdot 2^{k+4} \\ &\equiv \begin{cases} 0 & \mod 2^{(k+1)+3} \\ 1 & \mod 2^{(k+1)+4} \end{cases} \end{aligned}$$

which proves $S(k+1)$ and the truth of the statement by induction.

$$\begin{aligned} & 4 < \sqrt{18} < 5 \\ & \Longrightarrow -1 < 4 - \sqrt{18} < 0 \\ & \Longrightarrow \begin{cases} 0 < b_n < 1 & \text{for } n = 2k \\ -1 < b_n < 0 & \text{for } n = 2k+1 \end{cases} \\ & \Longrightarrow \begin{cases} a_n < c_n < a_n + 1 & \text{for } n = 2k \\ a_n - 1 < c_n < a_n & \text{for } n = 2k+1 \end{cases} \\ & \Longrightarrow \begin{cases} c_n - 1 < a_n < c_n & \text{for } n = 2k \\ c_n < a_n < c_n + 1 & \text{for } n = 2k+1 \end{cases} \\ & \Longrightarrow \begin{cases} [a_n] = c_n - 1 & \text{for } n = 2k \\ [a_n] = c_n & \text{for } n = 2k+1 \end{cases} \end{aligned}$$

The greatest power of 2 which divides $[a_n] = [(4 + \sqrt{18})^n]$ is thus for even n according to $S(2k)$ the number 0, because a_n is odd, and for odd n according to $S(2k+1)$ the number $2^{k+3} = 2^{(n+5)/2}$.

12 January 2021

1. Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. Then exactly one of the following two statements is true:

   - $Ax = b\,,\, x \geq 0\,,$ is solvable for a $x \in \mathbb{R}^n$.
   - $A^\tau y \leq 0\,,\, b^\tau y > 0\,,$ is solvable for some $y \in \mathbb{R}^m$.

   The ordering is meant componentwise.

   Reason: Farkas Lemma.

   Solution:

   Both statements cannot be simultaneously true, as

   $$0 < y^\tau b = y^\tau (Ax) = (y^\tau A)x = (A^\tau y)^\tau x \leq 0$$

   The statement can be proven with the **strict separation theorem**:

   Let $K \subseteq \mathbb{R}^n$ be convex, nonempty, closed, and $x \notin K$. Then there is a hyperplane $H = \{y \in \mathbb{R}^n \,|\, a^\tau y = \gamma\}$ with $a \in \mathbb{R}^n - \{0\}\,,\, \gamma \in \mathbb{R}$, which separates x and K, i.e. $a^\tau z \leq \gamma < a^\tau x$ for all $z \in K$. Moreover if K is additionally a cone, then we may choose $\gamma = 0$.

   or the **strong duality theorem**:

   Let $z = c^\tau x \longrightarrow \min!$ with $Ax = b\,,\, x \geq 0$ for $c, x \in \mathbb{R}^n$. the primal optimization problem, and $\tilde{z} := b^\tau y \longrightarrow \max!$ with $A^\tau y \leq c$ for $c \in \mathbb{R}^n\,,\, y \in \mathbb{R}^m$ its dual problem.

   The primal problem has a finite optimal solution if and only if its dual problem has a finite optimal solution, in which case $z_{\min} = \tilde{z}_{\max}\,.$

   (a) (strict separation theorem)

   Assume that the first statement is false. Then $b \notin K := \{Ax \,|\, x \in \mathbb{R}^n, x \geq 0\}$, which is a convex, polyhedral, closed cone. Thus we can separate b and K, i.e. there is a vector $y \in \mathbb{R}^m - \{0\}$ such that

   $$y^\tau Ax \leq 0 < y^\tau b$$

   If we choose subsequently all unit vectors $x = (0, \ldots, 0, 1, 0, \ldots, 0)$ then $A^\tau y \leq 0$ which had to be shown.

   (b) (strong duality theorem)

   Set $M := \{x \in \mathbb{R}^n \,|\, Ax = b, x \geq 0\}$. The statement can thus be rephrased by

$$M \neq \emptyset \Longleftrightarrow \forall\, y \in \mathbb{R}^m \,(A^\tau y \leq 0 \Longrightarrow b^\tau y \leq 0)$$

"$\Longrightarrow$":

If $M \neq \emptyset$ then there is a solution $x^* \in M$ to the primal problem

$$c^\tau x \;(x \in M) \longrightarrow \min! \quad (P)$$

with $c = 0$. This means by the strict duality theorem, that the dual problem

$$b^\tau y \;(y \in N := \{y \in \mathbb{R}^m \,|\, A^\tau y \leq c = 0\}) \longrightarrow \max! \quad (D)$$

has a solution $y^* \in N$, too, since $0 \in N$, i.e. $N \neq \emptyset$. Moreover $\min(P) = \max(D)$ and so for all $y \in \mathbb{R}^m$ with $A^\tau y \leq 0$

$$b^\tau y \leq b^\tau y^* = \max(D) = \min(P) = 0^\tau x^* = 0$$

"$\Longleftarrow$":

We consider again the dual problem (D), which is feasible since $0 \in N$ and $b^\tau y \leq 0$ for all $y \in N$. Hence $\sup(D) < \infty$. Assume that $M = \emptyset$. By the strong duality theorem we then had $\sup(D) = \infty$, a contradiction, so M cannot be empty.

2. Prove $\pi = \lim\limits_{n\to\infty} 2^n \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}}_{n \text{ square roots}}.$

Reason: Viète's formula.

Solution: We first prove that

$$\sin x = 2^n \sin \frac{x}{2^n} \left(\prod_{k=1}^{n} \cos \frac{x}{2^k} \right)$$

This is a simple induction on n. For $n = 1$ we have the known formula for half angles

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

and the induction step is

$$\begin{aligned}
2^{n+1}\sin\frac{x}{2^{n+1}}&\left(\prod_{k=1}^{n+1}\cos\frac{x}{2^k}\right)\\
&=2\cdot 2^n\sin\left(\frac{1}{2}\cdot\frac{x}{2^n}\right)\cos\left(\frac{1}{2}\cdot\frac{x}{2^n}\right)\left(\prod_{k=1}^{n}\cos\frac{x}{2^k}\right)\\
&=2^n\sin\frac{x}{2^n}\left(\prod_{k=1}^{n}\cos\frac{x}{2^k}\right)\\
&=\sin x
\end{aligned}$$

From the Taylor expansion of the exponential formula and Euler's formula $e^{\mathbf{i}x}=\cos x+\mathbf{i}\sin x$ we get the series expansion of the sine function $\sin x = x-\dfrac{x^3}{3!}\pm\ldots$ which shows that

$$\sin x=\lim_{n\to\infty}\sin x=\lim_{n\to\infty}\left(2^n\sin\frac{x}{2^n}\right)\cdot\lim_{n\to\infty}\prod_{k=1}^{n}\cos\frac{x}{2^k}=x\cdot\prod_{k=1}^{\infty}\cos\frac{x}{2^k}$$

For $x=\pi/2$ we get the formula

$$\frac{2}{\pi}=\prod_{k=1}^{\infty}\cos\frac{\pi}{2^{k+1}}$$

or

$$\pi=2\cdot\prod_{k=1}^{\infty}\left(\cos\frac{\pi}{2^{k+1}}\right)^{-1}$$

The half angle formula for the cosine function is

$$\begin{aligned}
\cos(2x)&=\cos^2 x-\sin^2 x=1-2\sin^2 x=2\cos^2 x-1\\
&\Longrightarrow\cos\frac{x}{2}=\sqrt{\frac{1}{2}+\frac{1}{2}\cdot\cos x}\Longrightarrow\left(\cos\frac{x}{2}\right)^{-1}=\sqrt{\frac{2}{1+\cos x}}
\end{aligned}$$

Set $a_0=0\,,\ a_n=\sqrt{\dfrac{1}{2}+\dfrac{1}{2}\cdot a_{n-1}}\,,\ 2a_n^2=1+a_{n-1}\quad(n\geq 1)$. We show

$$\prod_{k=1}^{n}\frac{1}{a_k}=2^n\sqrt{1-a_n^2}$$

which is true for $n = 0, 1$.

$$\begin{aligned}\prod_{k=1}^{n+1} \frac{1}{a_k} &= 2^n \sqrt{1-a_n^2} \cdot \frac{1}{a_{n+1}} = 2^n \sqrt{1-a_n^2} \cdot \sqrt{\frac{2}{1+a_n}} \\ &= 2^n \sqrt{2(1-a_n)} = 2^{n+1} \sqrt{1 - \left(\frac{1}{2} + \frac{1}{2} a_n\right)} \\ &= 2^{n+1} \sqrt{1 - a_{n+1}^2}\end{aligned}$$

Note that $\cos \dfrac{\pi}{2^{n+1}} = a_n$ since

$$\begin{aligned} a_0 &= 0 = \cos \frac{\pi}{2} \\ a_1 &= \sqrt{\frac{1}{2}} = \cos \frac{\pi}{4} \\ a_{n+1} &= \sqrt{\frac{1}{2} + \frac{1}{2} \cdot a_n} = \sqrt{\frac{1}{2} + \frac{1}{2} \cdot \cos \frac{\pi}{2^{n+1}}} = \cos \frac{\pi}{2^{n+2}} \end{aligned}$$

which combines to

$$\begin{aligned}\pi &= 2 \cdot \prod_{k=1}^{\infty} \left(\cos \frac{\pi}{2^{k+1}}\right)^{-1} = 2 \lim_{n\to\infty} \prod_{k=1}^{n} \left(\cos \frac{\pi}{2^{k+1}}\right)^{-1} = \lim_{n\to\infty} 2^{n+1} \sqrt{1-a_n^2} \\ &= \lim_{n\to\infty} 2^n \sqrt{4 - (2 + 2a_{n-1})} = \lim_{n\to\infty} 2^n \sqrt{2 - 2a_{n-1}} \\ &= \lim_{n\to\infty} 2^n \sqrt{2 - \sqrt{2 + 2a_{n-2}}} = \ldots \\ &= \lim_{n\to\infty} 2^n \underbrace{\sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}_{n \text{ square roots}}\end{aligned}$$

3. Let $z(t)$ be a non-negative continuous real function on the interval $[a, b]$ and $t_0 \in [a, b]$. Prove that if

$$z(t) \leq C + L \left| \int_{t_0}^{t} z(s)\, ds \right| \quad (*)$$

for all $t \in [a, b]$ with any constants $C, L \geq 0$, then

$$z(t) \leq C e^{L|t-t_0|} \quad (**)$$

for all $t \in [a, b]$.

Reason: Grönwall-Lemma.

Solution: The Grönwall-Lemma is often stated for $t_0 = a$ in which case all absolute values can be omitted as $t \geq t_0 = a$.

W.l.o.g. we assume $C > 0$. Indeed, if $C = 0$, then the condition $(*)$ holds for any positive $C > 0$ as well. Now if this implies $(**)$, then

$$z(t) \leq \lim_{C \searrow 0} C e^{L|t-t_0|} = 0$$

We define the function F on $[t_0, b]$ by

$$F(t) := C + L \int_{t_0}^{t} z(s)\, ds$$

which is strictly positive and differentiable with $F' = L \cdot z$. Condition $(*)$ means $z \leq F$ for $t \in [t_0, b]$ and so

$$\begin{aligned} F' = Lz \leq LF \Longrightarrow \frac{F'}{F} \leq L \\ &\Longrightarrow \log \frac{F(t)}{F(t_0)} = \int_{t_0}^{t} \frac{F'(s)}{F(s)}\, ds \leq \int_{t_0}^{t} L\, ds = L(t - t_0) \\ &\Longrightarrow z(t) \leq F(t) \leq F(t_0) e^{L(t-t_0)} = C e^{L(t-t_0)} \end{aligned}$$

Since $z \leq F$ we get the inequality $(**)$ for all $t \in [t_0, b]$.

We consider the function

$$G(t) = C + L \int_{t}^{t_0} z(s)\, ds$$

for the interval $[a, t_0]$. which is also positive and differentiable with $G' = -Lz$ and $z \leq G$ by $(*)$, so $G' \geq -LG$. Hence

$$\begin{aligned} \log \frac{G(t_0)}{G(t)} = \int_{t}^{t_0} \frac{G'(s)}{G(s)}\, ds \geq -\int_{t}^{t_0} L\, ds = -L(t_0 - t) = -L|t - t_0| \\ \Longrightarrow z(t) \leq G(t) \leq G(t_0) e^{L|t-t_0|} = C e^{L|t-t_0|} \end{aligned}$$

4. Solve the partial differential equation

$$\begin{gathered} u : D \longrightarrow \mathbb{R}\,,\; D \subseteq \mathbb{R}^3 \\ x u_x + y u_y + (x^2 + y^2) u_z = 0 \\ u(1,0,0) = 0\,,\; u_x(1,0,1) = 0 \\ u_y(-1,1,(\pi+2)/2) = 1\,,\; u_z(-1,1,(\pi+2)/2) = -1 \end{gathered}$$

Reason: Differential Equation. PDE.

Solution: The characteristic system of this PDE is

$$\dot{x} = x \, , \, \dot{y} = y \, , \, \dot{z} = x^2 + y^2$$

with the general (characteristic) solutions

$$x(t) = \alpha e^t \, , \, y(t) = \beta e^t \, , \, z(t) = \frac{1}{2}\left(\alpha^2 + \beta^2\right) e^{2t} + \gamma$$

The solution of the equation is thus

$$u(x(t), y(t), z(t)) = u\left(\alpha e^t, \beta e^t, \frac{1}{2}\left(\alpha^2 + \beta^2\right) e^{2t} + \gamma\right) = \text{ constant.}$$

For the characteristic flows we have the relations:

$$e^t = \frac{x(t)}{\alpha} = \frac{y(t)}{\beta} \Longrightarrow \frac{y(t)}{x(t)} = \frac{\alpha}{\beta} =: a \in \mathbb{R}$$

$$z(t) = \frac{1}{2}\left(x^2 + y^2\right) + \gamma \Longrightarrow z(t) - \frac{1}{2}/(x(t)^2 + y(t)^2) =: b \in \mathbb{R}$$

i.e. the two constants a, b alone define the value of u along the characteristic flows. The representation of the solution is thus

$$u(x, y, z) = \Phi\left(\frac{y}{x}, z - \frac{1}{2}(x^2 + y^2)\right)$$

for any differentiable function $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

This means we have infinitely many possible solutions and the initial values are useless. Those initial values given here correspond to $\Phi(a, b) = a^2 \sin(b)$, but this is not a unique solution.

5. Let $A \in \mathbb{M}(n, \mathbb{R})$ be a real square matrix and $x : \mathbb{R} \longrightarrow \mathbb{R}^n$ a parameterized path. Prove that there exists a unique solution of the differential equation $\dot{x}(t) = Ax(t)$ for any initial condition $x(t_0) = x_0$.

   Reason: Differential Equation. Matrix.

   Solution: We first show that $x(t) := e^{A(t-t_0)}x_0$ is a solution which proves the existence part.

$$x(t_0) = e^0 x_0 = \sum_{k=0}^{\infty} 0^k x_0 \frac{t^k}{k!} = 1 \cdot x_0 = x_0$$

$$\dot{x}(t) = \frac{d}{dt}x(t) = \sum_{k=0}^{\infty} \frac{d}{dt} A^k x_0 \frac{(t-t_0)^k}{k!} = \sum_{k=1}^{\infty} A^k x_0 \cdot k \cdot \frac{(t-t_0)^{k-1}}{k!}$$

$$= A \sum_{m=0}^{\infty} A^m x_0 \frac{(t-t_0)^m}{m!} = Ae^{A(t-t_0)}x_0 = Ax(t)$$

We next show that
$$\left(e^{At}\right)^{-1} = e^{-At}$$
$y(t) = e^{-At}y_0$ is a solution to $\dot{y}(t) = -Ay(t)$ for $y_0 = y(0)$ and $z(t) = e^{At}z_0$ is a solution to $\dot{z}(t) = -Az(t)$ for $z_0 = z(0)$ as we just saw. Therefore
$$\begin{aligned}\frac{d}{dt}\left(e^{-At}e^{At}x_0\right) &= \left(\frac{d}{dt}e^{-At}\right)e^{At}x_0 + e^{-At}\left(\frac{d}{dt}e^{At}\right)x_0\\ &= -A\cdot e^{-At}\cdot e^{At}x_0 + e^{-At}\cdot A\cdot e^{At}x_0 = 0\end{aligned}$$
by the Leibniz rule and because $Ae^{p(A)} = e^{p(A)}A$ for any polynomial $p(s) \in \mathbb{R}[s]$. Thus $e^{-At}e^{At}x_0$ is constant in t, i.e. we have for all $t \in \mathbb{R}, x_0 \in \mathbb{R}^n$
$$e^{-At}e^{At}x_0 = e^{-A\cdot 0}e^{A\cdot 0}x_0 = x_0 \text{ and thus } \left(e^{At}\right)^{-1} = e^{-At}$$
Let $y(t)$ be another solution of the differential equation, i.e. $\dot{y}(t) = Ay(t)\,,\, y(t_0) = x_0$. Then
$$\begin{aligned}\frac{d}{dt}\left(e^{-A(t-t_0)}y(t)\right) &= \left(\frac{d}{dt}e^{-A(t-t_0)}\right)y(t) + e^{-A(t-t_0)}\frac{d}{dt}y(t)\\ &= -Ae^{-A(t-t_0)}y(t) + e^{-A(t-t_0)}Ay(t) = 0\end{aligned}$$
Hence $e^{-A(t-t_0)}y(t)$ is constant in t and thus
$$e^{-A(t-t_0)}y(t) = e^{-A(t_0-t_0)}y(t_0) = 1\,x_0 = x_0 \Longrightarrow y(t) = e^{A(t-t_0)}x_0 = x(t)$$
An important consequence is the following Corollary:

The matrix exponential function $tA \longmapsto e^{At}$ is the unique solution of the matrix differential equation
$$\dot{X}(t) = AX(t)\,,\, X(0) = 1_{\mathbb{M}(n,\mathbb{R})}\;,\; X\,:\,\mathbb{R}\longrightarrow\mathbb{R}^{n\times n}$$

6. Calculate
   (a) $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{x^4+2x^2+1}\,dx$
   (b) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin^2 t}\,dt$

   Reason: Function Theory.

   Solution:

(a) Let $R(z)=\dfrac{P(z)}{Q(z)}$ with $P(z)=z^2$ and

$$Q(z)=z^4+2z^2+1=(z^2+1)^2=(z+i)^2(z-i)^2$$

There are no zeros on the real axis and the degree of the denominator polynomial is larger than the degree of the numerator polynomial, so the integral exists.

We want to apply the residue theorem and choose as closed curve the interval $I(r)=[-r,r]$ and the upper half circle $C(r)$ around 0 with radius $r>1$ in the complex number plane to surround the zero at $z=i$. Hence we get from the residue theorem

$$\int_{I(r)} R(z)\,dz+\int_{C(r)} R(z)\,dz=2\pi i\operatorname{Res}_i(R(z))$$

Since $|R(z)|\leq M\cdot|z|^{-2}$ for large values of $|z|$, we have

$$\left|\int_{C(r)} R(z)\,dz\right|\leq \pi r\cdot Mr^{-2}=M\pi r^{-1}\stackrel{r\to\infty}{\longrightarrow}0$$

and with a singularity of order two at $z=i$

$$\begin{aligned}\int_{-\infty}^{\infty} R(z)\,dz&=\lim_{r\to\infty}\int_{I(r)} R(z)\,dz\\&=2\pi i\operatorname{Res}_i(R(z))-\lim_{r\to\infty}\int_{C(r)} R(z)\,dz\\&=2\pi i\operatorname{Res}_i(R(z))\\&=2\pi i\cdot\frac{1}{(2-1)!}\lim_{z\to i}\frac{d}{dz}\left[(z-i)^2R(z)\right]\\&=2\pi i\lim_{z\to i}\frac{d}{dz}\frac{z^2}{(z+i)^2}\\&=2\pi i\lim_{z\to i}\frac{2z(z+i)^2-z^2(2z+2i)}{(z+i)^4}\\&=\frac{\pi}{2}\end{aligned}$$

where we calculated the residue from the formula

$$\operatorname{Res}_a f=\frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]$$

for a n-th order singularity of f at a.

(b) With $\gamma(t)=e^{it}$ we have

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin^2 t}\,dt &= \frac{1}{4}\int_0^{2\pi}\frac{1}{1+\sin^2 t}\,dt\\ &= \frac{1}{4}\int_0^{2\pi}\left(1-\frac{1}{4}\left(e^{it}-e^{-it}\right)^2\right)^{-1}dt\\ &= \frac{1}{4}\int_\gamma\left(1-\frac{1}{4}(z^2-2+z^{-2})\right)^{-1}\frac{1}{iz}\,dz\\ &= \frac{1}{4}\int_\gamma\frac{4z}{6iz^2-iz^4-i}\,dz\\ &= i\int_\gamma \underbrace{\frac{z}{z^4-6z^2+1}}_{=:z/P(z)}\,dz\end{aligned}$$

$$P(z)=\left(z-\underbrace{\sqrt{3+2\sqrt{2}}}_{\notin\overline{B_0(1)}}\right)\left(z-\underbrace{\sqrt{3-2\sqrt{2}}}_{\in B_0(1)}\right)\left(z+\underbrace{\sqrt{3+2\sqrt{2}}}_{\notin\overline{B_0(1)}}\right)\left(z+\underbrace{\sqrt{3-2\sqrt{2}}}_{\in B_0(1)}\right)$$

By the residue theorem we get

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\frac{1}{1+\sin^2 t}\,dt &= -2\pi\operatorname{Res}_{\sqrt{3-2\sqrt{2}}}\left(\frac{z}{P(z)}\right)-2\pi\operatorname{Res}_{-\sqrt{3-2\sqrt{2}}}\left(\frac{z}{P(z)}\right)\\ &= -2\pi\left(\frac{\sqrt{3-2\sqrt{2}}}{P'(\sqrt{3-2\sqrt{2}})}+\frac{-\sqrt{3-2\sqrt{2}}}{P'(-\sqrt{3-2\sqrt{2}})}\right)\\ &= \frac{-4\pi}{4\sqrt{3-2\sqrt{2}}^{\,2}-12}=\frac{\pi}{2\sqrt{2}}\end{aligned}$$

where we calculated the residue from the formula

$$\operatorname{Res}_a\frac{g}{f}=\frac{g(a)}{f'(a)}$$

for a first order zero $f(a)=0$ and a holomorphic g in a.

7. Prove the following well known theorem by using topological and analytical tools only.

For every real symmetric matrix A there is a real orthogonal matrix Q such that $Q^\tau AQ$ is diagonal.

Hint: 'Topological and analytical tools only' forbids the words 'characteristic' and 'eigen'. You could start with Heine-Borel.

Reason: Linear Algebra by Calculus.

Solution: The groups of orthogonal real matrices $\mathrm{O}(n) \subseteq \mathbb{R}^{n^2}$ are compact subsets. They are bounded, because the columns of an orthogonal matrix $Q = (q_{ij}) \in \mathrm{O}(n)$ are unit vectors and so $|q_{ij}| < 1$ for all i, j, and they are closed, since they are solutions to the linear equations $x_{i1}x_{j1} + \ldots x_{in}x_{jn} = \delta_{ij}$. We define the sum of the squares of the off-diagonal entries $\mathrm{Od}(A) = \sum_{i\neq j} a_{ij}^2$ for any real square matrix.

Lemma: If A is a real symmetric $n \times n$ matrix that is not diagonal, i.e. $\mathrm{Od}(A) > 0$, then there exists $U \in \mathrm{O}(n)$ such that $\mathrm{Od}(U^\tau AU) < \mathrm{Od}(A)$.

The map $\varphi_A : \mathrm{O}(n) \longrightarrow \mathbb{R}^{n^2}$ defined by $\varphi_A(Q) := Q^\tau AQ$ is continuous. Its image $\varphi_A(\mathrm{O}(n))$ is thus a compact subset in $\mathbb{R}^{n^2}$. The continuous function $\mathrm{Od} : \varphi_A(\mathrm{O}(n)) \longrightarrow \mathbb{R}$ assumes therefore a minimum, say at $M = Q^\tau AQ \in \varphi_A(\mathrm{O}(n))$. This implies by the Lemma that $\mathrm{Od}(M) = 0$, hence M is diagonal which had to be proven. (If $\mathrm{Od}(M) > 0$ we first note that $M^\tau = (Q^\tau AQ)^\tau = Q^\tau A^\tau Q = Q^\tau AQ = M$ is symmetric and not diagonal, so we can apply the Lemma on M, and find an $U \in \mathrm{O}(n)$ such that $\mathrm{Od}(U^\tau MU) < \mathrm{Od}(M)$, which contradicts the minimality of M as $U^\tau MU$ is a feasible point:

$$U^\tau MU = U^\tau Q^\tau AQU = (QU)^\tau A(QU) = \varphi_A(QU) \in \varphi_A(\mathrm{O}(n)) \ , \ U, Q \in \mathrm{O}(n)$$

because $\mathrm{O}(n)$ is a group.)

Hence it remains to prove the Lemma.

Given a real symmetric matrix $A = (a_{ij})$. If A is diagonal, them we choose $U = 1 \in \mathrm{O}(n)$ and we are done, so let's assume $a_{rs} \neq 0$ for some $r \neq s$. In this case we set U to be a rotation matrix in the $(r, s)-$plane

$$U = (u_{ij}) := \begin{cases} u_{ij} = 0 & \text{if } i, j \notin \{r, s\} \wedge i \neq j \\ u_{ii} = 1 & \text{if } i \notin \{r, s\} \\ u_{rr} = u_{ss} = \cos\alpha \\ u_{rs} = -u_{sr} = \sin\alpha \end{cases}$$

which is clearly orthogonal. Let $U^\tau AU = (b_{kl}) = \left(\sum_{i,j} u_{ik}a_{ij}u_{jl}\right)$.

Then $b_{ij}=a_{ij}$ in all cases $i,j\notin\{r,s\}$.

$$\begin{aligned}
b_{kr} &= \sum_i u_{ik}\sum_j a_{ij}u_{jr} = \sum_i u_{ik}(a_{ir}\cos\alpha - a_{is}\sin\alpha)\\
&= a_{kr}\cos\alpha - a_{ks}\sin\alpha\\
b_{ks} &= a_{ks}\cos\alpha + a_{kr}\sin\alpha\\
b_{kr}^2+b_{ks}^2 &= a_{kr}^2+a_{ks}^2\\
b_{rl}^2+b_{sl}^2 &= a_{rl}^2+a_{sl}^2
\end{aligned}$$

Now we have for the symmetric matrices A and $U^\tau AU$

$$\operatorname{Od}(A)-\operatorname{Od}(U^\tau AU) = a_{sr}^2-b_{sr}^2+a_{rs}^2-b_{rs}^2 = 2(a_{rs}^2-b_{rs}^2) = 2a_{rs}^2>0$$

if we can choose α in a way such that $b_{rs}=0$.

$$\begin{aligned}
b_{rs}(\alpha) &= \sum_{i,j} u_{ir}a_{ij}u_{js}\\
&= u_{rr}a_{rs}u_{ss}+u_{rr}a_{rr}u_{rs}+u_{sr}a_{sr}u_{rs}+u_{sr}a_{ss}u_{ss}\\
&= a_{rs}\left(\cos^2\alpha-\sin^2\alpha\right)+(a_{rr}-a_{ss})\cos\alpha\sin\alpha
\end{aligned}$$

Now $b_{rs}(0)=a_{rs}$ and $b_{rs}(90^\circ)=-a_{rs}$. By the mean value theorem there must be a choice of α such that $b_{rs}(\alpha)=0$, since b_{rs} depends continuously on α and $a_{rs}\neq 0$.

8. We define $e=\sum_{k=0}^{\infty}\frac{1}{k!}$. Prove that e^2 is irrational.

   Reason: Irrationality.

   Solution: Assume $e^2=\frac{a}{b}\in\mathbb{Q}$. Then $be=ae^{-1}$ and with the series

$$\begin{aligned}
e &= 1+\frac{1}{1}+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\ldots\\
e^{-1} &= 1-\frac{1}{1}+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}\pm\ldots
\end{aligned}$$

   we have for sufficiently large even n

$$n!be = \underbrace{n!b\left(1+\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n!}\right)}_{:=\beta_0\in\mathbb{Z}}+\underbrace{n!b\left(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\ldots\right)}_{:=\beta_n}$$

$$n!ae^{-1} = \underbrace{n!a\left(1-\frac{1}{1}+\frac{1}{2}\mp\ldots+\frac{(-1)^n}{n!}\right)}_{:=\alpha_0\in\mathbb{Z}}+\underbrace{n!a\left(\frac{(-1)^{n+1}}{(n+1)!}+\frac{(-1)^{n+2}}{(n+2)!}+\ldots\right)}_{:=\alpha_n}$$

and

$$0 < \frac{b}{n+1} < \beta_n < \frac{b}{n+1} + \frac{b}{(n+1)^2} + \ldots = \frac{b}{n+1} \cdot \frac{1}{1 - \dfrac{1}{n+1}} = \frac{b}{n}$$

$$\begin{aligned} -\frac{a}{n} &\stackrel{(n\,\text{even})}{<} -\frac{a}{n+1} + \varepsilon = \alpha_n < -\frac{a}{n+1}\left(1 - \frac{1}{n+1} - \frac{1}{(n+1)^2} - \ldots\right) \\ &= -\frac{a}{n+1}\left(1 - \frac{1}{n+1}\sum_{k=0}^{\infty}\frac{1}{(n+1)^k}\right) \\ &= -\frac{a}{n+1}\left(1 - \frac{1}{n+1}\cdot\frac{1}{1-\dfrac{1}{n+1}}\right) = -\frac{a}{n+1}\cdot\left(1-\frac{1}{n}\right) < 0 \end{aligned}$$

This means for sufficiently large even n, that $n!ae^{-1}$ is a bit smaller than the integer α_0 and $n!be$ is a bit larger than the integer β_0, hence they cannot be equal.

9. Let $p < q$ be two primes, $b \in \mathbb{N}$, and G a group with p^2q^b elements. Show that:

   (a) If there is no normal $q-$Sylow subgroup in G, then $(p,q) = (2,3)$, and there is a non trivial homomorphism from G to S_4.

   (b) G is always solvable.

   Reason: Group Theory.

   Solution:

   (a) The number of $q-$Sylow subgroups in G is a divisor of p^2 and congruent 1 modulo q. If there is no normal $q-$Sylow subgroup, then a single one is excluded and there are p or p^2 many of them. However, p many of them is also excluded, because otherwise we had $q \,|\, p-1$ which is not possible for $q > p$. There are thus p^2 many $q-$Sylow subgroups in G and $p^2 \equiv 1 \mod q$. Hence $p + q\mathbb{Z}$ is a zero of $x^2 - 1 \in \mathbb{F}_q[x]$. This polynomial has two zeros: ± 1. However, $p \not\equiv 1 \mod q$ since $q \nmid p-1$, and $p \equiv -1 \mod q$, i.e. $0 < p+1 = m \cdot q \leq q$. This means $k = 1$ and $p + 1 = q$ which is only possible for the primes $(p,q) = (2,3)$.

   There are $4 = p^2$ $3-$Sylow subgroups in this case on which G operates transitive via conjugation. If we number them, we get a group homomorphism $\varphi : G \longrightarrow S_4$ whose image operates transitive on $\{1, 2, 3, 4\}$. Now 4 divides the cardinality of the orbits by the orbit-stabilizer theorem, so $\{1\} \subsetneq \operatorname{im}(\varphi)$ is a proper subset and φ is non trivial.

(b) If G has a normal $q-$Sylow subgroup N, then it is a $q-$group of index p^2 and G/N is a $p-$group. Since p, q are both primes, N and G/N are both solvable and G is solvable, too. If G has no normal $q-$Sylow subgroup, we define $N := \ker(\varphi) \triangleleft G$ with the homomorphism from the previous part. Since $4 \,|\, |G/N| = |\operatorname{im}(\varphi)|$, the kernel N has to be a $3-$group, which is solvable. But $\operatorname{im}(\varphi) \cong G/N$ is a subgroup of the solvable group S_4, hence itself solvable. Thus N and G/N are again solvable and therewith G.

10. Let $f(x) = 2x^5 - 6x + 6 \in \mathbb{Z}[x]$. In which of the following rings is f irreducible and why?

(a) $\mathbb{Z}[x]$

(b) $(S^{-1}\mathbb{Z})[x]$ with $S = \{2^n \,|\, n \in \mathbb{N}_0\}$

(c) $\mathbb{Q}[x]$

(d) $\mathbb{R}[x]$

(e) $\mathbb{C}[x]$

Reason: Ring Theory.

Solution: (a) $f(x) = 2 \cdot (x^5 - 3x - 3)$. Both factors are non units, since the units in $\mathbb{Z}[x]$ are $\{\pm 1\}$, i.e. f is reducible.

(c) 2 is a unit in this case, so it's sufficient to consider $g(x) = x^5 - 3x - 3$. By Eisenstein's criterion with the prime $p = 3$, we find that g is irreducible over $\mathbb{Q}[x]$ and so is f.

(b) It's again sufficient to consider $g(x) = x^5 - 3x - 3$. Assume $g = pq$ over $S^{-1}\mathbb{Z}$. This factorization is also valid in $\mathbb{Q}[x] \supseteq (S^{-1}\mathbb{Z})[x]$. But g is irreducible over $\mathbb{Q}$, so one of the factors is of degree 0, say p. This means $p \in S^{-1}\mathbb{Z}$ and p divides each coefficient of g, especially the leading coefficient 1, which means p is a unit in $S^{-1}\mathbb{Z}$ and f is irreducible.

(d) $\deg f > 1$ and odd, i.e. $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$. Thus f has a zero by the mean value theorem and f is reducible.

(e) $\deg f > 1$ so f is reducible by the fundamental theorem of algebra.

11. (HS-1) Given a set A of 32 pairwise distinct, positive integers less than

112. Decide right or wrong:

(a) There is a number which occurs at least five times among the differences between two numbers of A.

(b) There is a number which occurs at least six times among the differences between two numbers of A.

Hint: A difference in this context is always positive, and only counted once between any two numbers of A.

Reason: Combinatorics.

Solution: There are $\binom{|A|}{2} = \frac{32 \cdot 31}{2} = 496$ differences to be considered. All of them are pairwise distinct, positive integers less than 112. If there was at most four occurrences of a certain difference, then we had with $4 \cdot 111 < 496$ a deficit of 52 possible differences. This proves the first part to be true.

The same argument doesn't work for six occurrences, so we need a closer look. Let the elements of A be ordered as

$$1 \leq a_1 < a_2 < \ldots < a_{31} < a_{32} < 112\,.$$

If the values among all differences would occur at most five times, then this is particularly true for the values $\{1, 2, 3, 4, 5, 6\}$ among the 31 differences $d_n = a_{n+1} - a_n$. So there is at least $31 - 5 \cdot 6 = 1$ difference with $d_n > 7$. Thus

$$d_1 + \ldots + d_{31} \geq 5 \cdot 1 + 5 \cdot 2 + \ldots + 5 \cdot 6 + 7 = 5 \cdot 21 + 7 = 112$$

On the other hand we have

$$a_{32} = a_1 + d_1 + d_2 + \ldots + d_{31} \geq a_1 + 112 > 112$$

which is a contradiction. This means that we must have at least 6 equal differences among the numbers of A.

12. (HS-2) The harmonic numbers are

$$H_n := \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\ ,\ (n \in \mathbb{N})$$

We define

$$T_n := \sum_{k=1}^{n} \frac{1}{k \cdot H_k^2} = \frac{1}{H_1^2} + \frac{1}{2 \cdot H_2^2} + \frac{1}{3 \cdot H_3^2} + \ldots + \frac{1}{n \cdot H_n^2}\ ,\ (n \in \mathbb{N})$$

Show that $T_n < 2$ for all $n \in \mathbb{N}$.

Reason: No Induction necessary.

Solution: It is $0 < H_{k-1} < H_k$ for any $k > 1$. Hence

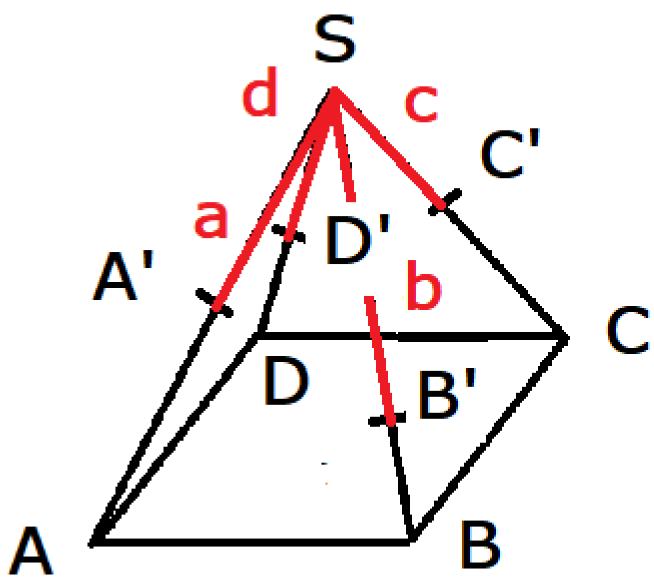
$$\frac{1}{k \cdot H_k^2} < \frac{\frac{1}{k}}{H_k \cdot H_{k-1}} = \frac{H_k - H_{k-1}}{H_k \cdot H_{k-1}} = \frac{1}{H_{k-1}} - \frac{1}{H_k}$$

and so

$$T_n = 1 + \sum_{k=2}^{n} \frac{1}{k \cdot H_k^2} < 1 + \sum_{k=2}^{n} \left(\frac{1}{H_{k-1}} - \frac{1}{H_k} \right) = 1 + \frac{1}{H_1} - \frac{1}{H_n} = 2 - \frac{1}{H_n} < 2$$

13. (HS-3) We have a four sided pyramid with summit S and a quadratic base A, B, C, D. Let A', B', C', D' be four points on the edges AS, BS, CS, DS, resp. with positive distances a, b, c, d from S, resp. Show that A', B', C', D' are coplanar if and only if

$$\frac{1}{a} + \frac{1}{c} = \frac{1}{b} + \frac{1}{d}$$



Reason: Geometry.

Solution: Without loss of generality we assume that $\overline{AS} = \overline{BS} = \overline{CS} = \overline{DS} = 1$. For the perpendiculars $A'F', AF$ from A, A' onto the plane B, C, S resp., we know that $A'F' \parallel AF$ and get from the intercept theorem $\overline{A'F'} : \overline{AF} = \overline{A'S} : \overline{AS} = a$. For the volumes of the pyramids $A'BCS$ and $ABCS$ we thus have

$$V(A'BCS) : V(ABCS) = a$$

and accordingly

$$\begin{aligned} V(A'B'CS) : V(A'BCS) &= b \\ V(A'B'C'S) : V(A'B'CS) &= c. \end{aligned}$$

Hence

$$V(A'B'C'S) : V(ABCS) = abc$$

and similarly

$$\begin{aligned} V(A'D'C'S) : V(ADCS) &= adc \\ V(A'B'D'S) : V(ABDS) &= abd \\ V(C'B'D'S) : V(CBDS) &= cbd \end{aligned}$$

The four points A', B', C', D' are coplanar if and only if $V(A'B'C'D') = 0$. This is equivalent to

$$V(A'B'C'S) + V(A'D'C'S) = V(A'B'D'S) + V(C'B'D'S) \quad (*)$$

because the difference of both sides of the equation is exactly $V(A'B'C'D')$. All areas of the triangles ABC, ABD, ADC, CBD of the square $ABCD$ have the same value, so $V(ABCS) = V(ABDS) = V(ADCS) = V(CBDS)$ and equation $(*)$ is equivalent to

$$abc + adc = abd + cbd \Longleftrightarrow \frac{1}{d} + \frac{1}{b} = \frac{1}{c} + \frac{1}{a}$$

Another way to solve the problem is by coordinates. We can choose a coordinate system such that A, B, C, S have the coordinates

$$A = (-t, -t, h)\,,\ B = (t, -t, h)\,,\ C = (t, t, h)\,,\ D = (-t, t, h)\,,\ S = (0, 0, 0)$$

for suitable $t, h > 0$. This means with the same assumption about the normed edges of the pyramid

$$A' = (-at, -at, ah)\,,\ B' = (bt, -bt, bh)\,,\ C' = (ct, ct, ch)\,,\ D' = (-dt, dt, dh)$$

Being coplanar is equivalent to the fact that

$$\det \begin{bmatrix} 1 & -at & -at & ah \\ 1 & bt & -bt & bh \\ 1 & ct & ct & ch \\ 1 & -dt & dt & dh \end{bmatrix} = 4t^2h(bcd - acd + abd - abc) = 0$$

14. (HS-4) Let $f(n) = [\,2\sqrt{n}\,] - \left[\,\sqrt{n-1} + \sqrt{n+1}\,\right]$ for $n \in \mathbb{N}$. Determine all values of n such that $f(n) = 1$ and all n such that $f(n) = 0$.

If $r \in \mathbb{R}$ with $s \leq r < s+1$ then $[r] = \lfloor r \rfloor = s$.

Reason: Cases.

Solution: We have $f(1) = [2] - [0 + \sqrt{2}] = 2 - 1 = 1$ and for $n \geq 2$

$$\begin{aligned} 5 &< 4n \\ 4n^2 - 4n + 1 &< 4n^2 - 4 < 4n^2 \\ 2n - 1 &< 2\sqrt{n^2-1} < 2n \\ 4n - 1 &< \left(\sqrt{n-1} + \sqrt{n+1}\right)^2 < 4n \end{aligned}$$

Assume there is an integer g such that $\sqrt{4n-1} < g \leq \sqrt{n-1} + \sqrt{n+1}$, then $4n - 1 < g^2 \leq \left(\sqrt{n-1} + \sqrt{n+1}\right)^2 < 4n$ which is not possible. Therefore $\left[\,\sqrt{n-1} + \sqrt{n+1}\,\right] = [\,\sqrt{4n-1}\,]$ and

$$f(n) = [\,\sqrt{4n}\,] - [\,\sqrt{4n-1}\,] \quad (*)$$

If $n = m^2$ is a square number, then $\sqrt{4n} = 2m \in \mathbb{N}$ and

$$\begin{aligned} 2 &\leq 2\sqrt{4n} \\ 4n - 2\sqrt{4n} + 1 &\leq 4n - 1 \\ \sqrt{4n} - 1 &\leq \sqrt{4n-1} < \sqrt{4n} \\ [\,\sqrt{4n-1}\,] &= \sqrt{4n} - 1 = [\,\sqrt{4n}\,] - 1 \end{aligned}$$

hence $f(n) = 1$ by $(*)$.

If n is not a square number, then there is no integer g such that $4n-1 < g^2 \leq 4n$ or $\sqrt{4n-1} < g \leq \sqrt{4n}$, which means $[\,\sqrt{4n}\,] = [\,\sqrt{4n-1}\,]$ and $f(n) = 0$.

We have shown that among all positive integers n

- exactly all positive square numbers fulfill $f(n) = 1$, and

- all positive non square numbers fulfill $f(n) = 0$.

15. (HS-5) Determine all pairs of non-negative integers (m, n) such that $2^m - 5^n = 7$.

    Reason: Modular Arithmetic.

    Solution: Among the numbers 2^m for integers $0 \leq m \leq 5$ are exactly the numbers 2^3 and 2^5 of the required form $5^n + 7$ with an integer $n \geq 0$, namely
    $$2^3 - 5^0 = 2^5 - 5^2 = 7.$$

    We will show that there are no other solutions than $(3, 0), (5, 2)$.

    Assume there were solutions (m, n) with $n \geq 0, m \geq 6$ such that $2^m = 5^n + 7$. Then $2^6 = 64 \,|\, (5^n + 7)$ and $5^n \equiv 57 \mod 64$. Possible remainders are periodically

n	0	1	2	3	4	5	6	7	8	...
$5^n \mod 64$	1	5	25	61	49	53	9	45	33	...
n	...	9	10	11	12	13	14	15	16	...
$5^n \mod 64$	...	37	$\underline{57}$	29	17	21	41	13	1	...

so $n = 16a + 10$ for some $a \in \mathbb{N}_0$.

Let's consider now the possible remainders modulo 17 which also have a periodicity of 16. Here we find

n	0	1	2	3	4	5	6	7	8	...
$5^n \mod 17$	1	5	8	6	13	14	2	10	16	...
$5^n + 7 \mod 17$	8	12	15	13	3	4	9	0	6	...
n	...	9	$\underline{10}$	11	12	13	14	15	16	...
$5^n \mod 17$	...	12	9	11	4	3	15	7	1	...
$5^n + 7 \mod 17$	...	2	$\underline{16}$	1	11	10	5	14	8	...

In order for $2^m \equiv 5^n + 7 \mod 17$ to hold, the remainders must be the same. For the left hand side we get the remainders

m	0	1	2	3	4	5	6	7	8	...
$2^m \mod 17$	1	2	4	8	$\underline{16}$	15	13	9	1	...

with periodicity 8, which all occurred as remainders of $5^n + 7$. We already know that $n = 16a + 10$, so only the entries for $n = 10, 26, 42, \ldots$

are relevant. Thus $2^m \equiv 5^n + 7 \equiv 16 \mod 17$ and $m = 8b + 4$ for some $b \in \mathbb{N}_0$. This means particularly that m, n are even, say $m = 2c, n = 2d$.

$$\begin{aligned}2^m &\equiv 2^{2c} \equiv 4^c \equiv 1^c \equiv 1 \mod 3\\ 5^n + 7 &\equiv 5^{2d} + 7 \equiv 25^d + 7 \equiv 1^d + 1 \equiv 2 \mod 3\end{aligned}$$

which cannot be simultaneously the case.

13 December 2020

1. Given a vector field

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 , \; F(x, y, z) = \begin{pmatrix} x - y^2 \\ x^2 \\ z \end{pmatrix}$$

Calculate the circulation of F along the mathematically positive oriented unit circle in the (x, y)-plane

(a) by using the curl of a vector field.

(b) directly by an appropriate path.

Reason: Circulation Of A Vector Field.

Solution: Let D be the unit disk around the origin with boundary ∂D. By Stoke's theorem we get

$$\int_{\partial D} F(r) \cdot dr = \int_D \operatorname{curl}(F) \cdot n \, dS$$

The curl of F is given by

$$\operatorname{curl}(F) = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \wedge \begin{pmatrix} x - y^2 \\ x^2 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2x + 2y \end{pmatrix}$$

and has only a z-component. Parametrization of the unit disk in polar coordinates $\psi(r, \varphi) = (r \cos \varphi, r \sin \varphi, 0)$ with a normal vector obeying the right hand rule, i.e. pointing to the positive z-direction

$$n(r, \varphi) = \partial_r \psi \wedge \partial_\varphi \psi = \begin{pmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{pmatrix} \wedge \begin{pmatrix} -r \sin \varphi \\ r \cos \varphi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

Thus the circulation can be expressed as surface integral

$$\begin{aligned}\int_{\partial D} F(r)\,dr &= \int_D \operatorname{curl}(F)\cdot n\,dS\\ &= \int_0^1 dr \int_0^{2\pi} d\varphi \begin{pmatrix}0\\0\\r\end{pmatrix}\cdot\begin{pmatrix}0\\0\\2r\cos\varphi+2r\sin\varphi\end{pmatrix}\\ &= \int_0^1 dr \int_0^{2\pi} d\varphi\ 2r^2(\cos\varphi+\sin\varphi)=0\end{aligned}$$

Alternatively, we can calculate the circulation directly, too. We therefore parameterize the boundary of the unit disk by $\gamma(\varphi)=(\cos\varphi,\sin\varphi)$, $\dot\gamma(\varphi)=(-\sin\varphi,\cos\varphi,0)$ and calculate

$$\begin{aligned}\int_{\partial D} F(r)\,dr &= \int_0^{2\pi} F(\gamma(\varphi))\cdot\dot{\gamma(\varphi)}\,d\varphi\\ &= \int_0^{2\pi}\begin{pmatrix}\cos\varphi-\sin^2\varphi\\ \cos^2\varphi\\ 0\end{pmatrix}\cdot\begin{pmatrix}-\sin\varphi\\ \cos\varphi\\ 0\end{pmatrix}d\varphi\\ &= \int_0^{2\pi}(\cos^3\varphi+\sin^3\varphi-\sin\varphi\cos\varphi)\,d\varphi\\ &= \frac{1}{3}\left[\cos^2\varphi\sin\varphi+2\sin\varphi\right]_0^{2\pi}\\ &-\frac{1}{3}\left[\sin^2\varphi\cos\varphi+2\cos\varphi\right]_0^{2\pi}\\ &-\frac{1}{2}\left[\sin^2\varphi\right]_0^{2\pi}\\ &= 0-0-\frac{2}{3}+\frac{2}{3}-0+0\\ &= 0\end{aligned}$$

2. An odd prime p can be written as $p=x^2+y^2$ with integers $x,y\in\mathbb{Z}$ if and only is $p\equiv 1 \mod 4$.

 Reason: Fermat's Theorem About The Sum Of Two Squares.

 Solution:

 - Let $p=4k+1$ and $S:=\{\,(x,y,z)\in\mathbb{N}^3 \mid x^2+4yz=p\,\}$. Then S has two involutions $(x,y,z)\longmapsto(x,z,y)$ whose fixed points (x,y,y)

correspond to representations of p as sum of two squares, and

$$(x,y,z) \longmapsto \begin{cases} (x+2z,z,y-x-z), & \text{if } x \leq y-z \\ (2y-x,y,x-y+z), & \text{if } y-z < x < 2y \\ (x-2y,x-y+z,y), & \text{if } x \geq 2y \end{cases}$$

which has exactly one fixed point $(1,1,k)$. Two involutions over the same finite set must have sets of fixed points with the same parity, which is odd in our case due to the second involution. (It is an easy proof by induction, that the order of the set of fixed points of an involution on a finite set has the same parity as the set has. Hence the parity is independent of a certain involution.) So the first involution has a nonzero fixed point (x_0,y_0,y_0) which means $x_0^2+(2y_0)^2=p$.

If conversely $p=x^2+y^2$, then $x^2+y^2 \equiv r \mod 4$ with $r \in \{0,1,2\}$ and $p=2n+1 \equiv s \mod 4$ with $s \in \{1,3\}$. Thus $p=x^2+y^2 \equiv 1 \mod 4$.

- The above theorem can also be proven with the help of Minkowski's lattice theorem:

  Let $\Gamma \subseteq \mathbb{R}^d$ be a lattice and $C \subseteq \mathbb{R}^d$ a convex, bounded and with respect to the origin symmetric area. If $\operatorname{vol}(C) > 2^d \cdot \operatorname{vol}(\Gamma)$, where the volume of the lattice is meant to be the volume of the primitive cell, then C contains besides the origin at least one more lattice point.

  Let $p=4k+1$. Then by Euler's criterion

$$\left(\frac{-1}{p}\right) \equiv (-1)^{\frac{p-1}{2}} \equiv ((-1)^2)^k \equiv 1 \mod p$$

  -1 is a quadratic residue modulo p, i.e. there is an integer m such that $-1 \equiv m^2 \mod p$ or $p \,|\, (m^2+1)$. Let $\hat{\imath},\hat{\jmath}$ be the standard basis of $\mathbb{R}^2$. Set $u=\hat{\imath}+m\hat{\jmath}$, $v=p\hat{\jmath}$ and consider the lattice $\Gamma=\mathbb{Z}u+\mathbb{Z}v$. If $w=\alpha u+\beta v=\alpha i+(\alpha m+\beta p)j \in \Gamma$ then

$$\|w^2\| \equiv \alpha^2+(\alpha m+\beta p)^2 \equiv \alpha^2(1+m^2) \equiv 0 \mod p$$

  and $p \,|\, \|w\|^2$ for any $w \in \Gamma$. Furthermore we have $\operatorname{vol}(\Gamma)=p$ and $\operatorname{vol}(C)=2\pi p > 2^2 \operatorname{vol}(\Gamma)$ for the open disc $C=U(0;\sqrt{2p})$. Then by Minkowski's theorem there exists a nonzero vector $w \in \Gamma$ with $w \in C$. Hence $\|w\|^2 < 2p$ and $p \,|\, \|w\|^2$ so

$$p=\|w\|^2=\alpha^2+(\alpha m+\beta p)^2$$

is the sum of two squares. The other direction follows as above.

3. Let $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ equipped with the quotient topology according to the projection
$$\pi \,:\, \mathbb{R}^n \longrightarrow \mathbb{T}^n \,,\, \pi(a) = a + \mathbb{Z}^n.$$
Show that $\mathbb{T}^n$ is a topological manifold.

Reason: Torus.

Solution: We have to show that $\mathbb{T}^n$ is Hausdorff, second countable, and that every point has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

$\mathbb{R}^n$ is second countable, because we can choose open balls of rational points with rational radius as basis. Quotient building doesn't change this property, i.e. the torus is second countable, too.

Let $P, Q \in \mathbb{T}^n$ be two distinct points. As π is surjective, there are $x, y \in \mathbb{R}^n$ such that $\pi(x) = P, \pi(y) = Q$. The preimages of Q under π is the set $\pi^{-1}(Q) = y + \mathbb{Z}^n$. Choose $\varepsilon > 0$ small enough, such that $B_\varepsilon(x) \subseteq \mathbb{R}^n$ and $B_\varepsilon(y+k) \subseteq \mathbb{R}^n$ are disjoint for all $k \in \mathbb{Z}^n$. Then $P \in \pi(B_\varepsilon(x))$ and $Q \in B_\varepsilon(y)$ are disjoint sets. Thus $\mathbb{T}^n$ is Hausdorff, if π is open.

Let $U \subseteq \mathbb{R}^n$ be open. Then $\pi(U)$ is open by the definition of the quotient topology, if and only if $\pi^{-1}(\pi(U)) \subseteq \mathbb{R}^n$ is open.

$$\begin{aligned} x \in \pi^{-1}(\pi(U)) &\Longleftrightarrow \pi(x) \in \pi(U) \\ &\Longleftrightarrow (\exists\, y \in U)\, \pi(x) = \pi(y) \\ &\Longleftrightarrow (\exists\, y \in U) x - y \in \mathbb{Z}^n \\ &\Longleftrightarrow (\exists\, y \in U) x \in y + \mathbb{Z}^n \\ &\Longleftrightarrow x \in U + \mathbb{Z}^n \\ &\Longleftrightarrow (\exists\, k \in \mathbb{Z}^n)\, x \in U + k \\ &\Longleftrightarrow x \in \bigcup_{k \in \mathbb{Z}^n} U + k \end{aligned}$$

Thus $\pi^{-1}(\pi(U)) = \bigcup_{k\in\mathbb{Z}^n} U + k$, and this is an open set in $\mathbb{R}^n$.

Now let us fix a point $a \in \mathbb{R}^n$ and set $V_a := B_{1/2}(a) \subseteq \mathbb{R}^n$, the open ball around a, and $U_a := \pi(V_a) \subseteq \mathbb{T}^n$ which is open as well by the previous argument that π is open. The mapping $\rho \,:\, V_a \longrightarrow U_a$ with $y \longmapsto \pi(y)$ is continuous, since it is the restriction of the continuous function π, surjective by definition, and open as restriction of an open function on open sets in domain and codomain. Hence we must show, that ρ is injective, too. Assume two points $x \neq y$ in V_a such that $\rho(x) = \pi(x) = x + \mathbb{Z}^n = y + \mathbb{Z}^n = \pi(y) = \rho(y.)$ Since the diameter of V_a is 1, we have $\|x - y\| < 1$. But $x - z \in \mathbb{Z}^n$ which implies $\|x - y\| \geq 1$, a contradiction. Thus ρ is continuous, open and bijective, i.e. a homeomorphism. Its inverse $\rho_a^{-1} : U_a \longrightarrow V_a$ is therefore a chart at a and $\mathcal{A} := \{ \rho_a \,|\, a \in \mathbb{R}^n \}$ an atlas, because each point in $\mathbb{T}^\ltimes$ is at least in one chart. Hence $\mathbb{T}^n$ is an n-dimensional topological manifold.

4. Let $\alpha \in \mathbb{R} - \{0\}$. Determine all functions $f : \mathbb{R}_{>0} \longrightarrow \mathbb{R}_{>0}$ which satisfy for all $x, y \in \mathbb{R}_{>0}$

$$f(f(x) + y) = \alpha x + \frac{1}{f\left(\dfrac{1}{y}\right)}$$

Reason: Functional Equation.

Hint: Use Cauchy's first functional equation.

Solution: We will show that only $\alpha = 1$ allows a solution, namely $f(x) = x$.

We obviously have $\alpha > 0$ for the expression on the left would otherwise become negative for large x. Moreover

$$\begin{aligned} f(x) = f(y) &\Longrightarrow \\ \alpha x + \frac{1}{f\left(\dfrac{1}{z}\right)} &= f(f(x) + z) = f(f(y) + z) = \alpha y + \frac{1}{f\left(\dfrac{1}{z}\right)} \\ &\Longrightarrow x = y \end{aligned}$$

$f(x)$ is not bounded from above, since $f(f(x) + 1) = \alpha x + \dfrac{1}{f(1)}$ and we can simply choose x large enough. Let $\beta \in (0, \infty)$. Then

$$f\left(f\left(\frac{\beta f(y^{-1}) - 1}{\alpha f(y^{-1})}\right) + y\right) = \alpha \cdot \frac{\beta f(y^{-1}) - 1}{\alpha f(y^{-1})} + \frac{1}{f(y^{-1})} = \beta$$

for any $y > 0$. With an y chosen such that $f(y^{-1}) > \beta^{-1}$ we get $\dfrac{\beta f(y^{-1}) - 1}{\alpha f(y^{-1})} > 0$ and so a preimage of β in the domain of $f(x)$.

This means that $f(x)$ is bijective, that is there is a one-to-one correspondence between all $y > 0$ and all $f(y)$. Hence we can write

$$f(f(x)+f(y)) = \alpha x + \frac{1}{f\left(\frac{1}{f(y)}\right)} = \alpha y + \frac{1}{f\left(\frac{1}{f(x)}\right)}$$

for the symmetry on the left hand side. If we fix y and set $C := \frac{1}{f\left(\frac{1}{f(y)}\right)} - \alpha y$ we get $\alpha x + C = \frac{1}{f\left(\frac{1}{f(x)}\right)} > 0$. We also have $C > 0$ since otherwise we could choose x small enough and get a negative function value. Now $f(f(x)+f(y)) = \alpha y + \alpha x + C$.

$$\begin{aligned} x+y=z+w &\Longrightarrow \alpha x + \alpha y + C = \alpha z + \alpha w + C \\ &\Longrightarrow f(f(x)+f(y)) = f(f(z)+f(w)) \\ &\Longrightarrow f(x)+f(y) = f(z)+f(w) \\ &\Longrightarrow f(x+1)+f(y+1) = f(x+y+1)+f(1) \end{aligned}$$

We thus get for the function $g : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ defined by $g(x) := f(x+1)$ the functional equation $g(x)+g(y) = g(x+y)+g(0)$. We now set $h(x) := g(x) - g(0) \geq -g(0)$ and get Cauchy's functional equation

$$h(x)+h(y) = g(x)-g(0)+g(y)-g(0) = g(x+y)-g(0) = h(x+y).$$

If there was a function value $h(t) < 0$ then $h(nt) = nh(t)$ would each any negative number, which is impossible as $-g(0)$ is a fixed lower bound. So $h(x) \geq 0$ for all $x \geq 0$, and for $0 < uyv$ we have $h(v) = h(v-u)+h(u) \geq h(u)$, i.e. $h(x)$ is a monotone function.The only solutions to Cauchy's functional equation which are monotone are linear functions $h(x) = cx$.

(see `https://en.wikipedia.org/wiki/Cauchy%27s_functional_equation`)

Hence we get for $x > 1$ that $f(x) = g(x-1) = h(x-1) - g(0) = cx + d$ for suitable constants c, d. Let $0 < x \leq 1$ and set $y = 3, z = 2, w = x+1$ so

$$f(x) = f(z)+f(w)-f(y) = (2c+d)+c(x+1)+d-3c-d = cx+d$$

which means that $f(x) = cx+d$ on its entire domain. As $f(x)$ reaches all positive values, and is positive itself, we conclude that $c > 0$ and

$d = 0$ for we would get negative or missing values otherwise. This means

$$f(f(x)+y) = f(cx+y) = c^2x + cy = \alpha x + \frac{1}{\frac{c}{y}} = \alpha x + \frac{y}{c}$$

Comparing coefficients yields $c^2 = \alpha$ and $c^2 = 1$, hence $f(x) = x$.

5. Show that the quaternion group $G = \{\pm 1, \pm i, \pm j, \pm k\}$ is a Hamilton group, and cannot be written as a semidirect product in a nontrivial way.

   Reason: Group Theory.

   Solution: Let's consider the subgroups of G. We clearly have the subgroup $\{\pm 1\}$, which is normal, since $m(-1)m^{-1} = -1$ for all $m \in G$. Any other subgroup U contains at least one pure quaternion or its negative, say w.l.o.g. $k \in U$. Then $k^2 = -1$ and $-k = k^3$ are also in U, i.e. $\langle k \rangle = \mathbb{Z}_4 \subseteq U$ and thus $U = \mathbb{Z}_4$, since any bigger subgroup would already have to be the entire group. Now $jkj^{-1} = ij^{-1} = -ij = -k \in U$ and analogue $iki^{-1} \in U$, i.e. U is a normal subgroup, too. Hence G is a Dedekind group (all subgroups are normal), and a Hamilton group (Dedekind and non abelian). Therefore any semidirect product in G is already a direct product. Thus $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ would be the only possible split, however, $\mathbb{Z}_2 \subseteq \mathbb{Z}_4$ in all possible combinations, and the product cannot be direct.

6. (a) Calculate

$$\frac{1}{2} - \frac{1}{12} + \frac{1}{40} - \frac{1}{112} \mp \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)2^{k+1}}\,.$$

   (b) Prove $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots = 1\,.$

   Reason: Trick For Infinite Series.

   Solution:

(a) We introduce a parameter and define

$$f(x) := \frac{x}{2} - \frac{x^3}{12} + \frac{x^5}{40} - \frac{x^7}{112} \mp \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)2^{k+1}}$$
$$f'(x) = \frac{1}{2} - \frac{x^2}{4} + \frac{x^4}{8} - \frac{x^6}{16} \mp \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^{k+1}} = \frac{1}{2+x^2}$$

As $f(0) = 0$ we get

$$\begin{aligned} f(1) &= \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)2^{k+1}} \\ &= f(1) - f(0) = \int_0^1 f'(x)\,dx \\ &= \int_0^1 \frac{dx}{2+x^2} = \left[\frac{1}{\sqrt{2}} \cdot \tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1 \\ &= \frac{1}{\sqrt{2}} \cdot \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 0.43521 \end{aligned}$$

(b)

$$\begin{aligned} F(x) &:= \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \ldots = \frac{1}{x}e^x - 1 - \frac{1}{x} \\ F'(x) &= -\frac{1}{x^2}e^x + \frac{1}{x}e^x + \frac{1}{x^2} \\ &\Longrightarrow \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots \\ &= \left(\frac{1}{2!} + \frac{2}{3!}x + \frac{3}{4!}x^2 + \ldots\right)(1) \\ &= F'(1) = \left(-\frac{1}{x^2}e^x + \frac{1}{x}e^x + \frac{1}{x^2}\right)(1) \\ &= 1 \end{aligned}$$

7. Prove

$$\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4 = 4\tan^{-1}(1/5) - \tan^{-1}(1/239)$$

and use the power series representation

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \mp \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

to determine how many terms would it take to compute π up to 100 digits by $\pi = 4\tan^{-1}(1)$ and by the formulas above.

Reason: Algorithmic Precision.

Solution:

$$z = a + ib = re^{i\varphi} = r(\cos\varphi + i\sin\varphi)\,,\ \tan\varphi = \frac{b}{a}$$

From $(2+i)(3+i) = 5(1+i)$ we get that the angles on the left $\tan^{-1}(1/2), \tan^{-1}(1/3)$ add up to the one on the right, $\pi/4$.

For the next identity we consider

$$\begin{aligned}(5+i)^4 &= (24+10i)^2 = 476+480i\\ (5+i)^4(-239+i) &= -114244(1+i)\\ 4\tan^{-1}(1/5) + (\pi - \tan^{-1}(1/239)) &= 5/4\pi\\ 4\tan^{-1}(1/5) - \tan^{-1}(1/239) &= \pi/4\end{aligned}$$

In order to compute π to 100 digits with the alternating series, we need the nth term being smaller than 10^{-100}. That is, with $x = 1$, $2k > 10^{100}$ or $k > \frac{1}{2}$ Googol.

$\tan^{-1}(1/2)\ :\ 2^{2k+1}\cdot 2k > 10^{100}$ or $(2k+1)\log(2) + \log(2k) > 100\log(10)$

We concentrate on the slower first term and get

$$k > \lceil -\frac{1}{2} + 50\frac{\log 10}{\log 2}\rceil = 166$$

and improve it to

$$k > 166 - \frac{1}{2}\log(2\cdot 166) > 166 - \lfloor\frac{\log 332}{2}\rfloor = 164$$

$$\begin{aligned}\tan^{-1}(1/3)\ &: 3^{2k+1}\cdot 2k > 10^{100} \text{ or } (2k+1)\log(3) + \log(2k) > 100\log(10)\\ k &> \lceil -\frac{1}{2} + 50\frac{\log 10}{\log 3}\rceil = 105\\ k &> 105 - \lfloor\frac{\log 210}{2}\rfloor = 103\end{aligned}$$

$$\begin{aligned}\tan^{-1}(1/5)\ &: 5^{2k+1}\cdot 2k > 10^{100} \text{ or } (2k+1)\log(5) + \log(2k) > 100\log(10)\\ k &> \lceil -\frac{1}{2} + 50\frac{\log 10}{\log 5}\rceil = 72\\ k &> 72 - \lfloor\frac{\log 144}{2}\rfloor = 70\end{aligned}$$

$$\tan^{-1}(1/239)\; : 239^{2k+1}\cdot 2k > 10^{100} \text{ or } (2k+1)\log(39)+\log(2k) > 100\log(10)$$
$$k > \lceil -\frac{1}{2} + 50\frac{\log 10}{\log 239}\rceil = 21$$
$$k > 21 - \lfloor\frac{\log 42}{2}\rfloor = 20$$

Thus we need a Googol steps to compute π with the standard tangent series up to 100 digits from $\tan^{-1}(1)$, $164+103=267$ steps with the second formula and $70+20=90$ steps with the third.

8. Calculate the following integrals

   (a) $\displaystyle\int_0^{2\pi} e^{(e^{it})}\,dt$

   (b) $\displaystyle\int_{|z|=1} \frac{\sin(z^2)}{(\sin z)^2}\,dz$

   (c) $\displaystyle\int_{|z|=1} \sin\left(e^{1/z}\right)\,dz$

   (d) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2-2x+2}\,dx$

   Reason: Function Theory.

   Solution:

   (a) Consider the closed path $\gamma\,:\,[0,2\pi]\longrightarrow\mathbb{C}\,,\;\gamma(t)=e^{it}$ such that

   $$\int_0^{2\pi} e^{(e^{it})}\,dt = \int_0^{2\pi}\frac{e^{(e^{it})}}{ie^{it}}\,ie^{it}\,dt = \frac{1}{i}\int_\gamma \frac{e^z}{z}\,dz = \frac{2\pi i}{i}\,e^0 = 2\pi$$

   by Cauchy's integral formula.

   (b) Consider the entire functions $f(z)=\sin(z^2)$ and $g(z)=(\sin z)^2$ where the zeros of $g(z)$ are all $\pi\mathbb{Z}$. Then f/g is holomorphic in $\{z\in\mathbb{C}\,:\,|z|<\pi\}-\{0\}$. Both functions have a twofold zero at $z=0$, since $f'(0)=g'(0)=0$ and $f''(0),g''(0)\neq 0$. Thus f/g has a removable singularity at $z=0$ and $\int_{|z|=1}(f(z)/g(z))dz=0$.

   (c) (Residue Theorem)

   $$\int_{|z|=1}\sin\left(e^{1/z}\right)\,dz = 2\pi i\;\operatorname{Res}(z=0)\;\sin\left(e^{1/z}\right)$$

   We develop $\sin\left(e^{1/z}\right)$ into a Laurent series at $z=0$ to calculate the residue. The function $w\longmapsto\sin\left(e^w\right)$ is holomorphic everywhere on $\mathbb{C}$ and can be developed into a power series with infinite radius of convergence, say $\sin(e^w) = \sum_{k=0}^{\infty} a_k w^k$. Thus $\sin\left(e^{1/z}\right) = \sum_{k=0}^{\infty} a_k z^{-k}$ and $\operatorname{Res}(z = 0) \sin\left(e^{1/z}\right) = a_1 = \sin(e^w)'\big|_{w=0} = \cos(1)$, all in all

$$\int_{|z|=1} \sin\left(e^{1/z}\right)\, dz = 2\pi i\, \cos(1).$$

(Cauchy's integral formula)

$$\begin{aligned}\int_{|z|=1} \sin\left(e^{1/z}\right)\, dz &= \int_{-\pi}^{\pi} \sin\left(e^{(e^{-is})}\right)\, ie^{is}\, ds = \int_{-\pi}^{\pi} \frac{\sin\left(e^{(e^{it})}\right)}{e^{2it}}\, ie^{it}\, dt \\ &= \int_{|w|=1} \frac{\sin(e^w)}{w^2}\, dw = 2\pi i\, (\sin(e^w))'\big|_{w=0} \\ &= 2\pi i\, \cos(1)\end{aligned}$$

(d) Let $f(z) = \dfrac{1}{z^2 - 2z + 2}$ where the denominator $z^2 - 2z + 2 = (z-1)^2 + 1 = (z - (1+i))(z - (1-i))$ has zeros $z_j = 1 \pm i$, hence f is holomorphic on $\mathbb{C} - \{z_1, z_2\}$ with first order poles z_j.

Let $r > 1$ and $\gamma_r : [0, \pi] \longrightarrow \mathbb{C}$ given as $\gamma_r(t) = 1 + re^{it}$. Then we get from the residue theorem

$$\int_{1-r}^{1+r} f(x)\, dx + \int_{\gamma_r} f(z)\, dz = 2\pi i\, \operatorname{Res}(z = 1+i) f(z).$$

As the pole $z = 1 + i$ is of first order, we have

$$\operatorname{Res}(z = 1+i) f(z) = \lim_{z\to 1+i} (z - 1 - i) f(z) = \lim_{z \to 1+i} \frac{1}{z - (1-i)} = \frac{1}{2i}$$

Furthermore

$$\int_{\gamma_r} f(z)\, dz = \int_0^{\pi} f(1 + re^{it}) rie^{it}\, dt = \int_0^{\pi} \frac{rie^{it}}{(1 + re^{it} - 1)^2 + 1}\, dt$$

so

$$\left| \int_{\gamma_r} f(z)\, dz \right| \leq \int_0^{\pi} \frac{r}{|r^2 e^{2it} + 1|}\, dt \leq \int_0^{\pi} \frac{r}{r^2 - 1}\, dt = \frac{\pi r}{r^2 - 1} \stackrel{r\to\infty}{\longrightarrow} 0$$

f is a rational real function which hasn't any real poles. The degree of the denominator polynomial is two less than that of the numerator, so the integral we are looking for exists as improper Riemannian integral, i.e.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2-2x+2}\,dx &= \lim_{r\to\infty}\int_{1-r}^{1+r} f(x)\,dx \\ &= 2\pi i\,\operatorname{Res}(z=1+i)f(z) - \lim_{r\to\infty}\int_{\gamma_r} f(z)\,dz \\ &= \pi\end{aligned}$$

9. Suppose someone gives you a coin and claims that this coin is biased; that it lands on heads only 48% of the time with an error margin of 2%. You decide to test the coin for yourself. If you want to be 95% confident that this coin is indeed biased, how many times must you flip the coin? Compare the estimations by the weak law of large numbers and by the central limit theorem!

   Reason: Coin Flips.

   Solution: Let X be the random variable such that $X = 1$ if the coin lands on heads and $X = 0$ for tails. Thus $\mu = 0.48 = p$ and $\sigma^2 = p(1-p) = 0.48 \cdot 0.52 = 0.2496$. To test the coin flip we flip it n times and allow for a 2% error of precision, i.e. $\varepsilon = 0.02$. This means we are testing the probability of the coin landing on heads being between $(0.46, 0.50)$.

   (a) (WLLN) By the law of large numbers, we want n such that

   $$P\left(\left|\bar{X}-0.48\right| > 0.02\right) \leq \frac{0.2496}{n(0.02)^2}$$

   So for a 95% confidence interval we need

   $$\frac{0.2496}{n(0.02)^2} = 0.05 \Longleftrightarrow n = 12,480$$

(b) (CLT)

$$\begin{aligned} P\left(\frac{S_n}{n};0.50\right) &= P\left(\frac{S_n-0.48n}{n}<0.02\right) \\ &= P\left(\frac{S_n-0.48n}{\sqrt{n\cdot 0.2496}<\dfrac{0.02\sqrt{n}}{\sqrt{0.2496}}}\right) \\ &\geq P\left(\frac{S_n-0.48n}{\sqrt{n\cdot 0.2496}}\leq 0.04\sqrt{n}\right) \\ &\approx \Phi(0.04\sqrt{n})\geq 0.95 \end{aligned}$$

which means $0.04\sqrt{n}=1.645$, i.e. $n=1,692$.

As we can see, the weak law of large numbers is not as powerful or accurate as the central limit theorem. However, it can still be used to a certain degree of accuracy.

10. A hat-check boy at a congress held at Hilbert's hotel completely loses track of which of hats belong to which owners, and hands them back at random to their owners as the latter leave. What is the probability P that nobody receives their own hat back?

Reason: Combinatorics.

Solution: Let D_n denote the number of derangements on a finite ordered set S of cardinality n. If s_m is the m-th element of S. If $A_m := \{\sigma \in \operatorname{Sym}(S)\,|\,\sigma(s_m)=s_m\}$, then the number W of permutations, with at least one fixed element is

$$W=\left|\bigcup_{m=1}^{n} A_m\right|$$

By the inclusion-exclusion-principle we get

$$\begin{aligned} W=\sum_{m_1=1}^{n}|A_{m_1}| \quad &- \sum_{1\leq m_1<m_2\leq n}|A_{m_1}\cap A_{m_2}| \\ + \sum_{1\leq m_1<m_2<m_3\leq n}&|A_{m_1}\cap A_{m_2}\cap A_{m_3}| \quad \mp \ldots \end{aligned}$$

Each value $A_{m_1}\cap\ldots\cap A_{m_k}$ represents the set of permutations which fix p values $m_1,\ldots,m_k$. Note that the number of permutations which fix

k values only depends on k, not on the particular values of m. There are thus $\binom{n}{k}$ terms in each summation

$$W = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} |A_1 \cap \ldots \cap A_k|$$

$|A_1 \cap \ldots \cap A_k|$ is the number of permutations fixing k elements in position. This is equal to the number of permutations which rearrange the remaining $n-k$ elements, which is $(n-k)!$, i.e.

$$W = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! = n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!}$$

So finally we have

$$D_n = |\operatorname{Sym}(S)| - W = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

and from that

$$P = \lim_{n\to\infty} \frac{D_n}{n!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} = \frac{1}{e} \approx 36.8\%$$

11. (HS-1) Prove that

$$\frac{1}{1+x+\frac{1}{y}} + \frac{1}{1+y+\frac{1}{z}} + \frac{1}{1+z+\frac{1}{x}} \leq 1$$

for all positive real numbers x, y, z

Reason: Inequality.

Solution: $0 \leq (xyz-1)^2 = x^2y^2z^2 - 2xyz + 1$ and thus $2 \leq xyz + \dfrac{1}{xyz}$.

Therefore

$$6+2x+2y+2z+\frac{2}{x}+\frac{2}{y}+\frac{2}{z}+xy+yz+zx$$
$$+\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}$$
$$\leq xyz+\frac{1}{xyz}+4+2x+2y+2z+\frac{2}{x}+\frac{2}{y}+\frac{2}{z}+xy+yz+zx$$
$$+\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}+\frac{y}{x}+\frac{z}{y}+\frac{x}{z}$$
$$\Longleftrightarrow$$
$$\left(1+x+\frac{1}{y}\right)\cdot\left(1+y+\frac{1}{z}\right)+\left(1+y+\frac{1}{z}\right)\cdot\left(1+z+\frac{1}{x}\right)$$
$$+\left(1+z+\frac{1}{x}\right)\cdot\left(1+x+\frac{1}{y}\right)$$
$$\leq\left(1+x+\frac{1}{y}\right)\cdot\left(1+y+\frac{1}{z}\right)\cdot\left(1+z+\frac{1}{x}\right)$$

12. (HS-2) Which is the smallest natural number greater than one such that the following statement holds:
    In any set of n natural numbers are at least two numbers, whose sum or difference is dividable by seven.

    Reason: Pigeon Hole Principle.

    Solution: We can exclude $n=2$ by $\{1,2\}$ and $n=3$ by $\{1,2,3\}$. We also exclude $n=4$ by $\{4,5,6,7\}$ which has sums $\{9,10,11,12,13\}$ and differences $\{3,2,1\}$. We will now show that $n=5$ has the required property. If a set of five natural numbers contains two numbers with the same remainder by division by seven, them their difference is dividable by seven. Hence we may assume that all remainders are pairwise different: five out of $\{0,1,2,3,4,5,6\}$. Hence there are at most two remainders in $R=\{1,2,3,4,5,6\}$ which do not occur. However, we have three pairs $(1,6),(2,5),(3,4)$ whose sum is dividable by seven. Since we can exclude at most two of them, the statement follows.

13. (HS-3) Determine

$$\left[\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\frac{1}{\sqrt{5}+\sqrt{6}}+\ldots+\frac{1}{\sqrt{n^2-4}+\sqrt{n^2-3}}+\frac{1}{\sqrt{n^2-2}+\sqrt{n^2-1}}\right]$$

for any odd natural number $n\geq 3$ where $[n]=\lfloor n\rfloor$ is the greatest integer smaller or equal n.

Reason: Arithmetics.

Solution: Set

$$a=\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\frac{1}{\sqrt{5}+\sqrt{6}}+\ldots+\frac{1}{\sqrt{n^2-4}+\sqrt{n^2-3}}+\frac{1}{\sqrt{n^2-2}+\sqrt{n^2-1}}$$

and

$$b=\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{4}+\sqrt{5}}+\frac{1}{\sqrt{6}+\sqrt{7}}+\ldots+\frac{1}{\sqrt{n^2-3}+\sqrt{n^2-2}}+\frac{1}{\sqrt{n^2-1}+\sqrt{n^2}}$$

Note that

$$\frac{1}{\sqrt{k}+\sqrt{k+1}}>\frac{1}{\sqrt{k+1}+\sqrt{k+2}}$$

for any positive k, so $a>b$ or

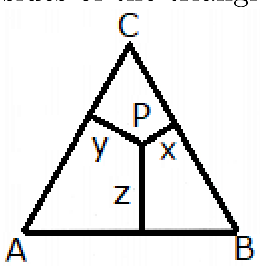
$$0<a-b=\frac{1}{\sqrt{1}+\sqrt{2}}-\left(\frac{1}{\sqrt{2}+\sqrt{3}}-\frac{1}{\sqrt{3}+\sqrt{4}}\right)-\left(\frac{1}{\sqrt{4}+\sqrt{5}}-\frac{1}{\sqrt{5}+\sqrt{6}}\right)-\ldots$$
$$\ldots-\left(\frac{1}{\sqrt{n^2-3}+\sqrt{n^2-2}}-\frac{1}{\sqrt{n^2-2}+\sqrt{n^2-1}}\right)-\frac{1}{\sqrt{n^2-1}+\sqrt{n^2}}$$
$$<\frac{1}{\sqrt{1}+\sqrt{2}}-\frac{1}{\sqrt{n^2-1}+\sqrt{n^2}}<\frac{1}{\sqrt{1}+\sqrt{2}}<1$$

On the other hand $\frac{1}{\sqrt{k}+\sqrt{k+1}}=\sqrt{k+1}-\sqrt{k}$ so

$$a=\left(\sqrt{2}-1\right)+\left(\sqrt{4}-\sqrt{3}\right)+\ldots+\left(\sqrt{n^2-1}-\sqrt{n^2-2}\right)$$
$$b=\left(\sqrt{3}-\sqrt{2}\right)+\left(\sqrt{5}-\sqrt{4}\right)+\ldots+\left(\sqrt{n^2}-\sqrt{n^2-1}\right)$$
$$a+b=n-1$$

Thus $n-1<2a<n$ or $\frac{n-1}{2}<a<\frac{n}{2}$, and since n is odd we get $[a]=\lfloor a\rfloor=\dfrac{n-1}{2}$.

14. (HS-4) Given a point P inside an equilateral triangle $\triangle ABC$ with area 1, show that for the lengths x,y,z of the perpendiculars of P onto the sides of the triangle holds



$$x+y+z=\sqrt[4]{3}$$

Reason: Geometry.

Solution: The area of an equilateral triangle of side length a is $F = 1 = \dfrac{a^2}{4} \cdot \sqrt{3}$, i.e. $a = \dfrac{2}{\sqrt[4]{3}}$. The straights $\overline{AP}, \overline{BP}, \overline{CP}$ divide the triangle into three smaller triangles with areas $\dfrac{az}{2}, \dfrac{ax}{2}, \dfrac{az}{2}$. Thus

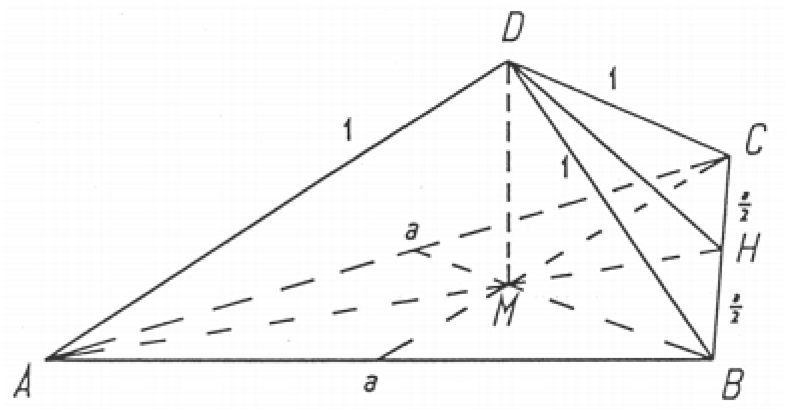
$$1 = \frac{az}{2} + \frac{ax}{2} + \frac{az}{2} = \frac{x+y+z}{\sqrt[4]{3}}$$

15. (HS-5) Prove: If for the edges of a tetrahedron $ABCD$ holds

$$\overline{AD} = \overline{BD} = \overline{CD} = 1 \text{ and } \overline{AB} = \overline{BC} = \overline{CA}\,,$$

then its surface is smaller than $\dfrac{3\sqrt{3}}{2}$.

Reason: Geometry.



Solution:

Set $a = \overline{AB} = \overline{BC} = \overline{CA}$ and $\overline{AH}$ the height of the triangle $\triangle ABC$ on $\overline{BC}$ with length $h = \dfrac{a}{2} \cdot \sqrt{3}$. Since the intersection M of all heights in $\triangle ABC$ is the barycenter which divides the height $2:1$, we have $\overline{AM} = \dfrac{2}{3}h = \dfrac{a}{3} \cdot \sqrt{3}$ and an area $F_1 = \dfrac{a}{2} \cdot h = \dfrac{a^2}{4} \cdot \sqrt{3}$.

Each of the triangles $\triangle ABD, \triangle BCD, \triangle CAD$ has isosceles of length 1 and a base line of length a. The corresponding height is also the median, so Pythagoras yields a height $\sqrt{1 - \dfrac{a^2}{4}}$, hence each of these triangles has an area $F_2 = \dfrac{a}{2}\sqrt{1 - \dfrac{a^2}{4}}$. The surface area of the tetrahedron is

thus

$$F = F_1 + 3F_2 = \frac{a^2}{4} \cdot \sqrt{3} + \frac{3a}{2}\sqrt{1 - \frac{a^2}{4}}$$

The points D and M have both the same distances to A, B, C, which means that DM is perpendicular to the plane $\triangle ABC$. This makes $\triangle AMD$ a right triangle at M and so

$$0 < \overline{AM} < \overline{AD} \iff 0 < \frac{a}{3} \cdot \sqrt{3} < 1 \iff 0 < a < \sqrt{3}$$

Hence

$$\begin{aligned} 0 < (3 - a^2)^2 &\Longrightarrow 0 < 36 - 24a^2 + 4a^4 \\ &\Longrightarrow -3a^4 + 12a^2 < 36 - 12a + a^4 = (6 - a^2)^2 \\ &\Longrightarrow 3a^2(4 - a^2) < (6 - a^2)^2 \end{aligned}$$

As all factors are positive ($0 < a^2 < 3$), we may apply the root function and get

$$a \cdot \sqrt{3} \cdot \sqrt{4 - a^2} = a \cdot 2 \cdot \sqrt{3} \cdot \sqrt{1 - \frac{a^2}{4}} < 6 - a^2$$
$$\Longrightarrow$$
$$\frac{a^2}{4} \cdot \sqrt{3} + a \cdot \frac{3}{2} \cdot \sqrt{1 - \frac{a^2}{4}} = F < \frac{3}{2} \cdot \sqrt{3}$$

14 November 2020

1. Let $u(x,t)$ satisfy the one dimensional diffusion equation $u_t = Du_{xx}$ in a space-time rectangle $R = \{0 \leq x \leq l, 0 \leq t \leq T\}$, then the maximum value of $u(x,t)$ is assumed either on the initial line ($t = 0$) or on the boundary lines ($x = 0 \, or \, x = l$). $D > 0$.

   Reason: (Weak) maximum principle for the diffusion equation.

   Solution: From Analysis we know: For a maximum in the inner of the definition area, the first derivatives have to vanish, and the second derivatives have to satisfy certain inequalities, e.g. $u_{xx} \leq 0$. If we knew (which is not the case), that $u_{xx} \neq 0$ at the maximum, then we have $u_{xx} < 0$ and simultaneously $u_t = 0$, i.e. $u_t \neq Du_{xx}$, a contradiction. But $u_{xx} = 0$ is possible, so we need some more effort.

   Let M be the maximum of $u(x,t)$ on the three boundaries $t = 0$, $x = 0$ and $t = l$. Note that a continuous function which is defined on a bounded, closed set, is bounded and assumes its maximum on this set, so M exists. We have to show that $u(x,t) \leq M$ on the whole rectangle R. Let $\varepsilon > 0$ and $v(x,t) := u(x,t) + \varepsilon x^2$. (Next goal is to show that $v(x,t) \leq M + \varepsilon l^2$ in R.) We have for $t = 0$, $x = 0$ and $x = l$

$$v(x,t) \leq M + \varepsilon l^2$$

   Furthermore

$$v_t - Dv_{xx} = u_t - D(u + \varepsilon x^2)_{xx} = u_t - Du_{xx} - 2\varepsilon D = -2\varepsilon D < 0$$

   which corresponds to a *diffusion inequality.* Assume that v assumes its maximum at an inner point (x_0, t_0), i.e. $0 < x_0 < l$ and $0 < t_0 < T$. Then $v_t = 0$ and $v_{xx} \leq 0$ at (x_0, t_0), but this contradicts the inequality above. Hence there is no maximum possible for $v(x,t)$ in the interior of R.

   Next assume that $v(x,t)$ has a maximum on the upper boundary of R $(t_0 = T, 0 < x < l)$. Again, $v_{xx}(x_0, t_0) \leq 0$. As $v(x_0, t_0) > v(x_0, t_0 - \delta)$, we get

$$v_t(x_0, t_0) = \lim_{\delta \downarrow 0} \frac{v(x_0, t_0) - v(x_0, t_0 - \delta)}{\delta} \geq 0$$

   and thus again a contradiction to the above inequality. But somewhere in R, there must be a maximum of $v(x,t)$. Thus, it has to be on the basic line or on the boundaries of R, and $v(x,t) \leq M + \varepsilon l^2$ is valid for the whole R. Thus

$$u(x,t) = v(x,t) - \varepsilon x^2 \leq M + \varepsilon(l^2 - x^2).$$

Since this is true for all $\varepsilon > 0$, we get for all $(x,t) \in R$

$$u(x,t) \leq M.$$

2. Show that $M = \{(a_n) \in \ell^2(\mathbb{C}) \,|\, \forall\, n : |a_n| \leq n^{-1}\} \subseteq \ell^2(\mathbb{C})$ is bounded and compact.

   Reason: Compactness in $\ell^2(\mathbb{C})$.

   Solution: Let $(x^{(n)})_n \subseteq M$ be a sequence, then $|x_k^{(n)}| \leq k^{-1}$ and $(x_k^{(n)})_n$ is a bounded sequence of complex numbers for any $k \in \mathbb{N}$. By Cantor's diagonalisation method we can choose a subsequence $(x^{(n_j)}) \subseteq (x^{(n)})$ such that for all $k \in \mathbb{N}$
$$\lim_{j\to\infty} x_k^{(n_j)} = x_k$$
   for some $x_k \in \mathbb{C}$. Since $|x_k^{(n_j)}| \leq k^{-1}$, we also have $|x_k| \leq k^{-1}$ for all $k \in \mathbb{N}$. This means $x := (x_k)_k \in M$ and

$$\|x^{(n_j)} - x\|_2^2 \leq \underbrace{\sum_{k=1}^{s} |x_k^{(n_j)} - x_k|^2}_{\stackrel{j\to\infty}{\longrightarrow} 0 \text{ for all } s\in\mathbb{N}} + \underbrace{\sum_{k=s+1}^{\infty} \left(\frac{2}{k}\right)^2}_{\stackrel{s\to\infty}{\longrightarrow} 0}$$

   Thus $\|x^{(n_j)} - x\|_2 \longrightarrow 0$ for $j \to \infty$ and M is sequence compact and therefore bounded.

3. Show by two different methods that the normed space $\mathcal{C} := (C^1([0,1]), \|.\|_\infty)$ is not a Banach space.

   Reason: Banach Space.

   Solution:

   (a) Solution 1.

$$\left[\left|x - \frac{1}{2}\right| + \sqrt{\frac{1}{n}}\right]^2 \geq \left(x - \frac{1}{2}\right)^2 + \frac{1}{n} \geq 0$$
$$\Longrightarrow$$
$$\left(\left(x - \frac{1}{2}\right)^2 + \frac{1}{n}\right)^{1/2} - \left|x - \frac{1}{2}\right| \leq \sqrt{\frac{1}{n}}$$

Therefore the differentiable functions $f_n(x) := \left(\left(x-\frac{1}{2}\right)^2+\frac{1}{n}\right)^{1/2} \in$ $C^1([0,1])$ converge to the function $f(x) = \left|x-\frac{1}{2}\right|$ which is not differentiable at $x = 1/2 \in [0,1]$. Hence $\mathcal{C}$ is not complete, i.e. no Banach space.

(b) Solution 2.

Let's consider the differential operator $D : (C^1([0,1]), \|.\|_\infty) \longrightarrow (C([0,1]), \|.\|_\infty)$ and a sequence $(f_n) \stackrel{n\to\infty}{\longrightarrow} f$ in $C^1([0,1])$ with $f'_n \stackrel{n\to\infty}{\longrightarrow} g \in C([0,1])$. Since we have a uniform convergence

$$\begin{aligned}\int_0^x g(s)\,ds &= \int_0^x \lim_{n\to\infty} f'_n(s)\,ds = \lim_{n\to\infty}\int_0^x f'_n(s)\,ds \\ &= \lim_{n\to\infty}(f_n(x)-f(0)) = f(x)-f(0)\end{aligned}$$

Thus $Df = f' = g$ for all $x \in (0,1]$, and because g is continuous, $Df(0) = f'(0) = g(0)$, i.e. $Df = g$. We have therefore shown that the graph

$$\Gamma(D) = \{(f, Df) \mid f \in (C^1([0,1]), \|.\|_\infty)\}$$

is closed, which is equivalent to the boundedness of D. However, the differential operator isn't bounded. This contradiction implies that $(C^1([0,1]), \|.\|_\infty)$ cannot be a Banach space.

4. Let

$$A := \begin{bmatrix} 5 & 0 & 1 & 6 \\ 3 & 3 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 6 & 0 & 3 & 0 \end{bmatrix} \in \mathbb{M}_4(\mathbb{Z}_7)$$

(a) Determine the characteristic polynomial $\chi_A(x)$ of A.

(b) Determine bases of the eigenspaces.

(c) Determine a matrix $S \in \mathrm{GL}_4(\mathbb{Z}_7)$ such that $S^{-1}AS$ is a diagonal matrix. Which one?

(d) Calculate A^{31}.

Reason: Finite Fields.

Solution:

(a)

$$\det(A - xI) = \det \begin{bmatrix} 5-x & 0 & 1 & 6 \\ 3 & 3-x & 5 & 2 \\ 0 & 0 & 3-x & 0 \\ 6 & 0 & 3 & -x \end{bmatrix}$$

$$= (3-x)\det \begin{bmatrix} 5-x & 1 & 6 \\ 0 & 3-x & 0 \\ 6 & 3 & -x \end{bmatrix}$$

$$= (3-x)^2 \det \begin{bmatrix} 5-x & 6 \\ 6 & -x \end{bmatrix}$$

$$= (3-x)^2((x-5)x-1) = (3-x)^2(x^2-5x-1)$$

$$= (3-x)^3(2-x)$$

(b) For the eigenvalue 3 we have to solve the linear equation system

$$A - 3I = \left[\begin{array}{cccc|c} 2 & 0 & 1 & 6 & 0 \\ 3 & 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 3 & 4 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and we see immediately the three eigenvectors $(0,1,0,0)^\tau$, $(1,0,5,0)^\tau$ and $(1,0,0,2)^\tau$ as basis of $E_A(3)$.

For the eigenvalue 2 we have to solve the linear equation system

$$A - 2I = \left[\begin{array}{cccc|c} 3 & 0 & 1 & 6 & 0 \\ 3 & 1 & 5 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 3 & 5 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

which is solved by the basis eigenvector $(2,3,0,6)^\tau$ of $E_A(2)$. Note that A is diagonalizable since the dimensions of the eigenspaces coincide with the algebraic multiplicities of the eigenvalues.

(c) The eigenvectors provide us the transformation matrix

$$S = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$S^{-1}AS = \begin{bmatrix} 3 & 1 & -3\cdot 5^{-1} & -3\cdot 2^{-1} \\ 0 & 0 & 5^{-1} & 0 \\ 3 & 0 & -3\cdot 5^{-1} & -1 \\ -1 & 0 & 5^{-1} & 2^{-1} \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 & 6 \\ 3 & 3 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 6 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 3 & 0 & 5 & 6 \\ 6 & 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 3 & 4 \\ 3 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} =: D$$

(d)

$$\begin{aligned} A^{31} &= (SDS^{-1})^{31} = SD^{31}S^{-1} = S\,\mathrm{diag}(3^{31},3^{31},3^{31},2^{31})S^{-1} \\ &= S\,\mathrm{diag}(3\cdot(3^6)^5, 3\cdot(3^6)^5, 3\cdot(3^6)^5, 2\cdot(2^6)^5)S^{-1} \\ &= S\,\mathrm{diag}(3\cdot 1^5, 3\cdot 1^5, 3\cdot 1^5, 2\cdot 1^5)S^{-1} = SDS^{-1} = A \end{aligned}$$

5. Let $f(x,y) = 34x^2 + 24xy + 41y^2 + 20x + 110y + 50$. Determine the Euclidean normal form of the conic section

$$Q_f = \{(x,y)^\tau \in \mathbb{R}^2 \mid f(x,y) = 0\}.$$

What are its foci and vertices in the normal form?

Reason: Quadratic Forms.

Solution: Two compute the normal form we consider the two matrices

$$A = \begin{bmatrix} 34 & 12 \\ 12 & 41 \end{bmatrix}, \; M = \left[\begin{array}{cc|c} 34 & 24/2 & 20/2 \\ 24/2 & 41 & 110/2 \\ \hline 20/2 & 110/2 & 50 \end{array}\right] = \left[\begin{array}{cc|c} 34 & 12 & 10 \\ 12 & 41 & 55 \\ \hline 10 & 55 & 50 \end{array}\right]$$

and compute the eigenvalues of A.

$$\chi_A(x) = (34-x)(41-x) - 144 = x^2 - 75x + 1250 = (x-50)(x-25)$$

To receive the eigenvector basis we solve

$$\left[\begin{array}{cc|c} 34-25 & 12 & 0 \\ 12 & 41-25 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 9 & 12 & 0 \\ 12 & 16 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and

$$\left[\begin{array}{cc|c} -16 & 12 & 0 \\ 12 & -9 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{cc|c} -4 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and get $E_A(25) = \mathbb{R}\cdot(-4,3)^\tau$, $E_A(50) = \mathbb{R}\cdot(3,4)^\tau$. Now we norm the basis and define the orthonormal matrix

$$T := \frac{1}{5}\cdot\begin{bmatrix}-4 & 3\\ 3 & 4\end{bmatrix} \in \mathrm{O}_2(\mathbb{R}).$$

Now we have

$$T^\tau AT = \frac{1}{25}\begin{bmatrix}-4 & 3\\ 3 & 4\end{bmatrix}\begin{bmatrix}34 & 12\\ 12 & 41\end{bmatrix}\begin{bmatrix}-4 & 3\\ 3 & 4\end{bmatrix} = \begin{bmatrix}-4 & 3\\ 6 & 8\end{bmatrix}\begin{bmatrix}-4 & 3\\ 3 & 4\end{bmatrix}$$
$$= \begin{bmatrix}25 & 0\\ 0 & 50\end{bmatrix} = \operatorname{diag}(25,50)$$

We set $D := \frac{1}{5}\left[\begin{array}{cc|c}-4 & 3 & 0\\ 3 & 4 & 0\\ \hline 0 & 0 & 5\end{array}\right] \in \mathrm{O}_3(\mathbb{R})$ and finally get

$$D^\tau MD = \frac{1}{25}\left[\begin{array}{cc|c}-4 & 3 & 0\\ 3 & 4 & 0\\ \hline 0 & 0 & 5\end{array}\right]\left[\begin{array}{cc|c}34 & 12 & 10\\ 12 & 41 & 55\\ \hline 10 & 55 & 50\end{array}\right]\left[\begin{array}{cc|c}-4 & 3 & 0\\ 3 & 4 & 0\\ \hline 0 & 0 & 5\end{array}\right]$$
$$= \left[\begin{array}{cc|c}-4 & 3 & 5\\ 6 & 8 & 10\\ \hline 2 & 11 & 10\end{array}\right]\left[\begin{array}{cc|c}-4 & 3 & 0\\ 3 & 4 & 0\\ \hline 0 & 0 & 5\end{array}\right] = \left[\begin{array}{cc|c}25 & 0 & 25\\ 0 & 50 & 50\\ \hline 25 & 50 & 50\end{array}\right]$$
$$= \frac{1}{25}\left[\begin{array}{cc|c}1 & 0 & 1\\ 0 & 2 & 2\\ \hline 1 & 2 & 2\end{array}\right] =: \frac{1}{25}M'$$

We now need a vector $(x_0,y_0)^\tau \in \mathbb{R}^2$ such that $\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\cdot\begin{bmatrix}x_0\\ y_0\end{bmatrix}+\begin{bmatrix}1\\ 2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ is as simple as possible, so we choose $x_0 = y_0 = -1$. Next we define $D' = \left[\begin{array}{cc|c}1 & 0 & -1\\ 0 & 1 & -1\\ \hline 0 & 0 & 1\end{array}\right]$ and compute

$$M'' = D'^{\,\tau}M'D' = \left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 1 & 0\\ \hline -1 & -1 & 1\end{array}\right]\left[\begin{array}{cc|c}1 & 0 & 1\\ 0 & 2 & 2\\ \hline 1 & 2 & 2\end{array}\right]\left[\begin{array}{cc|c}1 & 0 & -1\\ 0 & 1 & -1\\ \hline 0 & 0 & 1\end{array}\right]$$
$$= \left[\begin{array}{cc|c}1 & 0 & 1\\ 0 & 2 & 2\\ \hline 0 & 0 & -1\end{array}\right]\left[\begin{array}{cc|c}1 & 0 & -1\\ 0 & 1 & -1\\ \hline 0 & 0 & 1\end{array}\right] = \left[\begin{array}{cc|c}1 & 0 & 0\\ 0 & 2 & 0\\ \hline 0 & 0 & -1\end{array}\right]$$

From that we get that the normal form of Q_f is described by the equation $x^2 + 2y^2 - 1 = 0$ or

$$x^2 + \frac{y^2}{\left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

which is an ellipse with $a = 1$ and $b = \dfrac{1}{\sqrt{2}}$. Hence the foci are $(1/\sqrt{2}, 0), (-1/\sqrt{2}, 0)$, and the vertices are $(0, \pm\sqrt{a^2 - b^2}) = (0, \pm 1/\sqrt{2})$, and $(1, 0), (-1, 0)$.

6. Let $u(x, t)$ be a solution of the one dimensional diffusion equation $u_t = Du_{xx}$. Assume that

$$C := \int_{-\infty}^{\infty} u(x, t)\, dx$$

is independent of t, which corresponds to a constant *population*, and u is small at infinity, which means that

$$\lim_{x\to\pm\infty} xu(x, t) = 0 = \lim_{x\to\pm\infty} x^2 \frac{\partial}{\partial x} u(x, t)$$

If

$$\sigma^2(t) = \frac{1}{C} \int_{-\infty}^{+\infty} x^2 u(x, t)\, dx$$

then

$$\sigma^2(t) = 2Dt + \sigma^2(0)$$

In the special case of an initial population (i.e. for $t = 0$) which is concentrated near $x = 0$ (like a δ-function) then we get $\sigma^2(t) \approx 2Dt$.

Reason: Population Distribution.

Solution: Using the fact that u is a solution of the diffusion equation and integrating by parts twice yields:

$$\begin{aligned}\frac{C}{D}\frac{d\sigma^2}{dt} &= \frac{1}{D}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} x^2 u\, dx = \frac{1}{D}\int_{-\infty}^{\infty} x^2 \frac{\partial u}{\partial t}\, dx = \int_{-\infty}^{\infty} x^2 \frac{\partial^2 u}{\partial x^2} dx \\ &= \underbrace{\left[x^2 \frac{\partial u}{\partial x}\right]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} 2x \frac{\partial u}{\partial x}\, dx = \underbrace{-[2xu]_{-\infty}^{\infty}}_{=0} + \int_{-\infty}^{\infty} 2u\, dx \\ &= 2\int_{-\infty}^{\infty} u(x, t)\, dx = 2C\end{aligned}$$

thus

$$\frac{d\sigma^2}{dt} = 2D \Longrightarrow \sigma^2(t) = 2Dt + \sigma^2(0)$$

Last, we consider the special case that the particles start in a small interval around $x = 0$, e.g. such that $u(x, 0) = 0$ for all $|x| > \varepsilon$. Then we get automatically

$$\int_{-\infty}^{\infty} x^2 u(x,0)\,dx = \int_{-\varepsilon}^{\varepsilon} x^2 u(x,0)\,dx \leq \varepsilon^2 \int_{-\varepsilon}^{\varepsilon} u(x,0)\,dx = \varepsilon^2 C\,,$$

i.e. $\sigma^2(0) = \varepsilon^2 \approx 0$.

7. Let G be a group of order 351. Show that G has a non trivial normal subgroup.

   Reason: Sylow Subgroups.

   Solution: $351 = 13 \cdot 3^3$. By the third Sylow theorem we get that the number n_{13} of $13-$Sylow subgroups P is congruent one modulo 13 and a divisor of $[G : P] = 351 : 13 = 27$. Thus $n_{13} \in \{1, 27\}$. In case $n_{13} = 1$ we are done, since this is equivalent to $P \trianglelefteq G$ being normal by the second Sylow theorem. Let's consider the case $n_{13} = 27$. Each of the 27 $13-$Sylow subgroups is of prime order, so they intersect each other only trivially. This means we have $27 \cdot 12$ elements of order 13, and the remaining 27 elements generate the $3-$Sylow subgroups. Each of these subgroups Q has the order 27 by the first Sylow theorem, i.e. the number n_3 of $3-$Sylow subgroups is $n_3 = 1$ which again by the second Sylow theorem means, that $Q \trianglelefteq G$ is a normal subgroup.

   In any case, their is a normal subgroup in G.

8. Show that the diffusional Lotka-Volterra system ($a > 0$)

$$u_t = u(1-v) + D\Delta u \qquad (1)$$

$$v_t = av(u-1) + D\Delta v \qquad (2)$$

   with equal diffusion coefficient $D > 0$ and homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial n}(x,t) = 0 = \frac{\partial v}{\partial n}(x,t)$$

   for $x \in \partial\Omega\,,\, \Omega \subseteq \mathbb{R}^n$ of finite volume and n outward normal, Δ the Laplace operator, tends to a spatially uniform state for $t \to \infty$, i.e.

$$\lim_{t\to\infty} \nabla u = \lim_{t\to\infty} \nabla v = 0$$

Hint: Consider the *energy* of the system $s = a(u - \log u) + (v - \log v)$.

Reason: Murray, 1975, Lotka-Volterra.

Solution: Let the initial conditions be $u(x,0) = u_0(x)$ and $v(x,0) = v_0(x)$ for $x \in \Omega$. We define $s(x,t)$ as

$$s = a(u - \log u) + (v - \log v)$$

i.e. for $D = 0$ it would satisfy

$$\begin{aligned} s_t &= a\left(u_t - \frac{u_t}{u}\right) + \left(v_t - \frac{v_t}{v}\right) = au_t(1-u^{-1}) + v_t(1-v^{-1}) \\ &= au(1-v)(1-u^{-1}) + av(u-1)(1-v^{-1}) \\ &= a(1-v)(u-1) + a(u-1)(v-1) = 0 \end{aligned}$$

so the question is: How does the corresponding differential equation for s look like for $D > 0$?

In this case we get by differentiation

$$\begin{aligned} s_t &= a(1-u^{-1})(u(1-v) + D\Delta u) + (1-v^{-1})(av(u-1) + D\Delta v) \\ &= aD(1-u^{-1})\Delta u + D(1-v^{-1})\Delta v \\ &= aD\left(\Delta u - \frac{\Delta u}{u}\right) + D\left(\Delta v - \frac{\Delta v}{v}\right) \\ \Delta s &= a(\Delta u - \Delta \log u) + (\Delta v - \Delta \log v) \\ &= a\left(\Delta u - \frac{\Delta u}{u} + \frac{|\nabla u|^2}{u^2}\right) + \left(\Delta v - \frac{\Delta v}{v} + \frac{|\nabla v|^2}{v^2}\right) \end{aligned}$$

Thus

$$s_t - D\Delta s = -D\left(a\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2}\right) \leq 0$$

This can be interpreted in such a way that the energy is dissipated by the diffusion terms. The boundary conditions for s are

$$\frac{\partial s}{\partial n}(x,t) = 0 \text{ for } x \in \partial\Omega\,,$$

the initial conditions $s_0(x) := s(x,0) = a(u_0 - \log u_0) + (v_0 - \log v_0)$.

Via integration over Ω we can define the *total amount of energy* in the system at time t by

$$S(t) = \int_\Omega s(x,t)\,dx$$

Using the Neumann boundary condition and Green formula yields

$$\begin{aligned}\dot{S}(t) = \frac{dS}{dt} &= \int_\Omega s_t\,dx = \int_\Omega D\Delta s - D\left(a\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2}\right)dx \\ &= \underbrace{D\int_{\partial\Omega}\frac{\partial S}{\partial n}\,dS}_{=0} - D\int_\Omega a\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2}\,dx \leq 0\end{aligned}$$

Obviously, S is monotone non-increasing; there are two possibilities: it tends to a finite limit or it tends to $-\infty$ for $t \to \infty$. By definition, $s(x,t) \geq a+1$, so

$$S(t) = \int_\Omega s(x,t)\,dx \geq (a+1)|\Omega|\,,$$

so S indeed tends to a finite limit, which requires

$$\lim_{t\to\infty}\dot{S}(t) = -D\lim_{t\to\infty}\int_\Omega a\frac{|\nabla u|^2}{u^2} + \frac{|\nabla v|^2}{v^2}\,dx = 0$$

The only possibility to satisfy this, is that $\nabla u, \nabla v$ both tend to 0 for $t \to \infty$, i.e. the system tends to a spatially uniform state.

9. (a) Let R be a Notherian local commutative ring with 1 and maximal ideal M. If $A \trianglelefteq R$ is an ideal in R such that $A/MA \cong_R \{0\}$, then $A = (0)$.
   (b) Let R be an integral domain, and $\dim R_P = 0$ for all $P \in \operatorname{Spec}(R)$, then R is a field. The dimension is the Krull dimension.

   Reason: Ring Theory.

   Solution:

   (a) The Jacobson radical of a local ring is its maximal ideal. A as ideal of a Noetherian ring is a finitely generated submodule. Thus we can conclude by Nakayama's Lemma that $MA \neq A$ or $A = \{0\}$. We excluded the first possibility, so $A = \{0\}$.
   (b) The Krull dimension of an integral domain is defined by

$$\begin{aligned}\dim R &= \max\{n \in \mathbb{N} \mid P_0 \subsetneq \ldots \subsetneq P_n\,,\, P_j \trianglelefteq R \text{ prime ideal }\} \\ &\stackrel{(*)}{=} \sup\{\dim R_M \mid M \trianglelefteq R \text{ maximal ideal }\} \\ &= 0\end{aligned}$$

   per given condition. So every prime ideal is maximal. Particularly $\{0\} \subsetneq R$ is a prime, hence maximal, and $R = R/\{0\}$ a field.

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(*) locality of the dimension:
https://www.mathematik.uni-kl.de
/~gathmann/class/commalg-2013/commalg-2013-c11.pdf
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10. Let $\alpha \in \mathbb{C}$ a root of the polynomial $f(x) = x^3 - 3x - 1 \in \mathbb{Q}[x]$. Show that $f(x)$ is irreducible, and that there is an automorphism $\sigma \in \operatorname{Aut}(\mathbb{Q}(\alpha)/\mathbb{Q})$ with $\sigma(\alpha) = 2 - \alpha^2$. If α is chosen closest to zero, what is $+\sqrt{12-3\alpha^2}$ in the splitting field of $f(x)$? This means in terms of a polynomial in α, not numerical.

Reason: Field Extension.

Solution: Assume we have a rational root of $f(x)$. Then we get $p^3 - 3pq^2 = q^3$ with coprime integers $p, q \in \mathbb{Z}$, which cannot be both odd or both even. p even and q odd is also impossible, hence $p = 2k+1$ and $q = 2l$. But now we get $(2k+1)^3 \equiv 0 \mod 4$ which is not possible. This shows that $f(x)$ is irreducible over $\mathbb{Q}$.

$$\begin{aligned} f(2-\alpha^2) &= (2-\alpha^2)^3 - 3(2-\alpha^2) - 1 \\ &= 8 - 12\alpha^2 + 6\alpha^4 - \alpha^6 - 6 + 3\alpha^2 - 1 \\ &= -(\alpha^6 - 6\alpha^4 - 2\alpha^3 + 9\alpha^2 + 1 + 6\alpha) - 2\alpha^3 + 6\alpha + 2 \\ &= -(\alpha^3 - 3\alpha - 1)^2 - 2(\alpha^3 - 3\alpha - 1) \\ &= 0 \end{aligned}$$

Assume $\alpha = 2 - \alpha^2$. Then $2\alpha = -1 \pm \sqrt{3}$ and $f(\alpha) \neq 0$. Hence we have found two distinct roots $\alpha, 2-\alpha^2$ of $f(x)$, i.e. $\mathbb{Q}(a)$ is the decomposition field of $f(x)$, because $\mathbb{Q}(\alpha)[x]$ contains two of three and therewith all linear factors of $f(x)$. Thus $\mathbb{Q}(\alpha) \supsetneq \mathbb{Q}$ is a Galois extension and the automorphism group operates transitive on the roots of $f(x)$, which proves the existence of σ.

$f(x)$ has local extrema at $x = \pm 1$ with $f(-1) = 1$ and $f(1) = -3$. This implies that all roots are real. With $f(-1/3) = -1/27 \approx 0$ we have a root near $x = -1/3$. The other roots must be greater than 1 and less than -1. Long division by $x - \alpha$ yields

$$x^3 - 3x - 1 = (x-\alpha)\cdot\left(x + \frac{1}{2}\left(\alpha + \sqrt{12-3\alpha^2}\right)\right)\cdot\left(x + \frac{1}{2}\left(\alpha - \sqrt{12-3\alpha^2}\right)\right)$$

Since we know that $1/3 \approx \alpha$ and $f(2-\alpha^2) = 0$, we have

$$2 - \alpha^2 = -\frac{1}{2}\left(\alpha \pm \sqrt{12-3\alpha^2}\right) \Longleftrightarrow \pm\sqrt{12-3\alpha^2} = 2\alpha^2 - 4 - \alpha$$

Since the right hand side is negative for our choice of α, we have

$$+\sqrt{12-3\alpha^2} = -2\alpha^2+\alpha+4\,.$$

11. (HS-1) Determine all $a \in \mathbb{R}$ such that

$$x(x+1)(x+2)(x+3) = a$$

has no real solution, a unique real solution, exactly two, three, or four real solutions, more than four real solutions.

Reason: Equation Solving.

Solution: $(0,2,4,3,2)$. If we set $z := x + (0+1+2+3)/4 = x + 3/2$ then the equation reads

$$a = x(x+1)(x+2)(x+3) = \left(z-\frac{3}{2}\right)\left(x-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)\left(z+\frac{3}{2}\right)$$
$$= \left(z^2-\frac{9}{4}\right)\left(z^2-\frac{1}{4}\right) = z^4-\frac{5}{2}z^2+\frac{9}{16}$$
$$\Longleftrightarrow$$
$$z^2 = \frac{5}{4} \pm \sqrt{\frac{25}{16}-\frac{9}{16}+a} = \frac{5}{4} \pm \sqrt{1+a}$$

(a) $a = 9/16 \Longrightarrow z \in \{0, -\sqrt{5/2}, +\sqrt{5/2}\} \Longrightarrow x \in \left\{-\frac{3}{2}, -\frac{3\pm\sqrt{10}}{2}\right\}$

(b) $a < -1$ doesn't allow any real solution.

(c) $a = -1 \Longrightarrow x \in \left\{-\frac{3\pm\sqrt{5}}{2}\right\}$

(d) $-1 < a < 9/16 \Longrightarrow x = -\frac{3}{2} \pm \sqrt{\frac{5}{4} \pm \sqrt{1+a}}$

(e) $a > 9/16 \Longrightarrow z^2 = \frac{5}{4} + \sqrt{1+a} \Longrightarrow x = -\frac{3}{2} \pm \sqrt{\frac{5}{4} + \sqrt{1+a}}$

12. (HS-2) An international conference has 30 scientists who speak English, Russian or Spanish. The number of people who speak exactly two languages is more than twice as big, but less than thrice as much as the number of people who speak only one language, which are as many as speak all three languages. Those who speak only English are more than those who speak only Russian, but less than those who speak

only Spanish. The number of those who speak only English is less than thrice the number of people who speak only Russian. How many people do speak only English, Russian, Spanish, and how many all three languages? (The conference language is French.)

Reason: Combinatorics.

Solution: Let's denote the number of people who speak only Russian by R, only English by E, only Spanish by S, and people who speak one language by U, two languages by T and all three languages by A. Thus we are given the conditions:

(a) $3U > T > 2U$
(b) $U = A$
(c) $S > E > R$
(d) $3R > E$

We are not interested in T, so we eliminate it by the condition $30 = U + T + A = T + 2U$ and get $5U > 30 > 4U$ which is only possible for $U = E + R + S = A = 7$. Since $S > E > R$ we must have $S = 4 > E = 2 > R = 1$ which can be seen by checking $R = 1$ first.

13. (HS-3) Calculate (manually!)

$$z = \frac{65533^3 + 65534^3 + 65535^3 + 65536^3 + 65537^3 + 65538^3 + 65539^3}{32765 \cdot 32766 + 32767 \cdot 32768 + 32768 \cdot 32769 + 32770 \cdot 32771}$$

Reason: Calculation.

Solution: Set $n := 2^{15} = 32768$

$$z = \frac{(2n-3)^3 + (2n-2)^3 + (2n-1)^3 + (2n)^3 + (2n+1)^3 + (2n+2)^3 + (2n+3)^3}{(n-3)(n-2) + (n-1)n + n(n+1) + (n+2)(n+3)}$$

$$= \frac{7 \cdot (2n)^3 + 3 \cdot (2n)^2 \cdot (-3-2-1+1+2+3)}{4n^2 + n \cdot (-3-2-1+1+2+3) + 6 + 6}$$

$$+ \frac{3 \cdot (2n) \cdot ((-3)^2 + (-2)^2 + (-1)^2 + 1^2 + 2^2 + 3^2) - 3^3 - 2^3 - 1^3 + 1^3 + 2^3 + 3^3}{4n^2 + n \cdot (-3-2-1+1+2+3) + 6 + 6}$$

$$= \frac{56n^3 + 168n}{4n^2 + 12} = \frac{4 \cdot 14 \cdot n \cdot (n^2+3)}{4 \cdot (n^2+3)} = 14n = 7 \cdot 2^{16} = 458752$$

14. (HS-4) Show that ($n \in \mathbb{N}_0$)

$$f_n(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

has at most one real zero.

Reason: Exponential Function.

Solution: We show by induction the stronger statement: If n is even, then $f_n(x) > 0$; and if n is odd, then $f_n(x)$ has exactly one zero. This is true for $n = 0, 1$ so we may assume $n \geq 2$. Note that $f_n'(x) = f_{n-1}(x)$.

If n is odd, then by induction hypothesis $f_n'(x) = f_{n-1}(x) > 0$ and $f_n(x)$ is strictly monotone increasing. But $\lim_{x\to-\infty} f_n(x) = -\infty$ and $\lim_{x\to+\infty} f_n(x) = +\infty$, so there is exactly one zero for $f_n(x)$.

If n is even, then by induction hypothesis $f_n'(x_0) = f_{n-1}(x_0) = 0$ for exactly one point x_0. Now $f_n''(x_0) = f_{n-2}(x_0) > 0$, hence x_0 is a global minimum. As $f_n(x_0) = f_{n-1}(x_0) + \dfrac{x_0^n}{n!} = \dfrac{x_0^n}{n!} > 0$ for even n, we have shown that $f_n(x) > 0$ everywhere.

15. (HS-5) Find all $\lambda \in \mathbb{R}$ such that

$$\sin^4 x - \cos^4 x = \lambda(\tan^4 x - \cot^4 x)$$

has no, exactly one, exactly two, more than two real solutions in $\left(0, \dfrac{\pi}{2}\right)$

Reason: Trigonometry.

Solution: The equation holds for all $\lambda \in \mathbb{R}$ in case $x = \pi/4$. Furthermore we have an invariance $x \longleftrightarrow (\pi/2) - x$ on the interval given, i.e. every solution in $\left(0, \dfrac{\pi}{4}\right)$ corresponds uniquely to a solution in $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$. This already excludes the possibilities of *no solution* and *exactly two solutions*, plus we may assume $x \in \left(0, \dfrac{\pi}{4}\right)$.

Define $L : \left(0, \dfrac{\pi}{4}\right) \longrightarrow \mathbb{R}$ by the quotient

$$L(x) = \frac{\sin^4 x - \cos^4 x}{\tan^4 x - \cot^4 x} = \frac{\sin^4 x - \cos^4 x}{\dfrac{\sin^4 x}{\cos^4 x} - \dfrac{\cos^4 x}{\sin^4 x}} = \frac{\sin^4 x \cos^4 x}{\sin^4 x + \cos^4 x}$$

for $\sin^4 x \neq \cos^4 x$ which is given on the interval $\left(0, \dfrac{\pi}{4}\right)$. Now

$$L(x) = \frac{\sin^4(2x)}{16(1 - 2\sin^2 x \cos^2 x)} = \frac{\sin^4(2x)}{8(2 - \sin^2(2x))}$$

This shows that $L(x)$ is strictly monotone increasing on $\left(0, \dfrac{\pi}{4}\right)$ and assumes every value in $\left(0, \dfrac{1}{8}\right)$ exactly once.

We have exactly three solutions for any value $\lambda \in \left(0, \dfrac{1}{8}\right)$, and exactly one solution for any value $\lambda \in \mathbb{R} - \left(0, \dfrac{1}{8}\right)$.

15 October 2020

1. Let $(a_n) \subseteq \mathbb{R}$ be a sequence of real numbers such that $a_n \leq n^{-3}$ for all $n \in \mathbb{N}$. Given the family $\mathcal{A}$ of functions $f_n : [0,1] \longrightarrow \mathbb{R}$ defined by $f_n(x) = \sum_{k=n}^{\infty} a_k \sin(kx)$ for $n \in \mathbb{N}$, show that every sequence $(g_n) \subseteq \mathcal{A}$ contains a subsequence (g_{n_k}) which converges uniformly on $[0,1]$.

   Reason: Arzelà-Ascoli.

   Solution:

   $$|g_n(x)| \leq \sum_{k=n}^{\infty} \left|\frac{\sin(kx)}{k^3}\right| \leq \sum_{k=1}^{\infty} \frac{kx}{k^3} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} =: L < \infty$$

   i.e. $\|g_n\|_\infty \leq L$ for all $n \in \mathbb{N}$.

   We show that (g_n) is uniformly continuous. Let $\varepsilon > 0$ and $\delta = \varepsilon/L$. Then we have for all $x, y \in [0,1]$ with $|x-y| < \delta$ and all $n \in \mathbb{N}$

   $$|g_n(x)-g_n(y)| \leq \sum_{k=n}^{\infty} \left|\frac{\sin(kx)-\sin(ky)}{k^3}\right| \leq \sum_{k=1}^{\infty} \frac{|x-y|}{k^2} = \frac{\pi^2}{6}|x-y| < \varepsilon$$

   Since $g_n \in C([0,1])$ for all $n \in \mathbb{N}$, we may apply the theorem of Arzelà-Ascoli. Thus there is a subsequence $(g_{n_k}) \subseteq (g_n)$ which converges uniformly on $[0,1]$.

2. Let $\pi : \mathbb{R}^n \longrightarrow \mathbb{T}^n$ be the canonical projection and $f := \pi|_{[0,1]^n}$ its restriction on the closed unit cube. Show with the help of $f : [0,1]^n \longrightarrow \mathbb{T}^n$, that a quotient map in general doesn't have to be open.

   Solution: The set $U_0 := \left]-\frac{1}{2}, \frac{1}{2}\right[^n \subseteq \mathbb{R}^n$ is open, so $U := U_0 \cap [0,1]^n = \left[0, \frac{1}{2}\right[^n \subseteq [0,1]^n$ is open in the subspace topology. However, $f(U) \subseteq \mathbb{T}^n$ is not open, since $\pi^{-1}(f(U)) = \pi^{-1}\left(\left[0, \frac{1}{2}\right[^n + \mathbb{Z}^n\right) = \left[0, \frac{1}{2}\right[^n \subseteq \mathbb{R}^n$ is not open. Hence f isn't open.

   It remains to show that $f : [0,1]^n \longrightarrow \mathbb{T}^n$ is a quotient map, i.e that a set $U \subseteq \mathbb{T}^n$ is open if and only if $f^{-1}(U) \subseteq [0,1]^n$ is open. Hence we must show

   $$f^{-1}(U) \subseteq [0,1]^n \text{ open } \Longleftrightarrow \pi^{-1}(U) \subseteq \mathbb{R}^n \text{ open}$$

   which is equivalent to

   $$f^{-1}(U) \subseteq [0,1]^n \text{ closed } \Longleftrightarrow \pi^{-1}(U) \subseteq \mathbb{R}^n \text{ closed}$$

The implication that $f^{-1}(A) \subseteq [0,1]^n$ is closed for closed sets $A \subseteq \mathbb{T}^n$ is the continuity of π and the definition of the subspace topology. So let us conversely assume that $f^{-1}(A) \subseteq [0,1]^n$ is closed for some set $A \subseteq \mathbb{T}^n$. Then $f^{-1}(A)$ is compact, because $[0,1]^n$ is compact. Since continuous functions map compact sets on compact sets,

$$\pi(f^{-1}(A) = f(f^{-1}(A)) = A \subseteq \mathbb{T}^n$$

is also compact. However, $\mathbb{T}^n$ is Hausdorff, so compact subsets are closed. Hence A is closed what had to be shown.

3. Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ be the complex open unit disk and let $0 < a < 1$ be a real number. Suppose $f : D \longrightarrow \mathbb{C}$ is a holomorphic function such that $f(a) = 1$ and $f(-a) = -1$.

   (a) Prove that $\sup_{z\in D}\{|f(z)|\} \geq \dfrac{1}{a}$.

   (b) Prove that if f has no root, then $\sup_{z\in D}\{|f(z)|\} \geq \exp\left(\dfrac{1-a^2}{4a}\,\pi\right)$.

   Reason: Holomorphic Function.

   Solution:

   (a) Consider $g(z) = \dfrac{f(z)-f(-z)}{2z}$ for $z \neq 0$ and let $g(0) = f'(0)$. Then g is a holomorphic function, too, with $g(a) = 1/a$. By triangle inequality and the maximum principle we have for $a < r < 1$

$$\begin{aligned}\sup_{z\in D}|f(z)| &\geq \max_{|z|=r}|f(z)| \geq r \cdot \max_{|z|=r}\frac{|f(z)|+|f(-z)|}{2r}\\ &\geq r\cdot\max_{|z|=r}|g(z)| = r\cdot|g(a)| = \frac{r}{a}\end{aligned}$$

   from which the statement follows for $r \to 1-0$.

   (b) Let $M := \sup_{z\in D}|f(z)|$. Since f is not constant, $|f| < M$ everywhere in D. And from $f(a) = 1$ we know, that $M > 1$. The function f is nonzero on the simply connected set D, so it has a logarithm; i.e. there is a holomorphic function $g(z) : D \longrightarrow \mathbb{C}$ such that $f(z) = \exp(g(z))$. W.l.o.g. we assume $g(a) = 0$. From $f(-a) = -1$ we get $g(-a) = k\pi\, i$ with some odd integer k, and from $|f| < M$ we get $\Re(g) < \log M$. Denote by H the half-plane $\Re(z) < \log M$. Then $g : D \longrightarrow H$. Next we define the linear fractional transformations

$$\varphi : D \longrightarrow D,\;\; \varphi(z) = \frac{z+a}{1+az},\;\; \varphi^{-1}(z) = \frac{z-a}{1-az}$$

and
$$\psi \, : \, H \longrightarrow D, \;\; \psi(z) = \frac{z}{2\log M - z}.$$

Now $h := \psi \circ g \circ \varphi \; : \; D \longrightarrow D$ with $h(0) = 0$. Schwarz's lemma applied to h and the point $\varphi^{-1}(-a) = \dfrac{-2a}{1+a^2}$ gives us $\left| h\left(\dfrac{-2a}{1+a^2}\right)\right| \leq \dfrac{2a}{1+a^2}$. Thus

$$\begin{aligned}\frac{2a}{1+a^2} &\geq \left|h(\varphi^{-1}(-a))\right| = |\psi(g(-a))| = \left|\frac{k\pi\, i}{2\log M - k\pi i}\right| \\ &= \frac{1}{\sqrt{\left(\dfrac{2\log M}{|k|\pi}\right)^2 + 1}}\end{aligned}$$

So

$$\begin{aligned}\log M &\geq \frac{|k|\pi}{2}\sqrt{\left(\frac{1+a^2}{2a}\right)^2 - 1} \\ &= \frac{|k|\pi}{2} \cdot \frac{1-a^2}{2a} \geq \frac{1-a^2}{4a}\,\pi.\end{aligned}$$

Remark: The estimates in the problem are sharp. For example, we have equality for $f(z) = \dfrac{z}{a}$ in part (a), and in part (b) for

$$f(z) = -i\,\exp\left(\frac{iz - a^2}{iz+1} \cdot \frac{\pi}{2a}\right).$$

4. Let $0 < p \leq a, b, c, d, e \leq q$ and show that

$$(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right) \leq 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2.$$

This is a special case of a general inequality. Which is the general case and how is it proven?

Reason: Inequality.

Solution:

$$f(a,b,c,d,e) := (a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)$$

is a convex function of each of the variables. Hence the maximum is taken on one of the 32 vertices of the 5-cube given by $p \leq a, b, c, d, e \leq q$. If there are $n\ \ p's$ and $5-n\ \ q's$, then we have to maximize the quadratic function

$$f(n) = (np + (5-n)q)\left(\frac{n}{p} + \frac{5-n}{q}\right) = 25 + n(5-n)\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2.$$

So $f(n)$ takes its maximum at $n = 5/2$, i.e. $n \in \{2, 3\}$ where it has the value $25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2$.

The general theorem (Kantorovich's inequality) is: Let $x_1, \ldots, x_n \in [a, b]$, where $0 < a < b$, then

$$(x_1 + \ldots + x_m)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_m}\right) \leq \frac{(a+b)^2}{4ab}\, m^2.$$

The same argumentation as above results in the quadratic function

$$f(n) = -n^2\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + mn\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + m^2$$

with a maximum at $n = m/2$ and a value

$$\begin{aligned} f(m/2) &= -\frac{m^2}{4}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + \frac{m^2}{2}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + m^2 \\ &= \frac{m^2}{4}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + \frac{4m^2}{4} \\ &= \frac{m^2}{4}\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right)^2 \\ &= \frac{m^2}{4} \cdot \frac{(a+b)^2}{ab} \end{aligned}$$

5. Let $n > 1$ be an integer. There are n lamps $L_0, \ldots, L_{n-1}$ arranged in a circle. Each lamp is either ON (1) or OFF (0). A sequence of steps $S_0, \ldots, S_i, \ldots$ is carried out. Step S_j affects the state of L_j only (leaving the states of all other lamps unaltered) as follows:

   If L_{j-1} is ON, S_j changes the state of L_j from ON to OFF or from

OFF to ON;

If L_{j-1} is OFF, S_j leaves the state of L_j unchanged.

The lamps are labeled modulo n, that is $L_{-1} = L_{n-1}, L_0 = L_n$, etc. Initially all lamps are ON. Show that

(a) there is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are ON again;

(b) if $n = 2^k$, then all lamps are ON after $(n^2 - 1)$ steps;

(c) if $n = 2^k + 1$, then all lamps are ON after $(n^2 - n + 1)$ steps.

Reason: Algorithm.

Solution:

(a) Let $x_j \in \{0, 1\}$ represent the state of lamp L_j. Operation S_j affects the state of L_j, which in the previous round has been set to the value x_{j-n}. At the moment when S_j is being performed, lamp L_{j-1} is in the state x_{j-1}. Consequently,

$$x_j \equiv x_{j-n} + x_{j-1} \mod 2, \quad (1)$$

This is true for all $j \geq 0$. Note that the initial state (all lamps ON) corresponds to

$$x_{-n} = x_{-n+1} = \ldots = x_{-2} = x_{-1} = 1. \quad (2)$$

The state of the system at instant j can be represented by the vector $v_j = (x_{j-n}, \ldots, x_{j-1})$, $v_0 = (1, \ldots, 1)$. Since there are only n feasible vectors, repetitions must occur in the sequence $v_0, v_1, v_2, \ldots$ The operation that produces v_{j+1} from v_j is invertible. Hence, the equality $v_{j+m} = v_j$ implies $v_m = v_0$; the initial state recurs in at most 2^n steps proving the first part.

Let's consider equation (1):

$$\begin{aligned} x_j &\equiv x_{j-n} + x_{j-1} \mod 2 \\ &\equiv (x_{j-2n} + x_{j-n-1}) + (x_{j-1-n} + x_{j-2}) \mod 2 \\ &\equiv x_{j-2n} + 2x_{j-n-1} + x_{j-2} \mod 2 \\ &\equiv x_{j-3n} + 3x_{j-2n-1} + 3x_{j-n-2} + x_{j-3} \mod 2 \\ &\equiv \ldots \end{aligned}$$

After r iterations we arrive at the equality

$$x_j = \sum_{i=0}^{r} \binom{r}{i} x_{j-(r-i)n-i} \mod 2 \quad (3)$$

holding for all j, r such that $j - (r-i)n - i \geq -n$.
If $r = 2^k$, then the binomial coefficients are all even except the two outer ones, and we obtain

$$x_j \equiv x_{j-rn} + x_{j-r} \quad (\text{for } r = 2^k), \quad (4)$$

provided the subscripts do not go below $-n$, i.e., for $j \geq (r-1)n$.

(b) If $n = 2^k$, choose $j \geq n^2 - n = (2^k - 1)2^k$, and with $r = n$, we obtain from (4)

$$x_j \equiv x_{j-n^2} + x_{j-n} \equiv x_{j-n^2} + (x_j - x_{j-1}) \mod 2.$$

Hence $x_{j-n^2} \equiv x_{j-1} \mod 2$, showing that the sequence (x_j) is periodic with period $n^2 - 1$.

(c) If $n = 2^k + 1$, choose $j \geq n^2 - n = (2^k + 1)2^k$, and set in (4) $r = n - 1$, obtaining

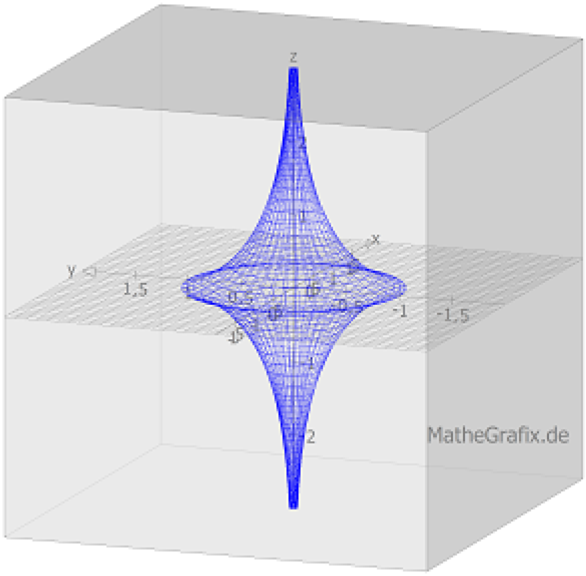
$$\begin{aligned} x_j &\equiv x_{j-rn} + x_{j-r} \mod 2 \\ &\equiv x_{j-n^2+n} + x_{j-n+1} \mod 2 \\ &\equiv x_{j-n^2+n} + (x_{j+1} - x_j) \mod 2 \\ &\equiv x_{j-n^2+n} + x_{j+1} + x_j \mod 2 \end{aligned}$$

Hence $x_{j-(n^2-n+1)} \equiv x_j$, showing the sequence is periodic with period $n^2 - n + 1$.

6. The pseudosphere is the rotational surface of the tractrix, e.g. parameterized by

$$f \,:\, \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \,,\; f(x,y) = \begin{bmatrix} \cos(y)/\cosh(x) \\ \sin(y)/\cosh(x) \\ x - \tanh(x) \end{bmatrix} .$$

Show that the pseudosphere has a constant negative Gauß curvature.



Reason: Curvature Pseudo-Sphere.

Solution:

$$f_x = \begin{bmatrix} -\dfrac{\cos(y)\sinh(x)}{\cosh^2(x)} \\ -\dfrac{\sin(y)\sinh(x)}{\cosh^2(x)} \\ \dfrac{\sinh^2(x)}{\cosh^2(x)} \end{bmatrix}, \; f_y = \begin{bmatrix} \dfrac{-\sin(y)}{\cosh(x)} \\ \dfrac{\cos(y)}{\cosh(x)} \\ 0 \end{bmatrix}, \; f_x \times f_y = \begin{bmatrix} -\dfrac{\cos(y)\sinh^2(x)}{\cosh^3(x)} \\ -\dfrac{\sin(y)\sinh^2(x)}{\cosh^3(x)} \\ -\dfrac{sinh(x)}{\cosh^3(x)} \end{bmatrix}$$

Thus we get

$$\|f_x \times f_y\|^2 = \frac{\sinh^4(x)}{\cosh^6(x)} + \frac{\sinh^2(x)}{\cosh^6(x)} = \frac{\sinh^2(x)}{\cosh^4(x)} \left(\frac{\sinh^2(x)+1}{\cosh^2(x)} \right) = \frac{\sinh^2(x)}{\cosh^4(x)}$$

and in case $x \geq 0$

$$N = \frac{f_x \times f_y}{\|f_x \times f_y\|} = \begin{bmatrix} -\dfrac{\cos(y)\sinh(x)}{\cosh(x)} \\ -\dfrac{\sin(y)\sinh(x)}{\cosh(x)} \\ -\dfrac{1}{\cosh(x)} \end{bmatrix} = \begin{bmatrix} -\cos(y)\tanh(x) \\ -\sin(y)\tanh(x) \\ -\dfrac{1}{\cosh(x)} \end{bmatrix}$$

Now we calculate

$$N_x = \begin{bmatrix} -\dfrac{\cos(y)}{\cosh^2(x)} \\ -\dfrac{\sin(y)}{\cosh^2(x)} \\ \dfrac{\sinh(x)}{\cosh^2(x)} \end{bmatrix} \, , \; N_y = \begin{bmatrix} \sin(y)\tanh(x) \\ -\cos(y)\tanh(x) \\ 0 \end{bmatrix}$$

Observe that $D_pN = D_pf \cdot A_p$ with $A_p = \begin{bmatrix} \dfrac{1}{\sinh(x)} & 0 \\ 0 & -\sinh(x) \end{bmatrix}$ such that the Gauß curvature is given by

$$\kappa_p = \det A_p = -1 \text{ for all } p \in \mathbb{R}^2.$$

The sign of the determinant of A_p does not change, if x changes its sign and so skips signs in D_pN, because A_p is a 2×2 matrix. The Gauß curvature is thus negative, too, in case $x < 0$.

7. Let $\mathfrak{g}$ be a Lie algebra with trivial center $\mathfrak{Z}(\mathfrak{g}) = \{0\}$ over a field of characteristic not equal two and

$$\begin{aligned} \mathfrak{A}(\mathfrak{g}) &= \{\varphi : \mathfrak{g} \stackrel{linear}{\longrightarrow} \mathfrak{g} \,|\, [\varphi(X), Y] = [\varphi(Y), X] \text{ for all } X, Y \in \mathfrak{g}\} \\ &= \operatorname{lin}\{\alpha, \beta \neq 0 \,|\, [\alpha, \beta] = \alpha\beta - \beta\alpha = \beta\} \end{aligned}$$

Show that image $\operatorname{im}\beta$ and kernel $\ker\beta$ of β are ideals in $\mathfrak{g}$.

Hint: $\mathfrak{A}(\mathfrak{g})$ is a $\mathfrak{g}$-module by $X.\varphi = [\operatorname{ad} X, \varphi]$.

Reason: Lie Algebras.

Solution: Let $L, K \in \ker\beta$, $B = \beta(A), D = \beta(C) \in \operatorname{im}\beta$.

$$\begin{aligned} \beta(\alpha(K)) &= \alpha(\beta(K)) - [\alpha,\beta](K) = -\beta(K) = 0 \\ \beta(A + \alpha(A)) &= B + \alpha(\beta(A)) - [\alpha,\beta](A) = B + \alpha(B) - \beta(A) = \alpha(B) \\ [L, B] &= [L, \beta(A)] = -[\beta(L), A] = 0 \\ [X, \beta([L, K])] &= -[\beta(X), [L, K]] = [\beta(L), [K, X]] + [\beta(K), [X, L]] = 0 \\ &\Longrightarrow \beta([L, K]) \in \mathfrak{Z}(\mathfrak{g}) = \{0\} \\ &\Longrightarrow [L, K] \in \ker\beta \\ [X, [\beta(A), \beta(C)]] &= -[X, [\beta^2(A), C]] = -[\beta^2(X), [A, C]] = -[X, \beta^2([A, C])] \\ &\Longrightarrow [\beta(A), \beta(C)] + \beta^2([A, C]) \in \mathfrak{Z}(\mathfrak{g}) = \{0\} \\ &\Longrightarrow [\beta(A), \beta(C)] = \beta(-\beta([A, C])) \in \operatorname{im}\beta \end{aligned}$$

The calculations show that $\mathfrak{K} := \ker \beta$ and $\mathfrak{I} := \operatorname{im} \beta$ are $\mathfrak{A}(\mathfrak{g})$ invariant, commuting subalgebras of $\mathfrak{g}$.

Let $\gamma \in \mathfrak{A}(\mathfrak{g})$ such that $X.\gamma = 0$ for all $X \in \mathfrak{g}$. Then

$$\begin{aligned} 0 &= (X.\gamma)(Y) = [X, \gamma(Y)] - \gamma([X, Y]) = [X, \gamma(Y)] + \gamma([Y, X]) \\ &= [X, \gamma(Y)] - (Y.\gamma)(X) + [Y, \gamma(X)] = 2[X, \gamma(Y)] \\ &\Longrightarrow \gamma(Y) \in \mathfrak{Z}(\mathfrak{g}) = \{0\} \Longrightarrow \gamma = 0 \end{aligned}$$

Hence $A.\alpha \neq 0$ and $B.\beta \neq 0$ for some $A, B \in \mathfrak{g}$. Since $X.\alpha, X.\beta \in \mathfrak{A}(\mathfrak{g})$ we can write

$$\begin{aligned} X.\alpha &= \lambda(X)\alpha + \mu(X)\beta \\ X.\beta &= \nu(X)\alpha + \omega(X)\beta \end{aligned}$$

$$\begin{aligned} X.\beta = X.[\alpha, \beta] = [X.\alpha, \beta] + [\alpha, X.\beta] = \lambda(X)\beta + \omega(X)\beta = \nu(X)\alpha + \omega(X)\beta \\ \Longrightarrow \nu(X) = \lambda(X) = 0 \Longrightarrow X.\alpha = \mu(X)\beta\,,\; X.\beta = \omega(X)\beta \end{aligned}$$

$$X.\alpha = \mu(X)\beta\,,\; X.\beta = \omega(X)\beta$$

This implies especially that $\mathfrak{I} \trianglelefteq \mathfrak{g}$ and $\ker \omega \supseteq \mathfrak{K} \trianglelefteq \mathfrak{g}$ are ideals:

$$\begin{aligned} [X, \beta(Y)] &= (X.\beta)(Y) + \beta([X, Y]) = \omega(X)\beta(Y) + \beta([X, Y]) \in \operatorname{im} \beta \\ \beta([X, K]) &= (X.\beta)(K) - [X, \beta(K)] = \omega(X)\beta(K) = 0 \\ \omega(K)\beta(X) &= (K.\beta)(X) = \beta([X, K]) = 0 \end{aligned}$$

8. (HS-1) Given a positive integer n. Assume that 4^n and 5^n start with the same digit in the decimal system. Show that this digit has to be 2 or 4.

   Reason: Numbers.

   Solution: Let z be the leading digit of 4^n and 5^n, so

$$\begin{aligned} z \cdot 10^r \leq 4^n = 2^{2n} < (z+1) \cdot 10^r \\ z \cdot 10^s \leq 5^n < (z+1) \cdot 10^s \end{aligned}$$

   If we square the second and multiply both, we get

$$z^3 \cdot 10^{r+2s} \leq 10^{2n} < (z+1)^3 \cdot 10^{r+2s} \Longrightarrow 1 \leq z^3 \leq 10^{2n-r-2s} < (z+1)^3 \leq 1000$$

   This means that $2n - r - 2s \in \{0, 1, 2\}$.

(a) $2n - r - 2s = 0$

In this case we have $z = 1$. If $1 \cdot 10^r < 4^n$, then we would get by the procedure above that $z^3 \cdot 10^{r+2s} = 1 \cdot 10^{2n} < 10^{2n}$ which is impossible. Hence $4^n = 10^r$ which is only possible for $n = r = 0$ since $5 \nmid 4^n$. But this contradicts our choice of n.

(b) $2n - r - 2s = 1$

Here we get $z^3 \leq 10^1 = 10$ which means $z = 2$.

(c) $2n - r - 2s = 2$

Here we get $z^3 \leq 10^2 = 100 < (z + 1)^3$ and so $z \leq \sqrt[3]{100} = 4.64\ldots < z + 1$ which means $z = 4$.

9. (HS-2) A parcel service charges a price proportional to the sum height plus length plus width per box. Could it be, that there is a case where it is cheaper to put a more expensive package into a cheaper box?

Reason: Optimization.

Solution: Assume we have boxes $B = B(a, b, c) \subseteq A = A(x, y, z)$. We define the sets $A_\delta := \{x \in \mathbb{R}^3 \,|\, d(A, x) \leq \delta\}$ and similar B_δ of all points not farther away from the boxes than δ. Each of these sets consists of the box itself, 6 boxes of height δ above each surface, 12 quarter cylinders of radius δ along each edge, and eight eighth of a ball of radius δ above each vertex. Hence the volumes are

$$|A_\delta| = xyz + 2(xy + xz + yz)\delta + \pi(x + y + z)\delta^2 + \frac{4}{3}\pi\delta^3$$
$$|B_\delta| = abc + 2(ab + ac + bc)\delta + \pi(a + b + c)\delta^2 + \frac{4}{3}\pi\delta^3$$

Since $B \subseteq A$ we have $B_\delta \subseteq A_\delta$ for any positive real number δ, too. Thus

$$\frac{abc}{\delta^2} + \frac{2(ab + ac + bc)}{\delta} + \pi(a + b + c) \leq \frac{xyz}{\delta^2} + \frac{2(xy + xz + yz)}{\delta} + \pi(x + y + z)$$

Since this has to hold for any δ, we can take the limit to infinity and see, that the inequality only holds if

$$a + b + c \leq x + y + z$$

which means our answer is NO: We cannot save money by using cheaper boxes.

10. (HS-3) Let a be a positive integer and $(a_n)_{n\in\mathbb{N}_0}$ the sequence defined by
$$a_0 := 1\ ,\ a_n := a + \prod_{k=0}^{n-1} a_k \quad (n \geq 1)$$

   (a) There are infinitely many primes which divide at least one number of the sequence.

   (b) There is a prime which does not divide any of the numbers in the sequence.

   Reason: Primes.

   Solution: $\gcd(a,a_0)=1$ and
$$\gcd(a,a_n) = \gcd\left(a, a+\prod_{k=0}^{n-1} a_k\right) = \gcd\left(a, \prod_{k=0}^{n-1} a_k\right) = 1$$
   by induction.

   (a) Let $p_1,\ldots,p_N$ be primes each dividing at least one a_n. Then there is a minimal M, such that all these primes are divisors of some numbers of $a_0,\ldots,a_M$. This means however, that all $p_i \mid \prod_{k=0}^{M-1} a_k$. Thus we get from the above consideration, that none of the p_i divides a, hence none of them divides $e_M > 1$. We thus get a prime factor p_{N+1} of e_M which wasn't on the list. But if we can always add a prime to the list, it cannot be finite.

   (b) If $a>1$ then it has a prime factor which does not divide any a_n because we saw that $\gcd(a,a_n)=1$.

   Now let $a=1$ and set $m_i := \prod_{k=0}^{i} a_k$. That is
$$m_0 = a_0 = 1\ ,\ m_{k+1} = m_k a_{k+1} = m_k(a+m_k) = m_k(1+m_k)$$
   We observe that $m_0 \equiv 1 \mod 5\,,\ m_1 \equiv 2 \mod 5\,,\ m_2 \equiv 1 \mod 5\,,\ \ldots$ As m_{k+1} only depends on m_k, we see that all remainders have to be 1 or 2, and the m_k are never divisible by 5. But $a_k \,|\, m_k$ so 5 can never be a divisor of any a_n.

11. (HS-4) Let a,b,c be positive real numbers such that $a+b+c+2=abc$. Show that $(a+1)(b+1)(c+1) \geq 27$. Under which condition does equality hold?

   Reason: Inequality.

Solution: We set $x = a+1, y = b+1, z = c+1$ and have to show that $xyz \geq 27$ under the assumption that

$$\begin{aligned} xyz &= (a+1)(b+1)(c+1) \\ &= (ab+a+b+1)(c+1) \\ &= abc+ac+bc+ab+a+b+c+1 \\ &= ac+bc+ab+2a+2b+2c+3 \\ &= (ab+a+b+1)+(ac+a+c+1)+(bc+b+c+1) \\ &= (a+1)(b+1)+(a+1)(c+1)+(b+1)(c+1) \\ &= xy+yz+xz \end{aligned}$$

With $AM \geq GM$ we get $xyz = xy+yz+xz \geq 3\sqrt[3]{x^2y^2z^2}$ which is equivalent to $xyz \geq 27$ since all numbers are positive.

Equality holds if and only if $xy = yz = xz$, i.e. $x = y = z$. This is true for $x = y = z = 3$ or $a = b = c = 2$.

16 September 2020

1. Given a group G then the intersection of all maximal subgroups of G is called Frattini subgroup $\Phi(G)$. If G hasn't a maximal subgroup, we set $\Phi(G) = G$. Show that $\Phi(G) \trianglelefteq G$ is a normal subgroup, and that $\Phi(G)$ is nilpotent in case G is finite.

   Reason: Frattini Subgroup.

   Solution: The intersection of all maximal subgroups of G is invariant under group automorphisms

$$\varphi(\Phi(G)) = \varphi\left(\bigcap_{\substack{M \lneq G \\ maximal}} M\right) \subseteq \bigcap_{\substack{M \lneq G \\ maximal}} \varphi(M) = \bigcap_{\substack{M \lneq G \\ maximal}} M = \Phi(G)$$

   and thus especially under inner automorphisms, i.e. conjugation, i.e. $\Phi(G) \triangleleft G$.

   Assume $|G| =: n$ is finite and $P \leq \Phi(G)$ a nontrivial p-group, i.e. the order of any element of P is a power of the prime $p \,|\, n$. Such subgroups exist by Sylow's first theorem for prime factors of n, or by Cauchy's theorem below.

   (a) Lemma: If a group H of order p^n (p prime) acts on a finite set S and if $S_0 := \{\, x \in S \,|\, h.x = x \text{ for all } h \in H \,\}$ denotes the set of fixed points of S under the action, then $|S| \equiv |S_0| \mod p$.

   Proof: An orbit $\tilde{x} = H.x$ contains exactly one element if and only if $x \in S_0$. Hence S can be written as a disjoint union $S = S_0 \cup \tilde{x}_1 \cup \tilde{x}_2 \cup \ldots \cup \tilde{x}_m$ with $|\tilde{x}_k| > 1$ for all k. Hence $|S| = |S_0| + |\tilde{x}_1| + |\tilde{x}_2| + \cdots + |\tilde{x}_m|$. Now $p \,|\, |\tilde{x}_k|$ for each k since $|\tilde{x}_k| > 1$ and $|\tilde{x}_k| = [H : H.x_k] \,|\, |H| = p^n$ by the orbit-stabilizer theorem. Thus $|S| \equiv |S_0| \mod p$.

   (b) Cauchy's Theorem.

   If G is a finite group whose order $|G| = n$ is divisible by a prime p, then G contains an element of order p.

   Proof: Let S be the set of p-tuples of group elements

$$\{\, (a_1, a_2, \ldots, a_p) \,|\, a_k \in G \text{ and } a_1 a_2 \cdots a_p = 1 \,\}.$$

   Since $a_p = (a_1 a_2 \cdots a_{p-1})^{-1}$ is uniquely determined by the other elements, if follows that $|S| = n^{p-1}$. As $p \,|\, n$, $|S| \equiv 0 \mod p$. The cyclic group $\mathbb{Z}_p$ acts on S by

$$k.(a_1,a_2,\ldots,a_p)=(a_{1+k},a_{2+k},\ldots,a_p,a_1,\ldots,a_k) \quad (k\in\mathbb{Z}_p)$$

(With $ba=(a^{-1}a)(ba)=a^{-1}(ab)a=1$ if $ab=1$ we see by induction, that $(a_{k+1},a_{k+2},\ldots,a_k)\in S$. It's easy to verify for $x\in S,\ 0,k,k'\in\mathbb{Z}_p$ that $0.x=x$ and $(k+k').x=k.(k'.x)$, and that the action is well-defined.)

Now $x=(a_1,\ldots,a_p)\in S_0$ is a fixed point if and only if $a_1=a_2=\ldots=a_p$ and $(1,1,\ldots,1)\in S_0$, so $S_0\neq 0$. By the previous Lemma we get $|S_0|\equiv|S|\equiv 0 \mod p$ and at least p elements in S_0, that is, there is $a\neq 1$ such that $(a,a,\ldots,a)\in S_0$ and hence $a^p=1$. Since p is prime, $|a|=p$.

(c) Corollary: A finite group P is a p-group if and only if $|P|$ is a power of p.

Proof: If P is a p-group and $q\,|\,|P|$ a prime, then P contains an element of order q by Cauchy's theorem. Since every element has order a power of p, $q=p$. Hence $|P|$ is a power of p. The converse is an immediate consequence of Lagrange's theorem, that the order of every group element divides the order of the (finite) group.

(d) P has a nontrivial center: $C(P)\neq\{1\}$.

Proof: Consider the class equation of P:

$$|P|=|C(P)|+\sum|P:C_P(x_i)|$$

where $C_P(x)=\{\,p\in P\,|\,px=xp\,\}$ is the centralizer of $x\in P$, and the action is conjugation. If $|P:C_P(x_i)|=1$ then $P=C_P(x_i)$ and $x_i\in C(P)$, and we are done. Otherwise each $|P:C_P(x_i)|>1$ and divides $|P|=p^n(n\geq 1)$, so p divides each $|P:C_P(x_i)|$ and divides $|P|$, and therefore divides $|C(P)|$. As $|C(P)|\geq 1$ because $1\in C(P)$, $C(P)$ has at least p elements.

(e) Every finite p-group P is nilpotent.

Proof: Let G be a group. The center $C(G)$ of G is a normal subgroup. Let $C_2(G)$ be the inverse image of $C(G/C(G))$ under the canonical projection $G\twoheadrightarrow G/C(G)$. Then $C_2(G)$ is normal in G and contains $C(G)$. Continue this process by defining inductively: $C_1(G)=C(G)$ and $C_i(G)$ is the inverse image of $C(G/C_{i-1}(G))$ under the canonical projection $G\twoheadrightarrow G/C_{i-1}(G)$. Thus we obtain a sequence of normal subgroups of G, called the ascending central series of $G\,:\,1\leq C_1(G)\leq C_2(G)\leq\ldots$ A group is called *nilpotent* if $G=C_n(G)$ for some $n\in\mathbb{N}$.

An equivalent definition can be given by commutator groups: Let $G^{(0)}=G$ and $G^{(i)}=[G,G^{(i-1)}]$. Then G is nilpotent if $G^{(n)}=1$ for some $n\in\mathbb{N}$. The series $G\geq G^{(1)}\geq G^{(2)}\geq\ldots$ is called descending central series.

Now P and all its nontrivial quotients are p-groups, and therefore have nontrivial centers. If P is Abelian, then it is nilpotent. Otherwise $P\neq C(P)$. If $P\neq C_i(P)$, then $C_i(P)$ is strictly contained in $C_{i+1}(P)$. Since P is finite, $C_n(P)$ must equal P for some n.

(f) Frattini's argument: $\Phi(G)N_G(P)=G$.

Proof: $N_G(P)=\{\,g\in G\,|\,gPg^{-1}\subseteq P\,\}$ is the normalizer of P in G. Recall $P\leq\Phi(G)$ has been chosen as a p-subgroup of $\Phi(G)$.

Let $g\in G$. Then gPg^{-1} is a subgroup of $\Phi(G)$. By Sylow's second theorem there is an element $f\in\Phi(G)$ such that $f(gPg^{-1})f^{-1}\subseteq P$. So $x:=fg\in N_G(P)$ and $G\ni g=f^{-1}x\in\Phi(G)N_G(P)$.

(g) $P\trianglelefteq\Phi(G)$ is normal.

Proof: Let $N_G(P)\subseteq M\subsetneq G$ be contained in a proper subgroup M of G. Then $\Phi(G)\subseteq M\cap\Phi(G)N_G(P)\subseteq M$ which is a contradiction. Hence $G=N_G(P)$ and P is normal in G, and especially normal in $\Phi(G)$.

(h) $\Phi(G)$ is nilpotent.

Proof: We will show that $\Phi(G)$ is the direct sum of its p-groups. Thus we have a direct sum of normal, nilpotent subgroups, which is therefore nilpotent, too. This follows e.g. from Fitting's theorem, but can also be proven directly.

Proof: Let $p_1,p_2,...,p_s$ be the distinct primes dividing the order of $\Phi(G)$, and let P_i be p_i-groups for $1\leq i\leq s$. For any t, $1\leq t\leq s$ we show inductively that $P_1P_2\cdots P_t$ is isomorphic to $P_1\times P_2\times\cdots\times P_t$. As each P_i is normal in $\Phi(G)$ so $P_1P_2\cdots P_t$ is a subgroup of $\Phi(G)$. Let H be the product $P_1P_2\cdots P_{t-1}$ and let $K=P_t$,so by induction H is isomorphic to $P_1\times P_2\times\cdots\times P_{t-1}$. In particular, $|H|=|P_1|\Delta|P_2|\cdots|P_{t-1}|$. Since $|K|=|P_t|$, the orders of H and K are relatively prime. Lagrange's Theorem implies the intersection of H and K is equal to 1. By definition, $P_1P_2\cdots P_t=HK$, hence HK is isomorphic to $H\times K$ which is equal to $P_1\times P_2\times\cdots\times P_t$. This completes the induction. Now we take $t = s$ to obtain the result.

Remark: The subgroup of a group G which is generated by all nilpotent normal subgroups is called Fitting subgroup $F(G)$. If G is finite, we have

$$[F(G), F(G)] \leq \Phi(G) \leq F(G)$$
$$F(G)/\Phi(G) = F(G/\Phi(G))$$

2. The n-th Fermat number $F_n = 2^{2^n} + 1$ is prime for $n \in \mathbb{N}$ if and only if

$$3^{(F_n-1)/2} \equiv -1 \mod F_n.$$

3 is a primitive root modulo F_n in this case.

Reason: Fermat Primes (Pépin, 1877).

Solution: We see from $3^{2^{2^n-1}} \equiv -1 \mod F_n$ that the remainder class of 3 in $(\mathbb{Z}/(F_n))^*$ has the order 2^{2^n}, i.e. $(\mathbb{Z}/(F_n))^*$ has at least $2^{2^n} = F_n - 1$ elements. This is only possible, if F_n is prime and 3 is a primitive root modulo F_n.

Now we show that this condition is necessary, too. Let F_n be prime. From $F_n \equiv 1 \mod 4$ we get

$$\left(\frac{3}{F_n}\right) = \left(\frac{F_n}{3}\right) = \left(\frac{2}{3}\right) = -1$$

where we used quadratic reciprocity and $F_n \equiv 2^{2^n} + 1 \equiv (-1)^{2^n} + 1 \equiv 2 \mod 3$. With Euler's criterion we now find

$$3^{(F_n-1)/2} \equiv -1 \mod F_n.$$

3. Show that none of the numbers

$$11\,,\,111\,,\,1111\,,\,11111\,,\,111111\,,\,\ldots$$

can be written as a sum of two squares.

Reason: Number Theory.

Solution: The numbers $n = 11\,,\,111\,,\,1111\,\ldots$ are all congruent 11 mod 100 and so congruent $3 \mod 4$. Such a number has at least one odd prime factor $p \equiv 3 \mod 4$ which occurs in an odd power, since otherwise we would only have prime factors congruent $1 \mod 4$ and even powers of prime factors congruent $3 \mod 4$. Pairing two prime factors congruent $3 \mod 4$ results in a factor $1 \mod 4$. But all factors congruent $1 \mod 4$ remain congruent $1 \mod 4$ by multiplication. In order for $n \equiv 3 \mod 4$, n has to have a prime factorization

$$n = p \cdot \prod_{i=1}^{I} p_i^{2\nu_i} \cdot \prod_{j=1}^{J} q_j^{\mu_j} \qquad (*)$$

with primes $p, p_i \equiv 3 \mod 4\,,\ q_j \equiv 1 \mod 4$. Let's assume now n can be written as $n = x^2 + y^2$. If $d = \gcd(x, y) > 1$, then $d^2 \,|\, n$ and we can cancel it out without affecting p since all prime divisors of d occur twice. Hence we may assume that x, y are coprime, to the expense that we changed the value of n, but we still have $n = x^2 + y^2$ with x, y coprime, and a factorization $(*)$.

Since $p \nmid x$, because otherwise $p \,|\, n - x^2 = y^2$ and $p \,|\, y$, but we assumed them to be coprime, x is a unit modulo p, say $tx \equiv 1 \mod p$. From $p \,|\, n = x^2 + y^2$ we get $y^2 \equiv -x^2 \mod p$ and thus

$$(ty)^2 = t^2y^2 \equiv -t^2x^2 \equiv -1 \mod p.$$

With Euler's criterion we calculate

$$\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = (-1)^{\frac{4k+2}{2}} = (-1)^{2k+1} = -1$$

i.e. -1 is no quadratic residue modulo p, which means there is no number $z^2 \equiv -1 \mod p$, contradicting $z = ty$ which we just found.

4. Let $G = \langle a, b \,|\, a^p = b^q = 1, (aba) = b^r, a^s = b^t \rangle$ be a group of order twelve which operates on $\mathbb{R}^4$ by

$$a.v = \frac{1}{2} \cdot \begin{bmatrix} 1 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 1 & 0 & 0 \\ 0 & 0 & 1 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 1 \end{bmatrix} .v, \quad b.v = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} .v.$$

   (a) Determine the group G and its presentation (p, q, r, s, t).

   (b) Which group is

$$H = \langle a, b \,|\, a^6 = b^2 = 1, (aba) = b \rangle\,?$$

   (c) The above groups are obviously not Abelian. There is another non Abelian group L of order twelve. Which one and what is (p, q, r, s, t) in that case?

Reason: Group Theory.

Solution:

(a) Since $a^{\operatorname{ord} a}(v) = 1(v)$ for all $v \in \mathbb{R}^4$ we have to calculate the order of the given, regular matrices:

$$\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}^2 = \begin{bmatrix} -2 & 2\sqrt{3} \\ -2\sqrt{3} & -2 \end{bmatrix}, \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}^3 = \begin{bmatrix} -8 & 0 \\ 0 & -8 \end{bmatrix}$$
$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}^2 = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}$$

so $\operatorname{ord} a = 6$ and $\operatorname{ord} b = 4$ and $a^3 = b^2$. Now

$$\frac{1}{4}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \cdot \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \cdot \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & AB \\ -BA & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

hence $aba.v = b.v$ for all $v \in \mathbb{R}^4$ and thus $aba = b$ and

$$G = \langle a, b \,|\, a^6 = b^4 = 1, (aba) = b^1, a^3 = b^2 \rangle.$$

The elements are $G = \{ 1, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b \}$ This means that $G = \operatorname{Dic}_3$, the dicyclic group of order 12.

(b) We have elements of order 6 and order 2 in

$$H = \langle a, b \,|\, a^6 = b^2 = 1, (aba) = b \rangle$$

The subgroup generated by a is normal: $ba^nb^{-1} = a^{-n}$, whereas the subgroup generated by b is not: $aba^{-1} = aba^5 = ba^4 \notin \langle b \rangle$. Hence $H \cong \mathbb{Z}_6 \rtimes \mathbb{Z}_2 \cong D_6$, the dihedral group of order 12.

(c) The third non Abelian group of order 12 is the alternating group $L = A_4 = \langle a, b \,|\, a^p = b^q = 1, (aba) = b^r, a^s = b^t \rangle$. It contains all even permutations of $\{ 1, 2, 3, 4 \}$. $(123), (234) \in A_4$, so there are at least two elements of order 3, from which we can choose one as generator, say $a = (123)$. The cycles of the Klein subgroup V_4 are all of order two. Now $(12)(34)(123)(12)(34) = (142)$ and $(123)(12)(34)(132) = (14)(23)$ show that $V_4 \triangleleft A_4$ is a normal subgroup, and that a $3-$cycle generates no normal subgroup, hence $A_4 \cong V_4 \rtimes \mathbb{Z}_3$. It can be shown that a and $(12)(34)$ generate A_4, but for our desired presentation, we need a generator b such that $aba = b^r$ is a relation. We can rule out $r = 0$ since it would imply $aba = 1 \Longrightarrow ab = a^{-1} = a^2 \Longrightarrow a = b$. But $aba = b$

cannot be true either, as $aV_4a \notin V_4$. Thus we have to choose another 3-cycle; $b := (234)$. Since $ab = (123)(234) = (12)(34)$ we already *know* by the omitted calculation above, that a and b generate the group. In addition we have $a^3 = b^3 = 1$ and $aba = (12)(34)(123) = (243) = b^{-1} = b^2$, so finally we get

$$A_4 \cong V_4 \rtimes \mathbb{Z}_3 \cong \langle a = (123), b = (234) | a^3 = b^3 = 1, (aba) = b^2, a^3 = b^3 \rangle$$

5. Let A be an associative, finite dimensional algebra with 1 over a field $\mathbb{F}$, $M \neq 0$ an A-module, and $0 \neq P \subseteq A_A$ a submodule of A as right A-module. Show that
   (a) M is irreducible if and only if 0 and 1 are the only idempotent elements of the endomorphism ring $\operatorname{End}_A(M)$.
   (b) P is a direct summand of A_A if and only if there is an idempotent element $e \in A$ such that $P = eA$.

   Reason: Modules.

   Solution:

   (a) If M is reducible, then there are submodules $0 \neq L, K \subseteq M$ such that $L \oplus K = M$. The projection $\pi_K \in \operatorname{End}_A(M)$ on K is idempotent and $\pi_K \notin \{0, 1\}$. If conversely $e \in \operatorname{End}_A(M)$ is idempotent, then $M = e(M) \oplus (1_M - e)(M)$. If $e \notin \{0, 1\}$, then $e(M), (1_M - e)(M) \neq 0$ and M is reducible.

   (b) Let P be a direct summand and $A = P \oplus Q$. Then we can write $A \ni 1 = e + f$.

$$e - e^2 = (1 - e)e = fe \in P \cap Q = 0$$

hence $e = e^2$ and $fe = 0$. By the same argument we get $f^2 = f$ and $ef = 0$. Moreover we have $eA \subseteq PA = P$ and for $p \in P$

$$p = 1 \cdot p = (e + f) \cdot p = ep + fp = ep \in eA$$

since $fp \in P \cap Q = 0$.
Let conversely be $e \in A$ an idempotent, and set $P := eA$. Then

$$f := 1 - e = 1 - 2e + e = 1 - 2e + e^2 = (1 - e)^2 = f^2$$

is also an idempotent and we have $1 = e + f$ and $ef = fe = 0$. Hence $A = eA \oplus fA = P \oplus fA$.

6. We consider the topological space $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ equipped with distance

$$\chi(x,y) := \begin{cases} \dfrac{\|x-y\|_2}{\sqrt{1+\|x\|_2^2}\sqrt{1+\|y\|_2^2}} & \text{if } x,y \neq \infty \\ \dfrac{1}{\sqrt{1+\|x\|_2^2}} & \text{if } x \neq \infty, y = \infty \\ \dfrac{1}{\sqrt{1+\|y\|_2^2}} & \text{if } x = \infty, y \neq \infty \\ 0 & \text{if } x = y = \infty \end{cases}$$

Show that χ defines a metric such that $\mathcal{C} := (\mathbb{C}_\infty, \chi)$ is a compact topological space.

Reason: Compact Space.

Solution: The chordal metric χ as defined is half the Euclidean distance in $\mathbb{R}^3$ under the stereographic projection

$$\pi \,:\, \mathbb{R}^3 \supset \mathbb{S}^2_{(0,0,1)} - \{(0,0,2)\} \longrightarrow \mathcal{C}\ ,\ \pi(a,b,c) = \left(\frac{2a}{2-c} + i \cdot \frac{2b}{2-c}\right)$$

of the Riemann sphere: $2\chi(x,y) \stackrel{(*)}{=} \|\pi^{-1}(x) - \pi^{-1}(y)\|_2$.

Let us consider the stereographic projection. A point $P = u + iv = (u,v,0)$ on the complex plane corresponds to the point on the sphere $\{\,(x,y,z)\,|\,x^2+y^2+(z-1)^2 \stackrel{(**)}{=} 1\,\}$ which is part of the line through P and the north pole $N = (0,0,2)$, hence the point $\pi^{-1}(P) = (x,y,z)$ that satisfies $(**)$ and $(x,y,z) = (0,0,2) + \lambda(u,v,-2)$ for some $\lambda \in \mathbb{R}$. Solving these for λ yields $\lambda = 4/(4+u^2+v^2)$ and

$$\pi^{-1}(P) = \pi^{-1}(u+iv) = \frac{1}{4+u^2+v^2} \cdot (4u, 4v, 2u^2+2v^2)$$

If we reduce the last coordinate by 1, we will get a sphere of radius 1 and the center at the origin. Thus

$$\bar{\pi} \,:\, \mathbb{R}^3 \supset \mathbb{S}^2_{(0,0,0)} - \{(0,0,1)\} \longrightarrow \mathcal{C}\ ,\ \bar{\pi}(a,b,c) = \left(\frac{a}{1-c} + i \cdot \frac{b}{1-c}\right)$$

and

$$\bar{\pi}^{-1}(P) = \bar{\pi}^{-1}(u+iv) = \frac{1}{1+u^2+v^2} \cdot (2u, 2v, -1+u^2+v^2)$$

which doesn't affect the distances on $\mathbb{S}^2 \subseteq \mathbb{R}^3$. So in order to prove $(*)$ we set $P = u + iv$, $Q = x + iy$, $p := x^2 + y^2 + 1$, $q := u^2 + v^2 + 1$ and calculate

$$\begin{aligned}
\|\bar{\pi}^{-1}(P) - \bar{\pi}^{-1}(Q)\|_2^2 &= \frac{1}{(pq)^2}\,\|(2up - 2xq, 2vp - 2yq, (p-2)q - (q-2)p)\|_2^2 \\
&= \frac{4}{(pq)^2}\left((up - xq)^2 + (vp - yq)^2 + (p - q)^2\right) \\
&= \frac{4}{(pq)^2}(u^2p^2 + v^2p^2 + x^2q^2 + y^2q^2 - 2uxpq - 2vypq \\
&\qquad + p^2 + q^2 - 2pq) \\
&= \frac{4}{(pq)^2}(p^2(u^2 + v^2 + 1) + q^2(x^2 + y^2 + 1) \\
&\qquad - 2pq(ux + vy + 1)) \\
&= \frac{4}{(pq)^2}(p^2q + q^2p - 2pq(ux + vy + 1)) \\
&= \frac{4}{pq}(p + q - 2(ux + vy + 1)) \\
&= \frac{4}{pq}(x^2 + y^2 + u^2 + v^2 - 2ux - 2vy) \\
&= \frac{4}{pq}\,\|(x - u) + i(y - v)\|_2^2 \\
&= 4\chi(P, Q)^2
\end{aligned}$$

As χ can be expressed as a Euclidean distance, it is clear that the triangle inequality holds. χ is positive definite and symmetric which is more or less obvious. It is also clear that

$$\chi(x, y) \leq \chi(x, z) + \chi(z, y)$$

as soon as at least two of the points are at infinity. Hence it remains to check the cases $x = \infty$ or $z = \infty$.

- $\chi(x,y) \leq \chi(x,\infty) + \chi(\infty,y)$.

$$\begin{aligned}\chi(x,y) &\leq \frac{\|x\|_2}{\sqrt{1+\|x\|_2^2}\sqrt{1+\|y\|_2^2}} + \frac{\|y\|_2}{\sqrt{1+\|x\|_2^2}\sqrt{1+\|y\|_2^2}} \\ &= \frac{\|x\|_2}{\sqrt{1+\|x\|_2^2}} \cdot \frac{1}{\sqrt{1+\|y\|_2^2}} \\ &\quad + \frac{\|y\|_2}{\sqrt{1+\|y\|_2^2}} \cdot \frac{1}{\sqrt{1+\|x\|_2^2}} \\ &\leq \chi(y,\infty) + \chi(\infty,x)\end{aligned}$$

because $\dfrac{\|x\|_2}{\sqrt{1+\|x\|_2^2}}, \dfrac{\|y\|_2}{\sqrt{1+\|y\|_2^2}} \leq 1$.

- $\chi(\infty,z) \leq \chi(\infty,y) + \chi(y,z)$.

Set $y = a+ib,\ z = u+iv,\ p = \sqrt{1+|y|^2},\ q = \sqrt{1+|z|^2}$. Then

$$\begin{aligned}0 &\leq (av-bu)^2 + (a-u)^2 + (b-v)^2 \\ &= a^2+b^2+u^2+v^2+a^2v^2+b^2u^2-2abuv-2au-2bv \\ 1&+a^2u^2+b^2v^2+2abuv+2au+2bv \\ &\leq a^2+b^2+u^2+v^2+a^2v^2+b^2u^2+1+a^2u^2+b^2v^2\end{aligned}$$

So $(1+au+bv)^2 \leq (1+a^2+b^2)(1+u^2+v^2)$ and

$$\begin{aligned}1+au+bv &\leq pq \\ 2+a^2+b^2+u^2+v^2-2pq &\leq -2au-2bv+a^2+b^2+u^2+v^2 \\ (p-q)^2 = q^2+p^2-2pq &\leq (a-u)^2(b-v)^2 \\ p-q &\leq \|y-z\|_2 \\ \chi(\infty,z)-\chi(y,\infty) = \frac{1}{q}-\frac{1}{p} &\leq \frac{\|y-z\|_2}{pq} = \chi(y,z) \\ \chi(\infty,z) &\leq \chi(\infty,y)+\chi(y,z)\end{aligned}$$

$\mathcal{C}$ is compact if and only if it is sequentially compact. Let $(z_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}$.

- Case 1: There is an $N \in \mathbb{N}$ such that $(z_n)_{n\geq N}$ is a bounded sequence in $\mathbb{C}$.

In this case there is a convergent subsequence $(z_{n_k})_{n_k\geq N} \subseteq (z_n)_{n\geq N} \subseteq (\mathbb{C}, \|.\|_2)$, say $\lim_{k\to\infty} z_{n_k} = z$. Thus

$$0 \leq \chi(z_{n_k}, z) = \frac{\|z_{n_k}-z\|_2}{\sqrt{1+\|z_{n_k}\|_2^2}\sqrt{1+\|z\|_2^2}} \leq \|z_{n_k}-z\|_2 \stackrel{n\to\infty}{\longrightarrow} 0$$

and $\lim_{k\to\infty} \chi(z_{n_k}, z) = 0$ by the sandwich principle, so $(z_{n_k})_{k\in\mathbb{N}}$ is a convergent subsequence of $(z_n)_{n\in\mathbb{N}}$ in $\mathcal{C}$, which had to be shown.

- Case 2: $(z_n)_{n\geq N}$ is for each $N \in \mathbb{N}$ an unbounded sequence in $\mathbb{C}$ or contains $\{\infty\}$.

  In this case there is an $N \leq M_N \in \mathbb{N}$ such that $\|z_{M_N}\|_2 > N$ for all $N \in \mathbb{N}$. If $z_{M_N} = \infty$ we write $\|z_{M_N}\|_2 = \infty$. Now if we define $C_N := \max\{\, M_k \,|\, 1 \leq k \leq N \,\}$ we will get an increasing list of natural numbers. Each natural number can occur at most finitely often on this list, since $M_N \geq N$. Thus $\lim_{N\to\infty} C_N = \infty$.

$$0 \leq \chi(z_{C_N}, \infty) = \begin{cases} 0 & \text{if } z_{C_N} = \infty \\ \dfrac{1}{\sqrt{1 + \|z_{C_N}\|_2^2}} & \text{if } \|z_{C_N}\|_2 \neq \infty \end{cases}$$
$$\leq \frac{1}{\sqrt{1+N^2}}$$
$$\stackrel{N\to\infty}{\longrightarrow} 0$$

  and $(z_{C_N})_{N\in\mathbb{N}} \subseteq (z_n)_{n\in\mathbb{N}}$ is the subsequence we were looking for, and which converges to $0 \in \mathcal{C}$.

7. (a) Calculate $\displaystyle\int_{|z|=5} \frac{e^z}{z^2+\pi^2}\,dz\,.$

   (b) Determine all $z \in \mathbb{C}$ such that
$$f(z) = e^{z^7 (\sin z)^{16}} + \bar{z}^2$$
   is complex differentiable.

Reason: Complex Integration And Differentiability.

Solution:

(a) We write $D(z_0, r) = \{\, z \in \mathbb{C} \,|\, |z - z_0| < r \,\}$ for a disk around z_0 with radius r. The zeros of the denominator are $z = \pm i\pi$ inside $D(0, 5)$. Let
$$\Omega := D(0,5) - (D(i\pi, 1) \cup D(-i\pi, 1))$$

Then

$$\begin{aligned}\int_{|z|=5}\frac{e^z}{z^2+\pi^2}\,dz &= \int_{|z|=5}\frac{e^z}{(z-i\pi)(z+i\pi)}\,dz\\ &= \int_{\partial\Omega}\frac{e^z}{(z-i\pi)(z+i\pi)}\,dz\\ &+\int_{|z-i\pi|=1}\frac{e^z}{(z-i\pi)(z+i\pi)}\,dz+\int_{|z+i\pi|=1}\frac{e^z}{(z-i\pi)(z+i\pi)}\,dz\\ &= 0+\int_{|z-i\pi|=1}\frac{\frac{e^z}{z+i\pi}}{z-i\pi}\,dz+\int_{|z+i\pi|=1}\frac{\frac{e^z}{z-i\pi}}{z+i\pi}\,dz\\ &= 2i\pi\left(\frac{e^{i\pi}}{i\pi+i\pi}+\frac{e^{-i\pi}}{-i\pi-i\pi}\right)\\ &= e^{i\pi}-e^{-i\pi}\\ &= 0\end{aligned}$$

(b) If $f(z)$ is complex differentiable, then $g(z) := f(z)-e^{z^7(\sin z)^{16}} = \bar{z}^2$ is complex differentiable at z, too, since it is the composition of two on $\mathbb{C}$ holomorphic functions.

$$g(z)=g(x+iy)=\overline{x+iy}^2=(x-iy)^2=\underbrace{x^2-y^2}_{=:u(x,y)}+i\underbrace{(-2xy)}_{=:v(x,y)}$$

For the Cauchy Riemann equations we check

$$u_x=2x\ ,\ u_y=-2y\ ,\ v_x=-2y\ ,\ v_y=-2x$$

Now $u_x=v_y$ implies $x=0$ and $u_y=-v_x$ implies $y=0$. Since all derivatives are continuous on $\mathbb{R}^2$, the Cauchy Riemann equations are not only necessary, but sufficient as well. Hence $g(z)$ is only complex differentiable at $z=0$ and so is $f(z)$.

8. Calculate

$$\int_1^\infty \frac{1+x^2-2x^2\log(x)}{x(1+x^2)^2}\,dx$$

Reason: Catalan's Constant.

Solution: The square in the denominator reminds us on the quotient rule, so we consider

$$\frac{f(x)}{g(x)}=\frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$$

and set $g(x) = (x^2+1)$. Hence

$$\begin{aligned}\frac{1+x^2-2x^2\log(x)}{x(1+x^2)^2} &= \frac{\frac{1}{x}+x-2x\log(x)}{(1+x^2)^2}\\ &= \frac{\frac{1}{x}(1+x^2)-2x\log(x)}{(1+x^2)^2}\\ &= \frac{f'(x)(x^2+1)-f(x)2x}{(x^2+1)^2}\end{aligned}$$

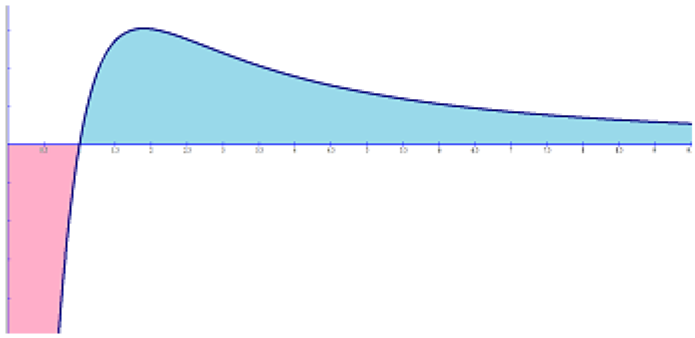
so $f(x) = \log x$ gives the desired solution and

$$\int_1^\infty \frac{1+x^2-2x^2\log(x)}{x(1+x^2)^2}\,dx = \left[\frac{\log(x)}{x^2+1}\right]_1^\infty = 0$$

The function $f(x) = \dfrac{\log(x)}{x^2+1}$ has an interesting property:

$$\left|\int_0^1 f(x)\,dx\right| = \int_1^\infty f(x)\,dx = C = 0.915965594177219 01\ldots$$

where C is Catalan's constant A006752 in the OEIS.



9. Determine the square root and the inverse matrix of

$$A = \begin{pmatrix} 5 & -4 & 2 \\ -4 & 7 & -8 \\ 1 & -4 & 6 \end{pmatrix}$$

What is the dimension of the simple Lie algebra whose Cartan matrix $\sqrt{A}$ is?

Reason: Matrix Calculations.

Solution:

$$\sqrt{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} \;,\; A^{-1} = \frac{1}{4} \cdot \begin{pmatrix} 10 & 16 & 18 \\ 16 & 28 & 32 \\ 9 & 16 & 19 \end{pmatrix}$$

The Cartan matrix $\sqrt{A}$ belongs to the simple Lie algebra of type B_3 which is the 21 dimensional orthogonal Lie algebra $\mathfrak{o}(7,\mathbb{R}) = \mathfrak{so}(7,\mathbb{R})$.

10. Let R be a commutative ring with 1. We define the nilradical $N(R) = N \subseteq R$ as intersection of all prime ideals of R, and the Jacobson radical $J(R) = J$ as intersection of all maximal ideals.

    (a) Show that $N(R)$ contains exactly all nilpotent Elements of R.

    (b) Assume R is Artinian. Show that all prime ideals are maximal, hence $N(R) = J(R)$ in an Artinian ring.

    (c) Assume R is Artinian. Show that $N(R)$ is a nilpotent Ideal.

    (d) Give an example of $N(R) \neq J(R)$ if R is not Noetherian and thus not Artinian either.

    Reason: Ring Theory.

    Solution:

    (a) $N(R)$ is the set of all nilpotent elements of R.

    i. Let $r \in R$ be nilpotent and $P \subseteq R$ a prime ideal. Then $r^n = 0 \in P$ for some $n \in \mathbb{N}$ and $r \in P$ since P is prime. So all nilpotent elements of R are contained in all prime ideals.

    ii. Let $r \in R$ be not nilpotent. We consider the set of ideals

    $$\Sigma := \{\, I \trianglelefteq R \mid n > 0 \Longrightarrow r^n \notin I\}$$

    Since $0 \in \Sigma$ we have $\Sigma \neq \emptyset$ and a maximal element $M \in \Sigma$ by inclusion as order and Zorn's Lemma (AC). We must show that M is a prime ideal, because from $r \notin M$ we get that any non nilpotent element cannot be in all prime ideals. Let $x\,,\, y \notin M$. Then we have to show that $x \cdot y \notin M$.

    $M + (x), M + (y) \supsetneq M$ so they cannot belong to Σ for the maximality of M. Thus there are numbers $n, m > 0$ with $r^n \in M + (x)$ and $r^m \in M + (y)$, i.e. $r^{n+m} = r^n \cdot r^m \in M + (x \cdot y)$. As we have found a positive power of r which is in an ideal $M + (xy)$, we have shown that $M + (xy) \notin \Sigma$, i.e. $xy \notin M \in \Sigma$ which had to be shown. In other words, we have found a prime ideal M which doesn't contain r, so $r \notin N(R)$.

(b) Let R be Artinian and $P \subseteq R$ a prime ideal. Then $S := R/P$ is an Artinian integral domain. Let $s \in S - \{0\}$. Because of the descending chain condition on ideals, we have $(s^n) = s^{n+1}$ for some $n \in \mathbb{N}$ and thus $s^n = s^{n+1}r$ or $s^n(sr-1) = 0$. Since S is an integral domain, we may conclude $sr = 1$ as $s^n \neq 0$. But this means s is a unit, i.e. S is a field and P maximal.

(c) Let $N^k = N^{k+1} = \ldots =: A \triangleleft R$ for some $0 \neq k \in \mathbb{N}$. Now assume that $A \neq 0$, and consider the set Ξ of all ideals B such that $AB \neq 0$. Since $A^2 = A \neq 0$ we have $A \in \Xi \neq \emptyset$, and because R is Artinian we can choose a minimal ideal $C \in \Xi$. Then there is an element $x \in C$ such that $Ax \neq 0$. We even have $(x) = C$ by minimality of C. The same argument leads to $Ax = (x)$, because $A(Ax) = A^2x = Ax \neq 0$ means $Ax \in \Xi$ and $Ax \subseteq AC \subseteq C$, so minimality of C implies $x \in (x) = C = Ax$. Thus there is an element $a \in A$ such that $x = ax$, hence $x = ax = a^2x = a^3x = \ldots = a^nx = \ldots$
Now $a \in A = N^k \subseteq N$ is nilpotent by the previous part, so $x = 0$ which contradicts $Ax \neq 0$ and our assumption $A \neq 0$ was wrong, hence $0 = A = N^k$ and $N(R)$ is nilpotent.

(d) Let $R = \mathbb{R}[x_1, x_2, \ldots]$ be the ring of real polynomials with countably infinite many indeterminates. This ring is neither Noetherian (a.c.c.) as $(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \ldots$ shows, nor Artinian (d.c.c.) as $(x_1) \supsetneq (x_1^2) \supsetneq (x_1^3) \supsetneq \ldots$ shows. (x_1) is a prime ideal, $R/(x_1) \cong R$ is an integral domain, but no field and (x_1) not maximal. $M := (x_1, x_2, \ldots)$ is the unique maximal ideal, so $J(R) = M$ with $R/J \cong \mathbb{R}$. On the other hand R doesn't contain any nilpotent elements, so $N(R) = 0$. Another way to see it is, that $\bigcap_{n\in\mathbb{N}} (x_n) = 0$ because there is no polynomial which is divided by all indeterminates, or that the intersection of the two prime ideals $(x_1) \cap (1 + x_1) = 0$ is zero.

11. (HS-1) Prove that the geometric mean of two numbers is less or equal the arithmetic mean of these numbers by three different methods.

Reason: Geometry - Algebra - Calculus.

Solution:

(a) Geometry.

The height h in a right triangle is square root of the product pq of the two sections of the diameter it separates: $h = \sqrt{pq}$. The height is also less or equal the radius r of the surrounding circle, which is $r = \frac{p+q}{2}$ and thus $\sqrt{pq} \leq \frac{p+q}{2}$.

(b) Algebra.

$$\begin{aligned} 0 &\leq (p-q)^2 = p^2 - 2pq + q^2 \\ \Longrightarrow 4pq &\leq p^2 + 2pq + q^2 = (p+q)^2 \\ \Longrightarrow \sqrt{pq} &\leq \frac{p+q}{2} \end{aligned}$$

(c) Calculus.

Assume a fixed length L such that $L = p+q$. We want to minimize

$$f(p) := \frac{p+(L-p)}{2} - \sqrt{p(L-p)} = \frac{L}{2} - \sqrt{pL-p^2}$$

$$\begin{aligned} \frac{df}{dp} &= -\frac{1}{2} \cdot \frac{L-2p}{\sqrt{pL-p^2}} \\ \frac{d^2f}{dp^2} &= \frac{1}{4} \cdot \frac{L^2}{\sqrt{pL-p^2}^3} \end{aligned}$$

$f'(p) = 0$ for $p = L/2$ and $f''(L/2) > 0$ so $p = L/2$ is a minimum, i.e. $f(L/2) = 0 \leq f(p) = \dfrac{p+q}{2} - \sqrt{pq}$ for any p.

12. (HS-2) Calculate the formula for the tangent at the unit circle at $p = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ by three different methods, or better points of view.

Reason: Tangent Spaces.

Solution:

(a) The point lies in the first quadrant, so we can take $y = \sqrt{1-x^2}$ as function for the circle segment. Then $y' = -\dfrac{x}{\sqrt{1-x^2}}$ and $y'(1/2) = -1/\sqrt{3}$. Solving $y_T = -\frac{1}{\sqrt{3}} \cdot x + b$ for p results in

$$b = \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}} \cdot \frac{1}{2} \text{ and } y_T = -\frac{1}{\sqrt{3}} \cdot x + \frac{2}{\sqrt{3}}$$

(b) The tangent is perpendicular to the (normal) position vector $\vec{p}$, hence has the direction $\vec{t} = \vec{p}^{\perp} = (-\sqrt{3}/2\,,\, 1/2)$. This results in the straight

$$T : \vec{p} + s \cdot \vec{t} \Longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \frac{s}{2} \cdot \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

(c) The circle is parameterizable by $C = \left(\dfrac{1-s^2}{1+s^2} \, , \, \dfrac{2s}{1+s^2} \right)$. The corresponding parameter for p is $s_p = 1/\sqrt{3}$.

$$D_pC = \left(-4\frac{s}{(s^2+1)^2} \, , \, -2\frac{s^2-1}{(s^2+1)^2} \right)_p = \left(-\frac{3\sqrt{3}}{4} \, , \, \frac{3}{4} \right) \sim (-\sqrt{3} \, , \, 1)$$

Thus we get the tangent

$$T : \vec{p} + s \cdot D_pC \Longleftrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} + \frac{3s}{4} \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$$

13. (HS-3) We are looking for the number $n = abc$, where a is the maximal number of rotations which are necessary to solve Rubik's cube out of any state, b is the largest natural number of Chicken McNuggets which cannot be bought by the usual box sizes of $6, 9$ or 20, and c is the smallest three digit emirp number.

    Reason: Riddle.

    Solution: $a = 20 \, , \, b = 43 \, , \, c = 107 \, , \, n = 92020$.

14. (HS-4) Show that the following linear equation system with variables $x_1, \ldots, x_n$ has always a unique solution:

$$\begin{aligned} x_1 &= 2x_{n-m+1} + 3x_{n-m+2} + b_1 \\ x_2 &= 4x_{n-m+2} + 9x_{n-m+3} + b_2 \\ \cdots & \quad \cdots \\ x_{m-1} &= 2^{m-1}x_{n-1} + 3^{m-1}x_n + b_{m-1} \\ x_m &= 2^m x_n + b_m \\ x_{m+1} &= b_{m+1} \\ \cdots & \quad \cdots \\ x_n &= b_n \end{aligned}$$

    for all positive integers $1 \leq m < n$ and any real numbers $b_1, \ldots, b_n$.

    Reason: Nilpotent Matrix.

Solution: The coefficient matrix of the linear equation system is

$$\begin{bmatrix} 1 & 0 & \ldots & 0 & -2 & -3 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 & -4 & -9 & \ldots & 0 & 0 \\ & & \ldots & & & & & \ldots & & \\ 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & -2^{m-1} & -3^{m-1} \\ 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 0 & -2^m \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ & & \ldots & & & & & \ldots & & \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} = 1 + N$$

with a nilpotent matrix N, i.e. $N^m = 0$. Now

$$1 = \underbrace{(1 - N + N^2 - N^3 \pm \ldots + (-1)^{m-1}N^{m-1})}_{=:M} \cdot (1 + N)$$

The linear equation system now writes $(1 + N)\vec{x} = \vec{b}$ or $\vec{x} = M\vec{b}$.

15. (HS-5) Calculate the following derivatives:

(a) $\dfrac{dy}{dx}$ if $y = 1 + y^x$

(b) $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ if $y = x + \log y$

(c) $\left.\dfrac{dy}{dx}\right|_{x=1}$ and $\left.\dfrac{d^2y}{dx^2}\right|_{x=1}$ if $x^2 - 2xy + y^2 + x + y - 2 = 0$

Reason: Differentiation.

Solution:

(a)

$$\begin{aligned} y = 1 + y^x \Longrightarrow y' &= (y^x)' = (\exp(x \log y))' \\ &= (\exp(x \log y)) \cdot (x \log y)' = y^x \cdot \left(\log y + x \cdot \frac{y'}{y}\right) \\ &\Longrightarrow y'\left(1 - \frac{xy^x}{y}\right) = y^x \log y \\ &\Longrightarrow \frac{dy}{dx} = y' = \frac{y^x \log y}{1 - xy^{x-1}} \end{aligned}$$

(b)

$$
\begin{aligned}
y = x + \log y \Longrightarrow y' &= 1 + (\log y)' = 1 + \frac{y'}{y} \\
&\Longrightarrow \frac{dy}{dx} = y' = \frac{y}{y-1} \\
&\Longrightarrow y'' = (y(y-1)^{-1})' = y'(y-1)^{-1} - y(y-1)^{-2}y' \\
&= \frac{y(y-1)}{(y-1)^3} - \frac{y^2}{(y-1)^3} = -\frac{y}{(y-1)^3} \\
&\Longrightarrow \frac{d^2y}{dx^2} = y'' = \frac{y}{(1-y)^3}
\end{aligned}
$$

(c)

$$
\begin{aligned}
x^2 - 2xy + y^2 + x + y - 2 = 0 &\Longrightarrow (x-y)^2 + (x+y) = 2 \\
\Longrightarrow 0 = 2(x-y)(1-y') + 1 + y' &= y'(1 + 2(y-x)) + 1 + 2(x-y) \\
&\Longrightarrow y' = \frac{2(y-x)-1}{2(y-x)+1}
\end{aligned}
$$

At $x = 1$ we have $1 - 2y + y^2 + 1 + y - 2 = 0 = y^2 - y = y(y-1)$, i.e. $y = 0$ or $y = 1$.

This is $\left.\frac{dy}{dx}\right|_{x=1} = y'(1) = 3$ or $\left.\frac{dy}{dx}\right|_{x=1} = y'(1) = -1$.

$$
\begin{aligned}
0 &= y'(1 + 2(y-x)) + 1 + 2(x-y) \Longrightarrow \\
0 &= y''(1 + 2(y-x)) + y'(2(y'-1)) + 2(1-y') \\
y'' &= \frac{2(y'-1)^2}{2(x-y)-1}
\end{aligned}
$$

At $x = 1$ we have $(y, y') = (0, 3)$ or $(y, y') = (1, -1)$.

This is $\left.\frac{d^2y}{dx^2}\right|_{x=1} = y''(1) = 8$ or $\left.\frac{d^2y}{dx^2}\right|_{x=1} = y''(1) = -8$.

17 August 2020

1. Let F be a meromorphic function (holomorphic up to isolated poles) in $\mathbb{C}$ with the following properties:

   (a) F is holomorphic (complex differentiable) in the half plane $H(0) = \{z \in \mathbb{C} : \Re(z) > 0\}$.

   (b) $zF(z) = F(z+1)$.

   (c) F is bounded in the strip $\{z \in \mathbb{C} : 1 \leq \Re(z) \leq 2\}$.

   Show that $F(z) = F(1)\Gamma(z)$.

   Reason: Wielandt's theorem.

   Solution: The gamma function satisfies the first two properties and the third follows from $|\Gamma(z)| \leq \Gamma(\Re(z))$ and that $\Gamma(x)$ is bounded on the closed interval $1 \leq x \leq 2$ since it is continuous.

   Now consider
   $$F_0(z) := F(z) - F(1)\Gamma(z).$$
   Then F_0 fulfills all three conditions, too, and $F_0(1) = 0$. The functional equation $F_0(z) = F_0(z+1)/z$ implies, that F_0 is holomorphic at $z = 0$, and that F_0 is bounded on the strip $S_0 := \{z \in \mathbb{C} : 0 \leq \Re(z) \leq 1\}$. Hence the function
   $$\Phi(z) := F_0(z)F_0(1-z)$$
   is bounded in S_0.. We have
   $$\Phi(z+1) = F_0(z+1)F_0(-z) = zF_0(z)F_0(-z) = -F_0(z)F_0(-z+1) = -\Phi(z)$$
   This means that Φ is periodic with period 2 and bounded in entire $\mathbb{C}$. Now Φ is constant by Liouville's theorem. The constant must equal zero, as $\Phi(1) = -\Phi(0)$. Hence $0 = F_0(z)F_0(1-z)$ so $F_0 \equiv 0$ and $F_0(z) = 0 = F(z) - F(1)\Gamma(z)$.

2. Show that if f is any continuous real function and n any positive number,
   $$I := \int_{n^{-1}}^{n} f\left(x + \frac{1}{x}\right) \frac{\log x}{x}\, dx = 0.$$

   Reason: Integration Trick.

Solution:

$$\begin{aligned} I &\overset{y=1/x}{=} \int_n^{n^{-1}} f\left(\frac{1}{y}+y\right)\,\frac{-\log y}{1/y}\,\frac{-dy}{y^2} \\ &= \int_n^{n^{-1}} f\left(y+\frac{1}{y}\right)\,\frac{\log y}{y}\,dy \\ &= -\int_{n^{-1}}^{n} f\left(y+\frac{1}{y}\right)\,\frac{\log y}{y}\,dy \\ &= -I \end{aligned}$$

Since $\operatorname{char}\mathbb{R} = 0 \neq 2$ we get $I = 0$.

3. (HS-1) Let $a < b < c < d$ be real numbers. Sort $x = ab + cd$, $y = bc + ad$, $z = ac + bd$ and prove it.

   Reason: Arithmetics.

   Solution: We suppose from examples that $y < z < x$.

   (a) $y < z$

   This is equivalent to

   $$\begin{aligned} bc + ad < ac + bd &\Longleftrightarrow (b-a)c < (b-a)d \\ &\Longleftrightarrow (b-a)(c-d) < 0 \end{aligned}$$

   which is true as $b - a > 0$ and $c - d < 0$.

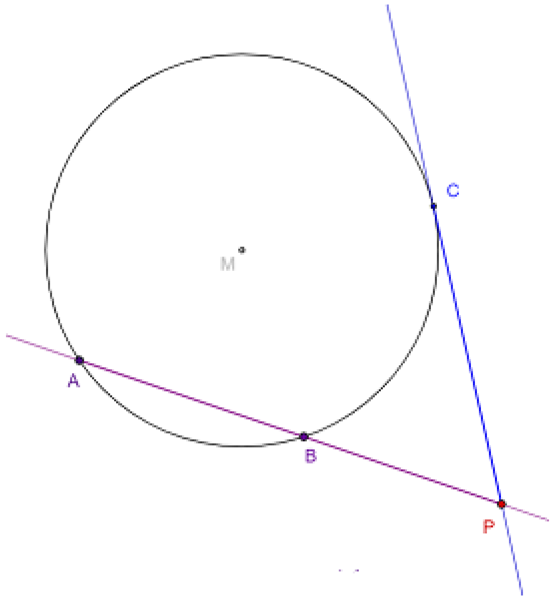
   (b) $z < x$

   This is equivalent to

   $$ac + bd < ab + cd \Longleftrightarrow (d-a)(b-c) < 0$$

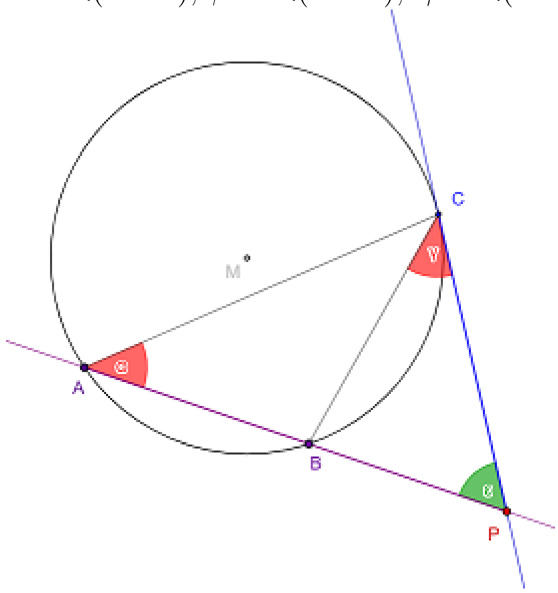
   which is true as $d - a > 0$ and $b - c < 0$.

4. (HS-2) Prove $\overline{CP}^2 = \overline{AP} \cdot \overline{BP}$



Reason: Geometry.

Solution: We connect the points AC and BC and define the angles $\alpha = \sphericalangle(BAC),\ \beta = \sphericalangle(BPC),\ \gamma = \sphericalangle(BCP)$



If we connect $\overline{CM}$ and elongate it to a diameter $\overline{CD}$ then $\gamma + \sphericalangle(DCB) = 90^\circ$. Thales' theorem now gives us $\sphericalangle(CDB) + \sphericalangle(DCB) = 90^\circ$ thus $\gamma = \sphericalangle(CDB)$. As all periphery angles over the same chord $\overline{CB}$ are equal $(= 1/2\sphericalangle(CMB))$, we get $\gamma = \alpha$. So with β we have two identical angles in $\triangle(PAC)$ and $\triangle(PBC)$, i.e. they are similar. This means

$$\frac{\overline{AP}}{\overline{CP}} = \frac{\overline{CP}}{\overline{BP}} \Longleftrightarrow \overline{AP} \cdot \overline{BP} = \overline{CP}^2$$

5. (HS-3) How big is the probability for two pocket aces in Texas Hold'em? Assume we have seen a show down in a heads-up. How many possible combinations are there, how many combinations of possible starting hands can the opponents have? How many possible community cards?

   Reason: Poker.

   Solution: There are $\binom{4}{2} = 6$ possible pocket aces among $\binom{52}{2} = 1,326$ possible hands, so the probability is $\frac{6}{1,326} = \frac{1}{221} \approx 0.45\%$. We have $\binom{52}{9} = 3,679,075,400$ combinations total after a show down in a heads-up. Community cards are $\binom{48}{5} = 1,712,304$ possibilities and $\binom{52}{2} \cdot \binom{50}{2} = 1,624,350$ possible starting hands in a heads-up.

6. (HS-4) Everybody knows that Schrödinger's cat is trapped in the box since 1935. Not well known is the fact, that the radioactive material was ten ^{14}C isotopes. Calculate the probability that the cat is still alive.

   Reason: Radiation.

   Solution: Half-life of ^{14}C are $5,730$ years. The decay rate is thus $\lambda = \dfrac{\log 2}{T_{1/2}} \approx 1.21 \cdot 10^{-4}\,\mathrm{a}^{-1}$. The probability for a single isotope to survive is $P = \exp(-\lambda T) \approx 98.977\%$. The cat survived, if all ten isotopes survived, i.e. with a probability of P^{10}. With significant figures provided by a modern calculator, this results in

$$P^{10} = \left(\exp\left(-\frac{85 \cdot \log 2}{5730}\right)\right)^{10} = 0.902286772193 \approx 90\%$$

   So the cat has a $9:10$ chance to be still alive.

7. (HS-5) Show that there is no rational solution for $p^2 + q^2 + r^2 = 7$.

   Reason: Modular Arithmetic.

   Solution: The equation can be transformed into an equivalent integer equation $x^2 + y^2 + z^2 = 7w^2$ where $\gcd(x^2 + y^2 + z^2, w^2) = 1$. Given any integer n, then $n^2 \in \{\bar{0}, \bar{1}, \bar{4}\} \mod 8$. If $w^2 \equiv \bar{1} \mod 8$ then $7w^2 \equiv \bar{7} \mod 8$ but there is no way to get $\bar{7}$ as a result of three sums of elements from $\{\bar{0}, \bar{1}, \bar{4}\}$. Hence $4|w^2$ and w is even. Not all x, y, z can thus be odd.

Assume x is even and y, z are odd: $4a^2+(2b+1)^2+(2c+1)^2 \equiv \bar{2} \mod 4$ but $7w^2 \equiv \bar{0} \mod 4$ which cannot be equal. So x, y, z are all even. But now we have a divider 2 of $x^2+y^2+z^2$ and of w^2, in contradiction to our assumption of a primitive equation.

18 July 2020

1. Calculate the electrostatic potential $U(a)$ of a surface $S = \{ (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = z^2,\, 0 \leq z \leq 1 \}$ charged with a field of homogeneous density ρ at the point $a = (0, 0, 1)$.

   Reason: Electrostatic Potential.

   Solution: The general formula (Coulomb) for the potential is

$$U(a) = \int\int_S \frac{\rho}{|x-a|}\, dO$$

   With the parameterization $\Phi(t, \varphi) = (t\cos\varphi, t\sin\varphi, t)$ we get $\Phi_t = (\cos\varphi, \sin\varphi, 1)$ and $\Phi_\varphi = (-t\sin\varphi, t\cos\varphi, 0)$. The fundamental quantities are

$$E = \Phi_t \cdot \Phi_t\,,\; F = \Phi_t \cdot \Phi_\varphi\,,\; G = \Phi_\varphi \cdot \Phi_\varphi$$

   These are in our case $E = 2$, $F = 0$, $G = t^2$ and the scalar surface element is $dO = \sqrt{EG - F^2}\, dt\, d\varphi = \sqrt{2}t\, dt\, d\varphi$ since $t \geq 0$. Thus

$$\begin{aligned}
U(a) &= \int_0^1 \int_0^{2\pi} \frac{\rho}{\sqrt{(t\cos\varphi - 0)^2 + (t\sin\varphi - 0)^2 + (t-1)^2}} \sqrt{2}t\, dt\, d\varphi \\
&= \int_0^1 dt \int_0^{2\pi} d\varphi\; \frac{\sqrt{2}\rho\, t}{\sqrt{t^2 + (t-1)^2}} = 2\,\pi\rho \int_0^1 \frac{t}{\sqrt{t^2 - t + \frac{1}{2}}}\, dt \\
&= 2\,\pi\rho \int_0^1 \frac{1}{2} \left(\frac{2t-1}{\sqrt{t^2 - t + \frac{1}{2}}} + \frac{1}{2\sqrt{t^2 - t + \frac{1}{2}}} \right) \\
&= \pi\rho \left[\sqrt{t^2 - t + \frac{1}{2}} \right]_0^1 + \pi\rho \int_0^1 \frac{dt}{\sqrt{t^2 - t + \frac{1}{2}}} \\
&\overset{\tau = t - 1/2}{=} \pi\rho \int_{-1/2}^{1/2} \frac{d\tau}{\sqrt{\tau^2 + \frac{1}{4}}} = \pi\rho \int_{-1/2}^{1/2} \frac{d(2\tau)}{\sqrt{(2\tau)^2 + 1}} \\
&= \pi\rho\, [\operatorname{arsinh}(2\tau)]_{-1/2}^{1/2} = \pi\rho \left[\log\left(2\tau + \sqrt{(2\tau)^2 + 1} \right) \right]_{-1/2}^{1/2} \\
&= \pi\rho \log\left(\frac{1 + \sqrt{2}}{-1 + \sqrt{2}} \right) = (\log(3 + 2\sqrt{2}))\pi\rho
\end{aligned}$$

2. (HS-1) Prove that the product of a finite number of sums of two integers

squares is again a sum of two integers squared.

$$(a_1^2 + b_1^2) \cdot (a_2^2 + b_2^2) \cdot \ldots \cdot (a_n^2 + b_n^2) = a^2 + b^2$$

Reason: Useful Trick.

Solution: We need to show that

$$\prod_{k=1}^{n} (a_k^2 + b_k^2) = \prod_{k=1}^{n} \det \left(\begin{bmatrix} a_k & -b_k \\ b_k & a_k \end{bmatrix} \right) = \det \left(\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \right) = a^2 + b^2$$

which is true as the matrices of this form multiplied with each another have again such a form:

$$\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \cdot \begin{bmatrix} \gamma & -\delta \\ \delta & \gamma \end{bmatrix} = \begin{bmatrix} \alpha\gamma - \beta\delta & -\alpha\delta - \beta\gamma \\ \alpha\delta + \beta\gamma & \alpha\gamma - \beta\delta \end{bmatrix}$$

and the determinant is a multiplicative function.

3. (HS-2) Given a positive integer in decimal representation without zeros. We build a new integer by concatenation of the number of even digits, the number of odd digits, and the number of all digits (the sum of the former two). Then we proceed with that number.

   Determine whether this algorithm always comes to a halt. What is or should be the criterion to stop?

   Reason: Algorithm.

   Solution: Let $|x|$ denote the number of digits of the integer x expressed in the decimal system, i.e. for $x = \sum_{k=0}^{n-1} x_k \cdot 10^k$ we have $|x| = n$. Say we have m even digits among the $\{x_i\}$, then one step of our algorithm transforms x to $f(x) = m \cdot 10^{|n|+|n-m|} + (n-m) \cdot 10^{|n|} + n$. Let us assume $n = |x| \geq 4$. Now $|x| \leq 1 + \log_{10} x$ so

$$\begin{aligned} f(x) &\leq n \cdot 10^{2|n|} + n \cdot 10^{|n|} + n \\ &\leq n \cdot 100^{1+\log_{10} n} + n \cdot 10^{1+\log_{10} n} + n \\ &\leq 100n + n^3 + 10n + n^2 + n \\ &= n^3 + n^2 + 111n \\ &< 10^{n-1} \\ &\leq x \end{aligned}$$

   if $n \geq 4$. Hence the algorithm decreases the input number on every single step, as long as there are at least four digits. If $n = 4$, then

$f(x) \leq 999$ and we have a three digit number. This means $f(x) = 100m + 10(3-m) + 3$ with $m \in \{1, 2, 3\}$. The last step is thus one of the following: $f(303) = f(213) = f(123) = 123$, and 123 the stopping criterion for the algorithm. Since this is an endless loop, the algorithm doesn't stop and needs a stopping command at 123.

It remains to show what will happen on numbers smaller than 100.

- $n = 2$: The only possibilities are

$$\begin{aligned} f(x_1x_0) &= (022) \longrightarrow (303) \longrightarrow (123) \\ \text{or } &= (\ \ 22) \longrightarrow (202) \longrightarrow (303) \longrightarrow (123) \\ f(x_1x_0) &= (202) \longrightarrow (303) \longrightarrow (123) \\ f(x_1x_0) &\longrightarrow (112) \longrightarrow (123) \end{aligned}$$

- $n = 1$: In this case we will get the previous cases, too. Either $f(x_0) = (101) \longrightarrow (123)$ or $f(x_0) = (011) \longrightarrow (123)$ or if we do not allow leading zeros $f(x_0) = (11) \longrightarrow (022)$ (see case $n = 2$).

4. (HS-3) List all real functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ with the following properties:

$$\begin{aligned} f(xy) &= f(x)f(y) - f(x) - f(y) + 2 \\ f(x+y) &= f(x) + f(y) + 2xy - 1 \\ f(1) &= 2 \end{aligned}$$

Reason: Real Function.

Solution: Let f be such that the conditions hold. Then

$$\begin{aligned} f(2) &= f(1+1) = 2f(1) + 1 = 5 \\ f(2x) &= f(2)f(x) - f(2) - f(x) + 2 = 4f(x) - 3 \\ f(2x) &= f(x) + f(x) + 2x^2 - 1 = 2f(x) + 2x^2 - 1 \\ 0 &= 2f(x) - 2x^2 - 2 \\ f(x) &= x^2 + 1 \end{aligned}$$

Conversely we have to check that this function fulfills the conditions:

$$\begin{aligned} (xy)^2 + 1 &= (x^2+1)(y^2+1) - x^2 - 1 - y^2 - 1 + 2 && \checkmark \\ (x+y)^2 + 1 &= x^2 + 1 + y^2 + 1 + 2xy - 1 && \checkmark \\ 1^2 + 1 &= 2 && \checkmark \end{aligned}$$

5. (HS-4) Find all real solutions (x,y) such that

$$\sin^4 x = y^4 + x^2y^2 - 4y^2 + 4\,,\ \cos^4 x = x^4 + x^2y^2 - 4x^2 + 1$$

Reason: Pigeon Hole Principle.

Solution:

$$\sin^4 x + \cos^4 x = (x^2 + y^2 - 2)^2 + 1 \geq 1$$

The values of sine and cosine are all in $[-1,1]$, so $\sin^4 x \leq \sin^2 x$ and $\cos^4 x \leq \cos^2 x$. Hence $\sin^4 x + \cos^4 x \leq \sin^2 x + \cos^2 x = 1$, which is only possible if equality holds everywhere:

$$x^2 + y^2 = 2\,,\ 0 = \sin^2 x(\sin^2 x - 1) = \cos^2 x(\cos^2 x - 1).$$

The equality of sines holds if and only if $x = k \cdot \pi/2$ for some $k \in \mathbb{Z}$. Now $k^2 \cdot \pi^2/4 = x^2 = 2 - y^2 \leq 2$ or $k^2 \leq 8/\pi^2 < 1$, i.e. $k = 0$. This $(0, \pm\sqrt{2})$ are the only possible solutions. It is easy to check that both points fulfill the given conditions.

6. (HS-5) Prove

$$\frac{(2n)!}{(n!)^2} > \frac{4^n}{n+1}$$

for all natural numbers $n > 1$.

Reason: Inequality.

Solution: We proceed by induction and check for $n = 2$ the inequality

$$\frac{(2\cdot 2)!}{(2!)^2} = \frac{4!}{4} = 3! = 6 > \frac{16}{3} = \frac{4^2}{2+1}.$$

Now let us assume that $\dfrac{(2k)!}{(k!)^2} > \dfrac{4^k}{k+1}$.

$$\begin{aligned}\frac{(2(k+1))!}{((k+1)!)^2} &= \frac{(2k+2)!}{(k!(k+1))^2}\\ &= \frac{(2k)!(2k+1)(2k+2)}{(k!)^2(k+1)^2}\\ &= \frac{(2k)!}{(k!)^2}\cdot\frac{2(2k+1)}{k+1}\end{aligned}$$

and

$$\frac{4^{k+1}}{(k+1)+1} = \frac{4^k}{k+1}\cdot\frac{4(k+1)}{k+2}$$

From $4k^2+10k+4=2(2k+1)(k+2)>4k^2+8k+4=4(k+1)^2$ we get

$$\frac{2(2k+1)}{k+1}>\frac{4(k+1)}{k+2}$$

Combining those we get

$$\begin{aligned}\frac{(2(k+1))!}{((k+1)!)^2}&=\frac{(2k)!}{(k!)^2}\cdot\frac{2(2k+1)}{k+1}\\&>\frac{4^k}{k+1}\cdot\frac{4(k+1)}{k+2}\\&=\frac{4^{k+1}}{k+2}\end{aligned}$$

what had to be shown.

19 June 2020

1. Let $S := \{ (x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2 = (2-z)^2 ,\, 0 \leq z \leq 2 \}$ be the surface of a cone C with a circular cross section and a peak at $(0,0,2)$. The orientation of S be such, that the normal vectors point outwards. Calculate the flux through S of the vector field

$$F\,:\,\mathbb{R}^3 \longrightarrow \mathbb{R}^3\,,\; F(x,y,z) = \begin{pmatrix} xy^2 \\ x^2y \\ (x^2+y^2)(1-z) \end{pmatrix}.$$

Reason: Flux.

Solution: In order to apply Gauß' theorem, we need to cover our cone with a disk $D := \{ (x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2 \leq 2 \}$. Then

$$\int_{S\cup D} F\cdot n\, dS = \int_S F\cdot n\,dS + \int_D F\cdot n\,dS = \int_C \operatorname{div}(F)\,dx\,dy\,dz$$

We have

$$\begin{aligned}\operatorname{div}(F) &= \partial_x(xy^2) + \partial_y(x^2y) + \partial_z((x^2+y^2)(1-z))\\ &= y^2+x^2+(x^2+y^2)(-1)\\ &= 0\end{aligned}$$

so our vector field is solenoidal and $\int_S F\cdot n\,dS = -\int_D F\cdot n\,dS$. We use the parametrization $\psi(r,\varphi) = (r\cos\varphi, r\sin\varphi, 0)$ with a normal vector $n(r,\varphi) = -(0,0,r)$ which has to point in negative z-direction.

$$\begin{aligned}\int_D F\cdot n\,dS &= \int_0^2 dr \int_0^{2\pi} d\varphi\, F(\psi(r,\varphi))\cdot n(r,\varphi)\\ &= \int_0^2 dr \int_0^{2\pi} d\varphi \begin{pmatrix} r^3\sin^2\varphi\cos\varphi \\ r^3\sin\varphi\cos^2\varphi \\ (1-0)(r^2\cos^2\varphi + r^2\sin^2\varphi)\end{pmatrix}\cdot\begin{pmatrix}0\\0\\-r\end{pmatrix}\\ &= \int_0^2 dr\int_0^{2\pi} d\varphi\,(-r^3)\\ &= -8\pi\end{aligned}$$

and $\int_S F\cdot n\,dS = 8\pi$.

2. Calculate for $|\alpha| \geq 1$

$$\int_0^\pi \log(1 - 2\alpha\cos(x) + \alpha^2)\,dx$$

(a) without using series expansions.

Hint: $\displaystyle\int_0^{\pi/2} \log(\sin(x))\,dx = -\frac{\pi}{2}\log(2)$

(b) by using serious expansions.

Reason: Feynman's Integration Trick.

Solution:

(a) Set $f(\alpha) = \int_0^\pi \log(1-2\alpha\cos(x)+\alpha^2)\,dx$ and assume $|\alpha| > 1$. Now we get by integration under the integral

$$\begin{aligned}
\frac{\partial f}{\partial \alpha} &= \int_0^\pi \frac{\partial}{\partial \alpha}\log(1 - 2\alpha\cos(x) + \alpha^2)\,dx \\
&= \int_0^\pi \frac{2\alpha - 2\cos(x)}{1 - 2\alpha\cos(x) + \alpha^2}\,dx \\
&= \frac{1}{\alpha}\int_0^\pi \left(\frac{2(\alpha^2 - \alpha\cos(x))}{1 - 2\alpha\cos(x) + \alpha^2} + 1 - 1\right)dx \\
&= \frac{1}{\alpha}\int_0^\pi \left(1 - \frac{1 - \alpha^2}{1 - 2\alpha\cos(x) + \alpha^2}\right)dx \\
&= \frac{\pi}{\alpha} - \frac{1 - \alpha^2}{\alpha}\int_0^\pi \frac{dx}{1 - 2\alpha\cos(x) + \alpha^2} \\
&= \frac{\pi}{\alpha} - \frac{1}{\alpha}\frac{1 - \alpha^2}{1 + \alpha^2}\int_0^\pi \frac{dx}{1 - \frac{2\alpha}{1+\alpha^2}\cos(x)}
\end{aligned}$$

Now we use the Weierstraß substitution $u := \tan\dfrac{x}{2}$, $|x| < \pi$ with

$$\sin(x) = \frac{2u}{1+u^2}\,,\ \cos(x) = \frac{1-u^2}{1+u^2}\,,\ dx = \frac{2du}{1+u^2}$$

and calculate

$$\begin{aligned}\int_0^\pi \frac{dx}{1-\frac{2\alpha}{1+\alpha^2}\cos(x)} &= \int_0^\infty \frac{2du}{(1+u^2)(1-\frac{2\alpha}{1+\alpha^2}\frac{1-u^2}{1+u^2})}\\ &= \int_0^\infty \frac{2du}{(1+u^2)-(1-u^2)\frac{2\alpha}{1+\alpha^2}}\\ &= \int_0^\infty \frac{2du}{(1-\frac{2\alpha}{1+\alpha^2})+u^2(1+\frac{2\alpha}{1+\alpha^2})}\\ &= \int_0^\infty \frac{2(1+\alpha^2)du}{(1+\alpha^2-2\alpha)+u^2(1+\alpha^2+2\alpha)}\\ &= \int_0^\infty \frac{2(1+\alpha^2)du}{(1-\alpha)^2+u^2(1+\alpha)^2}\\ &= \frac{2(1+\alpha^2)}{(1-\alpha)^2}\int_0^\infty \frac{du}{1+\left(\frac{1+\alpha}{1-\alpha}\right)^2 u^2}\end{aligned}$$

The last substitution will be

$$y=\frac{1+\alpha}{1-\alpha}\,u\,,\;dy=\frac{1+\alpha}{1-\alpha}\,du$$

and we continue

$$\begin{aligned}\int_0^\pi \frac{dx}{1-\frac{2\alpha}{1+\alpha^2}\cos(x)} &= \frac{2(1+\alpha^2)}{(1-\alpha)^2}\int_0^\infty \frac{du}{1+\left(\frac{1+\alpha}{1-\alpha}\right)^2 u^2}\\ &\overset{(*)}{=} \frac{2(1+\alpha^2)}{(1-\alpha)^2}\frac{(1-\alpha)}{(1+\alpha)}\int_0^{-\infty}\frac{dy}{1+y^2}\\ &= \frac{2(1+\alpha^2)}{1-\alpha^2}\left[\arctan(y)\right]_0^{-\infty}\\ &= -\pi\frac{1+\alpha^2}{1-\alpha^2}\end{aligned}$$

$(*)$ Note that y and u have different signs due to our choice of $|\alpha|>1$. Combined with what we omitted for simplification we have now

$$\begin{aligned}\frac{\partial f}{\partial\alpha} &= \frac{\pi}{\alpha}-\frac{1}{\alpha}\frac{1-\alpha^2}{1+\alpha^2}\cdot\left(-\pi\frac{1+\alpha^2}{1-\alpha^2}\right)\\ &= \frac{2\pi}{\alpha}\\ &\Longrightarrow\\ f(\alpha) &= 2\pi\log(|\alpha|)+C\end{aligned}$$

In order to calculate the integration constant, and deal with the case $\alpha = 1$, we compute

$$\begin{aligned} f(1) &= \int_0^\pi \log(2-2\cos(x))\,dx \\ &= \int_0^\pi \log\left(4\sin^2\left(\frac{x}{2}\right)\right)\,dx \\ &= \pi\log(4) + 4\int_0^{\pi/2} \log(\sin(y))\,dy \\ &\overset{Hint}{=} \pi\log(4) - 4\cdot\left(\frac{\pi}{2}\log(2)\right) \\ &= 0 \end{aligned}$$

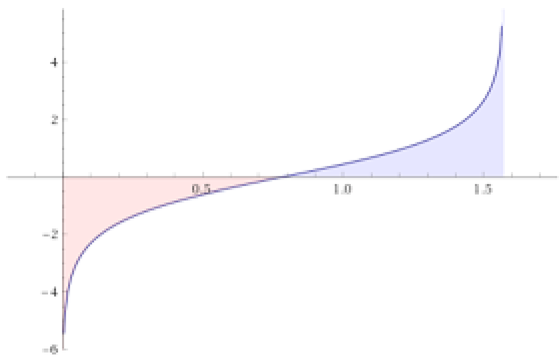
Thus we have $0 = f(1) = 2\pi\log(|1|) + C = C$ and the value of the integral is

$$\int_0^\pi \log(1-2\alpha\cos(x)+\alpha^2)\,dx = 2\pi\log(|\alpha|), \quad (|\alpha| \geq 1).$$

Proof of the hint: Let $I := \displaystyle\int_0^{\pi/2} \log(\sin(x))\,dx$.

For symmetry reasons we have

$$I = \int_0^{\pi/2} \log(\sin(x))\,dx = -\int_{\pi/2}^0 \log(\cos(x))\,dx$$



(The same sine curve in reverse order, and since the area is orientated, with a minus sign.)

Skipping the integration path yields

$$I = \int_0^{\pi/2} \log(\sin(x))\,dx = \int_0^{\pi/2} \log(\cos(x))\,dx$$

Hence

$$\begin{aligned}
2I &= \int_0^{\pi/2} \left(\log(\sin(x)) + \log(\cos(x))\right)\,dx \\
&= \int_0^{\pi/2} \log(\sin(x)\cos(x))\,dx \\
&= \int_0^{\pi/2} \log\left(\frac{\sin(2x)}{2}\right)\,dx \\
&= \int_0^{\pi/2} \log(\sin(2x))\,dx - \frac{\pi}{2}\log(2) \\
&= -\frac{\pi}{2}\log(2) + \frac{1}{2}\int_0^{\pi} \log(\sin(u))\,du \\
&= -\frac{\pi}{2}\log(2) + \frac{I}{2} + \frac{1}{2}\int_{\pi/2}^{\pi} \log(\sin(x))\,dx \\
&= -\frac{\pi}{2}\log(2) + \frac{I}{2} + \frac{1}{2}\int_0^{\pi/2} \log(\cos(x))\,dx \\
&= -\frac{\pi}{2}\log(2) + \frac{I}{2} + \frac{I}{2}
\end{aligned}$$

so $I = -\dfrac{\pi}{2}\log(2)$.

(b) We start with a value $|\alpha| \leq 1$ and $0 < x < \pi$. Then by Gradshteyn, Ryzhik 1.514

$$\log(1 - 2\alpha\cos(x) + \alpha^2) = -2\sum_{k=1}^{\infty} \frac{\cos(kx)}{k}\,\alpha^k$$

and we get

$$I(\alpha) := \int_0^{\pi} \log(1-2\alpha\cos(x)+\alpha^2)\,dx = -2\sum_{k=1}^{\infty} \frac{\alpha^k}{k}\int_0^{\pi} \cos(kx)\,dx = 0$$

This means for $|\alpha| \geq 1$

$$\begin{aligned}
\int_0^{\pi} \log(1 - 2\alpha\cos(x) + \alpha^2)\,dx &= \int_0^{\pi} \log\left(\alpha^2\left(\frac{1}{\alpha^2} - \frac{2}{\alpha}\cos(x) + 1\right)\right)\,dx \\
&= 2\pi\log(|\alpha|) + I\left(\frac{1}{\alpha}\right) \\
&= 2\pi\log(|\alpha|)
\end{aligned}$$

3. (HS-1)

   (a) Show that $\sqrt{i^i} \in \mathbb{R}$ where i is the imaginary unit $i = \sqrt{-1}$.

   (b) Which of the following equation signs is wrong and why?

$$-1 \overset{(1)}{=} i \cdot i \overset{(2)}{=} \sqrt{-1} \cdot \sqrt{-1} \overset{(3)}{=} \sqrt{(-1) \cdot (-1)} \overset{(4)}{=} \sqrt{1} \overset{(5)}{=} 1$$

   (c) Calculate all solutions of $z^3 = 1$ by three different methods.

Reason: Complex Numbers.

Solution:

   (a) We write $i = 0+1\cdot i = \cos(\pi/2)+i\sin(\pi/2) = e^{i\pi/2}$ by Euler's identity. Now $i^i = (e^{i\pi/2})^i = e^{-\pi/2}$ and $\sqrt{i^i} = e^{-\pi/4} \approx 0.45593813 \in \mathbb{R}$.

   (b) Equation (1) is the definition of the imaginary unit and can't be wrong. Equation (2) is only another (bad) way to write the imaginary unit i, so it's misleading but not wrong. Equations (4) and (5) are ordinary real arithmetic. Hence equation (3) must be wrong:

$$\begin{aligned}\sqrt{-1} \cdot \sqrt{-1} &\overset{\text{Gauß}}{=} (\cos(\pi/2) + i\sin(\pi/2)) \cdot (\cos(\pi/2) + i\sin(\pi/2)) \\ &\overset{\text{Euler}}{=} e^{i\pi/2} \cdot e^{i\pi/2} \\ &= e^{i\pi} \\ &= \cos(\pi) + i \cdot \sin(\pi) \\ &= -1 + i \cdot 0 \\ &= -1\end{aligned}$$

   whereas

$$\begin{aligned}\sqrt{(-1) \cdot (-1)} &= \sqrt{(\cos(\pi) + i \cdot \sin(\pi)) \cdot (\cos(\pi) + i \cdot \sin(\pi))} \\ &= \sqrt{e^{\pi} \cdot e^{\pi}} \\ &= \sqrt{e^{2\pi}} \\ &= \sqrt{\cos(2\pi) + i \cdot \sin(2\pi)} \\ &= \sqrt{1 + i \cdot 0} \\ &= 1\end{aligned}$$

   (c) One can use Euler's formula and calculate $e^{2i\pi} = (e^{i\varphi})^3 = e^{3i\varphi}$ and get $2\pi n = 3\varphi$ or $\varphi \in \{\,0, 2\pi/3, 4\pi/3\,\} \subseteq [0, 2\pi)$ which corresponds to

$$e^0 = 1\ ,\ e^{2i\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}\ ,\ e^{4i\pi/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2},$$

or use the fact that $z = 1$ is a solution and perform a long division:

$$z^3 : (z-1) = z^2 + z + 1 \Longrightarrow z_{1,2} = -\frac{1}{2} \pm \sqrt{-\frac{3}{4}}.$$

For the third method we write $z = r\cdot(\cos\varphi + i\sin\varphi)$. and calculate

$$\begin{aligned} 1 = z^3 &= r^3 \cdot (\cos\varphi + i\sin\varphi)^3 \\ &= r^3 \cdot \left(\cos^3\varphi + 3i\cos^2\varphi\sin\varphi - 3\cos\varphi\sin^2\varphi - i\sin^3\varphi\right) \\ &= r^3\left(\cos^3\varphi - 3\cos\varphi\sin^2\varphi\right) + i\cdot r^3\left(3\cos^2\varphi\sin\varphi - \sin^3\varphi\right) \end{aligned}$$

We see immediately that $\varphi = 0\,,\, r = 1$ is a solution, i.e. $z = 1$. The solution $\varphi = \pi\,,\, r = -1$ is identical, i.e. $z = 1$. Hence we may assume $\sin\varphi \neq 0$ now, and $\cos\varphi \neq 0$. By comparison of the coefficients we get

$$\begin{aligned} 0 &= 3\cos^2\varphi - \sin^2\varphi \\ 1 &= r^3 \cdot \cos\varphi \cdot \left(\cos^2\varphi - 3\sin^2\varphi\right) \end{aligned}$$

We get by substituting the first into the second equation

$$-\frac{1}{8} = r^3 \cdot \cos^3\varphi \Longrightarrow -\frac{1}{2} = r\cos\varphi$$

As $1 = |z|^3 = |r|^3$ the only real solutions for r are $r = \pm 1$, i.e. $\cos\varphi = \pm\frac{1}{2}$ and $\sin\varphi = \pm\frac{\sqrt{3}}{2}$. Checking all possibilities

φ	$\pi/3$	$2\pi/3$	$4\pi/3$	$5\pi/3$
$\cos\varphi$	$1/2$	$-1/2$	$-1/2$	$1/2$
r	-1	1	1	-1
$\sin\varphi$	$\sqrt{3}/2$	$\sqrt{3}/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$
z	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$	$-\frac{1}{2} - i\frac{\sqrt{3}}{2}$

4. (HS-2) Which is the smallest natural number $n \in \mathbb{N}_0$ such that there are no integers $a, b \in \mathbb{Z}$ with $3a^3 + b^3 = n$?

   Reason: Diophantine Equation.

   Solution: We start with the observation

$$\begin{array}{lll} 0 = 3\cdot 0^3 + 0^3 & 1 = 3\cdot 0^3 + 1^3 & 2 = 3\cdot 1^3 + (-1)^3 \\ 3 = 3\cdot 1^3 + 0^3 & 4 = 3\cdot 1^3 + 1^3 & 5 = 3\cdot(-1)^3 + 2^3 \end{array}$$

and show that such an equation is impossible for $n = 6$. Assume we had a solution, then $b^3 = 6 - 3a^3 \equiv 0 \mod 3$, i.e. $3 \,|\, b$ since 3 is prime. Hence we can write $b = 3c$ for some integer c, and get $6 = 3a^3 + 27c^3$ or $2 = a^3 + 9c^3$. This means, that $a^3 \equiv 2 \mod 3$, which is only possible, if $a = 3d + 2$ for some integer d. Now we have

$$2 = (3d + 2)^3 + 9c^3 = 27d^3 + 54d^2 + 36d + 8 + 9c^3 \equiv 8 \mod 9$$

which is impossible.

5. (HS-3) Is it possible to cover an equilateral triangle with two smaller equilateral triangles without a gap? It's not required that they are of equal area, nor that they won't overlap, only that they are smaller and together have a greater area than the original triangle.

   Reason: Pigeonhole Principle.

   Solution: It is impossible. Let us assume it could be done, and let the side length of the original triangle $\triangle A$ be a. Accordingly we set the side lengths of the smaller triangles $\triangle A', \triangle A''$ resp. to $a', a'' < a$.

   Now the three corners of $\triangle A$ have to be covered by two smaller triangles. W.l.o.g. we may assume that $\triangle A'$ covers two corners by the pigeonhole principle. But this means $a' \geq a$ as maximal possible distance in $\triangle A'$. But $a > a'$, a contradiction.

6. (HS-4) Given n different integers $\{ a_1, \ldots, a_n \}$, then there exists a subset $\{ a_{j_1}, \ldots, a_{j_m} \}$ with $1 \leq j_1 < \ldots < j_m \leq n$ such that n divides $a_{j_1} + \ldots + a_{j_m}$.

   Reason: Pigeonhole Principle.

   Solution: We consider the n sums $s_j := a_1 + \ldots + a_j$. If $s_i = s_j$ then $n \,|\, 0 = s_i - s_j = a_{j+1} + \ldots + a_i$ and we are done. So we may assume that all sums are pairwise different. Each of them can be written as $s_j = q_j \cdot n + r_j$ with $0 \leq r_j < n$. If one $r_j = 0$ then we are done again, so we may assume $1 \leq r_j < n$. But since we have n different s_j two remainders must be equal, say $r_i = r_j$. Thus n divides

$$a_{j+1} + \ldots + a_i = s_i - s_j = (q_i - q_j) \cdot n + (r_i - r_j) = (q_i - q_j) \cdot n$$

20 May 2020

1. Let $1 < p < 4$ and $f \in L^p((1,\infty))$ with the Lebesgue measure λ. We define $g : (1,\infty) \longrightarrow \mathbb{R}$ by

$$g(x) = \frac{1}{x}\int_x^{10x} \frac{f(t)}{t^{1/4}}\,d\lambda(t).$$

Show that there exists a constant $C = C(p)$ which depends on p but not on f such that $\|g\|_2 \leq C \cdot \|f\|_p$ so $g \in L^2((1,\infty))$.

Reason: L^p-Norms.

Solution: Let us first assume $p \neq 4/3$. By Hölder's inequality we get for a fixed $x \in \mathbb{R}_+$ with $\frac{1}{p}+\frac{1}{q} = 1$ and $q \neq 4$

$$\begin{aligned}\int_x^{10x} t^{-1/4}|f(t)|d\lambda(t) &\leq \|f\|_p \left(\int_x^{10x} t^{-q/4}\,dt\right)^{1/q}\\ &= \|f\|_p \left(\left[\frac{4}{4-q}\,t^{-\frac{q}{4}+1}\right]_x^{10x}\right)^{1/q}\\ &= \|f\|_p\left(\frac{4}{4-q}\left(10^{-\frac{q}{4}+1}-1\right)\right)^{1/q}\cdot x^{-\frac{1}{4}+\frac{1}{q}}\end{aligned}$$

Now we have

$$\begin{aligned}\|g\|_2^2 &\leq \int_1^\infty \left(\frac{1}{x}\int_x^{10x}\frac{|f(t)|}{t^{1/4}}\,d\lambda(t)\right)^2 d\lambda(x)\\ &\leq \|f\|_p^2 \underbrace{\left(\frac{4}{4-q}\left(10^{-\frac{q}{4}+1}-1\right)\right)^{2/q}}_{=:C_1(p)} \underbrace{\int_1^\infty \frac{1}{x^2}\cdot x^{-\frac{1}{2}+\frac{2}{q}}\,d\lambda(x)}_{=:I(p)}\end{aligned}$$

The integral $I(p) = \displaystyle\int_1^\infty \frac{1}{x^2}\cdot x^{-\frac{1}{2}+\frac{2}{q}}\,d\lambda(x) = \int_1^\infty x^{-\frac{5}{2}+\frac{2}{q}}\,d\lambda(x)$ only depends on q and therewith on p. It is finite if and only if $-\dfrac{5}{2}+\dfrac{2}{q} < -1$, i.e. $1/q < 3/4$ or $p < 4$ and thus

$$\|g\|_2 \leq \underbrace{\sqrt{C_1(p)\cdot I(p)}}_{=:C}\cdot\|f\|_p\,.$$

In case $p=\frac{3}{4}$ or $q=4$ we get $\displaystyle\int_x^{10x} t^{-1/4}|f(t)|d\lambda(t) \leq \|f\|_{3/4}\cdot(\log 10)^{1/4}$.

$$\begin{aligned}\|g\|_2^2 &\leq \int_1^\infty \left(\frac{1}{x}\int_x^{10x}\frac{|f(t)|}{t^{1/4}}\,d\lambda(t)\right)^2 d\lambda(x)\\ &\leq \|f\|_{3/4}^2\cdot(\log 10)^{1/2}\int_1^\infty \frac{dx}{x^2}\\ &= \|f\|_{3/4}^2\cdot\underbrace{(\log 10)^{1/2}}_{=:C^2}\end{aligned}$$

2. We define
$$\mathbb{R}^\infty=\mathbb{R}^{(\mathbb{N})}=\{\,(x_1,x_2,\ldots)\,|\,x_i\overset{a.a.}{=}0\,\}$$
and equip $\mathbb{R}^\infty$ with the Euclidean metric
$$d((x_1,x_2,\ldots),(y_1,y_2,\ldots))=\sqrt{\sum_{i=1}^\infty |x_i-y_i|^2}.$$
which defines a topology
$$\mathcal{S}:=\{\,U\subseteq\mathbb{R}^\infty\,|\,\forall\,p\in U\,\exists\,\varepsilon>0\,:\,B_\varepsilon(p)\subseteq U\,\}$$
with the open ball $B_\varepsilon(p)=\{\,q\in\mathbb{R}^\infty\,|\,d(p,q)<\varepsilon\,\}$.

(a) Show that the function
$$\begin{aligned}\alpha\,:\,(\mathbb{R}^\infty,\mathcal{S})&\longrightarrow(\mathbb{R},\mathcal{E})\\ (x_1,x_2,\ldots)&\longmapsto\sum_{i=1}^\infty 2^i\cdot x_i\end{aligned}$$
is not continuous, where $\mathcal{E}$ is the usual Euclidean topology on $\mathbb{R}$.

(b) Let B be the diagonal matrix where the diagonal entries are 2^i for $i=1,2,\ldots$, i.e.
$$B=\begin{bmatrix}2&0&0&\ldots\\0&4&0&\ldots\\0&0&8&\ldots\\\vdots&\vdots&\vdots&\ddots\end{bmatrix}$$
Show that $\beta\,:\,(\mathbb{R}^\infty,\mathcal{S})\longrightarrow(\mathbb{R}^\infty,\mathcal{S})$ defined by $\beta(x)=Bx$ is not continuous.

(c) Define a topology $\mathcal{T}$ on $\mathbb{R}^\infty$ such that the inclusion maps

$$\begin{aligned}\iota_n \,:\, (\mathbb{R}^n, \mathcal{E}) &\longrightarrow (\mathbb{R}^\infty, \mathcal{T})\\ (x_1,\ldots,x_n) &\longmapsto (x_1,\ldots,x_n,0,\ldots)\end{aligned}$$

are continuous for any $n \in \mathbb{N}_0$.

Reason: Topologies on $\mathbb{R}^\infty$.

Solution:

(a) We consider $U = (-1,1) \subseteq \mathbb{R}$ so $0 \in \alpha^{-1}(U)$. Suppose there is an $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq U$. We pick $i \in \mathbb{N}$ with $2^{-i} < \varepsilon$ and the point $q = (0,\ldots,0,2^{-i},0,\ldots)$. Then $d(0,q) = 2^{-i} < \varepsilon$ and $q \in B_\varepsilon(0)$. But $\alpha(q) = 1$, so $q \notin \alpha^{-1}(U)$, i.e. $B_\varepsilon(0) \not\subseteq U$, a contradiction.

(b) $U = B_1(0) \subseteq \mathbb{R}^\infty$ is an open set and $0 \in \beta^{-1}(U)$. Suppose there is an $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq U$. We pick again $i \in \mathbb{N}$ such that $2^{-i} < \varepsilon$. For $q = (0,\ldots,0,2^{-i},0,\ldots)$ we have $d(0,q) = 2^{-i} < \varepsilon$, so $q \in B_\varepsilon(0) \subseteq U$. But $\beta(q) = (0,\ldots,0,1,0,\ldots)$ so $d(0,\beta(q)) = 1$ which means that $q \notin U$, a contradiction.

(c) We define the so called weak topology on $\mathbb{R}^\infty$ as

$$\mathcal{T} := \{\, U \subseteq \mathbb{R}^\infty \,|\, \forall\, n \in \mathbb{N}_0 \,:\, U \cap \mathbb{R}^n \subseteq \mathbb{R}^n \text{ is open } \}.$$

This means: A map $f \,:\, (\mathbb{R}^\infty, \mathcal{T}) \longrightarrow (X, \mathcal{X})$ of topological spaces is continuous with respect to $\mathcal{T}$ if and only if for each $n \in \mathbb{N}_0$ the map

$$(\mathbb{R}^n, \mathcal{E}) \stackrel{\iota_n}{\hookrightarrow} (\mathbb{R}^\infty, \mathcal{T}) \stackrel{f}{\longrightarrow} (X, \mathcal{X})$$

is continuous.

3. Calculate

$$\int_{-\infty}^{+\infty} \frac{\cos(\alpha x)}{1+x^2}\,dx \quad (\alpha \geq 0).$$

Reason: Integral.

Solution: For $\alpha = 0$ we have

$$\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = [\arctan(x)]_{-\infty}^{+\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

so we may assume $\alpha > 0$ now, substitute $t = \alpha x, \beta = \alpha^{-1}$ and observe using integration by parts twice

$$\begin{aligned}\int_0^\infty e^{-t}\sin(\beta t)\,dt &= -e^{-t}\sin(\beta t)\Big|_0^\infty + \beta\int_0^\infty e^{-t}\cos(\beta t)\,dt \\ &= 0 + \beta\left(\left[-e^{-t}\cos(\beta t)\right]_0^\infty - \beta\int_0^\infty(-e^{-t})(-\sin(\beta t))\,dt\right) \\ &= \beta - \beta^2\int_0^\infty e^{-t}\sin(\beta t)\,dt\end{aligned}$$

and thus

$$\int_0^\infty e^{-t}\sin(\beta t)\,dt = \frac{\beta}{1+\beta^2}$$

This means

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\cos(\alpha x)}{1+x^2}\,dx &= \int_{-\infty}^{+\infty}\frac{\cos(\alpha x)}{x}\cdot\frac{x}{1+x^2}\,dx \\ &= 2\int_0^\infty\left(\frac{\cos(\alpha x)}{x}\int_0^\infty e^{-t}\sin(xt)\,dt\right)\,dx \\ &= \int_0^\infty\int_0^\infty e^{-t}\cdot\frac{2\sin(xt)\cos(\alpha x)}{x}\,dt\,dx \\ &= \int_0^\infty\int_0^\infty e^{-t}\cdot\frac{\sin(x(t+\alpha))+\sin(x(t-\alpha))}{x}\,dx\,dt \\ &= \int_0^\infty e^{-t}\cdot\left(\frac{\pi}{2}\operatorname{sgn}(t+\alpha)+\frac{\pi}{2}\operatorname{sgn}(t-\alpha)\right)\,dt \\ &= \pi\int_\alpha^\infty e^{-t}\,dt \\ &= \pi\cdot e^{-\alpha}\end{aligned}$$

4. Calculate

$$\int_0^1 \sin(\pi x)\,x^x\,(1-x)^{1-x}\,dx.$$

Hint: You may use calculators to determine residues.

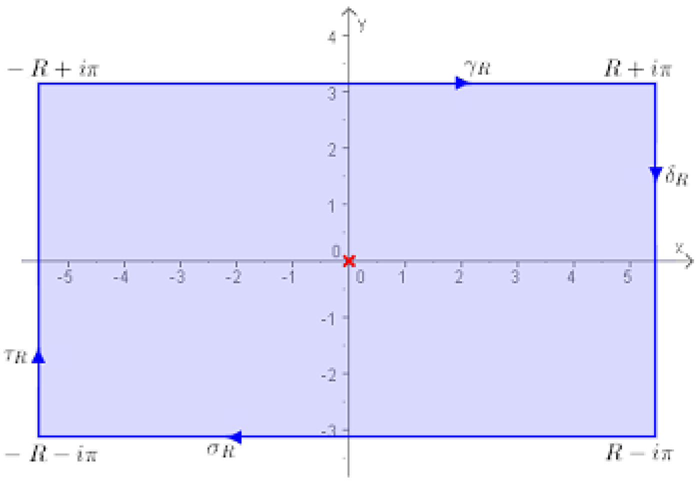
Reason: Ramanujan Integral.

Solution: We set $S := \displaystyle\int_0^1 e^{i\pi x}\,x^x\,\frac{1-x}{(1-x)^x}\,dx$ and substitute $t =$

$\log(x) - \log(1-x)$, i.e. $x = \dfrac{e^t}{e^t+1}$.

$$\begin{aligned} S &= \int_0^1 e^{i\pi x}\, e^{(\log(x))\,x}\, \frac{1-x}{e^{\log(1-x)\,x}}\, dx = \int_0^1 (1-x)\, e^{(i\pi + \log(x) - \log(1-x))x}\, dx \\ &= \int_{-\infty}^{\infty} \frac{1}{e^t+1}\, e^{(i\pi+t)\frac{e^t}{e^t+1}}\, \frac{e^t}{(e^t+1)^2}\, dt \\ &= \int_{-\infty+i\pi}^{\infty+i\pi} \frac{1}{-e^s+1}\, e^{s\frac{-e^s}{-e^s+1}}\, \frac{-e^s}{(-e^s+1)^2}\, ds \\ &= \int_{-\infty+i\pi}^{\infty+i\pi} \frac{e^t}{(e^t-1)^3}\, e^{t\,\frac{e^t}{e^t-1}}\, dt \end{aligned}$$

With $f(z) := \dfrac{e^z}{(e^z-1)^3}\, e^{z\,\frac{e^z}{e^z-1}}$ we have a meromorphic function on $D := \{\, z \in \mathbb{C} \mid -\pi \leq \Im(z) \leq \pi \,\}$.



The only singularity is at $z=0$ and we have the residue $\operatorname{res}(f,0) = -\dfrac{e}{24}$, see e.g. WolframAlpha.com with the input

```
residue of (e^(z(e^z/(e^z-1)))(e^z/(e^z-1)^3))at z=0)
```

Consider the path $\kappa_R = \gamma_R + \delta_R + \sigma_R + \tau_R$ around $z=0$ as in the graphic. Then $\oint_{\kappa_R} f(z)\, dz = -2\pi i \cdot \operatorname{res}(f(z),0) = 2i\,\dfrac{\pi e}{24}$.

$(f(z_n))_{n\in\mathbb{N}}$ converges to 0 for any sequence $(z_n)_{n\in\mathbb{N}} \subseteq D$ such that $|z_n| \longrightarrow \infty$, so

$$\lim_{R\to\infty} \int_{\delta_R} f\, dz = \lim_{R\to\infty} \int_{\tau_R} f\, dz = 0$$

The function $f(z)$ is odd, so $\int_{\sigma_R} f(z)\,dz = \int_{\gamma_R} f(z)\,dz$ and $2S = 2\lim_{R\to\infty}\int_{\gamma_R} f(z)\,dz = \lim_{R\to\infty}\oint_{\kappa_R} f(z)\,dz = 2i\,\dfrac{\pi e}{24}$ and our integral becomes:

$$\int_0^1 \sin(\pi x)\,x^x\,(1-x)^{1-x}\,dx = \Im(S) = \frac{\pi e}{24}.$$

5. The p-Prüfer group is defined as

$$G := \mathbb{C}_{p^\infty} = \{\,\exp(2n\pi i/p^m)\,|\,n\in\mathbb{Z}, m\in\mathbb{N}\,\} \cong \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$$

Show that G is isomorphic to the factor group F/R of the free Abelian group over an countably infinite basis $\{\,a_1,a_2,\ldots,a_n,\ldots\}$ with the subgroup of relations R generated by $\{\,pa_1, a_1-pa_2,\ldots,a_n-pa_{n+1},\ldots\}$, so

$$G = \langle x_1,x_2,\ldots\,|\,x_1^p=1, x_2^p=x_1, x_3^p=x_2,\ldots\rangle$$

Reason: Prüfer Group.

Solution: Let $F=\langle a_1,a_2,\ldots\rangle$ be the free Abelian group as defined. Then we consider the function

$$f\,:\,S\longrightarrow G\,,\,a_m\longmapsto \exp(2\pi i/p^m)$$

By the universal property of F there is a unique group homomorphism

$$\varphi\,:\,F\longrightarrow G\,,\,\varphi(a_m)=f(a_m)\;\forall\,m$$

Now $\exp(2n\pi i/p^m)=\exp(2\pi i/p^m)^n=\varphi(a_m)^n=\varphi(n\cdot a_m)$ shows that φ is surjective. By

$$\begin{aligned}\varphi(p\cdot a_1) &= \varphi(a_1)^p=\exp(2\pi i/p)^p=\exp(2\pi i)=1\\ \varphi(a_n-pa_{n+1}) &= \varphi(a_n)\varphi(a_{n+1})^{-p}=\exp(2\pi i/p^n)\exp(2\pi i/p^{n+1})^{-p}\\ &= \exp(2\pi i/p^n)/\exp(2p\pi i/p^{n+1})=1\end{aligned}$$

we see that $R\subseteq\ker\varphi$. Let $x=n_1a_{i_1}+\ldots+n_ra_{i_r}\in F$ be in the kernel of φ, and assume $1\leq i_1\leq i_2\leq\ldots\leq i_r=:s$. Then

$$\begin{aligned}1=\varphi(x)=\varphi(n_1a_{i_1}+\ldots+n_ra_{i_r})&=\varphi(a_{i_1})^{n_1}\cdots\varphi(a_{i_r})^{n_r}\\ =\prod_{k=1}^r\exp(2\pi i/p^{i_k})^{n_k}&=\prod_{k=1}^s\exp(2\delta_{ki_k}n_k\pi i/p^k)=\exp\left(\sum_{k=1}^s\frac{2\delta_{ki_k}n_k\pi i}{p^k}\right)\end{aligned}$$

So $2m\pi i = \sum_{k=1}^{s} \frac{2\delta_{ki_k} n_k \pi i}{p^k}$ or $0 = -mp^s + \sum_{k=1}^{s} \delta_{ki_k} n_k \cdot p^{s-k}$ or

$$
\begin{aligned}
n_s &= mp^s - \sum_{k=1}^{s-1} \delta_{ki_k} n_k \cdot p^{s-k} \\
x &= \sum_{k=1}^{s-1} \left(\delta_{ki_k} n_k a_k - \delta_{ki_k} n_k \cdot p^{s-k} a_s\right) + mp^s a_s \\
&= \sum_{k=1}^{s-1} \left(\delta_{ki_k} n_k (a_k - pa_{k+1})\right) \quad \in R \\
&+ \sum_{k=1}^{s-1} \left(p\delta_{ki_k} n_k a_{k+1} - \delta_{ki_k} n_k \cdot p^{s-k} a_s\right) + mp^s a_s \\
&\stackrel{mod\ R}{\equiv} \sum_{k=1}^{s-1} \left(p\delta_{ki_k} n_k (a_{k+1} - pa_{k+2})\right) \quad \in R \\
&+ \sum_{k=1}^{s-1} \left(p^2\delta_{ki_k} n_k a_{k+2} - \delta_{ki_k} n_k \cdot p^{s-k} a_s\right) + mp^{s-1} a_{s-1} \\
&\stackrel{mod\ R}{\equiv} \sum_{k=1}^{s-1} \left(p^2\delta_{ki_k} n_k (a_{k+2} - pa_{k+3})\right) \quad \in R \\
&+ \sum_{k=1}^{s-1} \left(p^3\delta_{ki_k} n_k a_{k+3} - \delta_{ki_k} n_k \cdot p^{s-k} a_s\right) + mp^{s-2} a_{s-2} \\
&\stackrel{mod\ R}{\equiv} \\
&\vdots \\
&\stackrel{mod\ R}{\equiv} \sum_{k=1}^{s-1} \left(p^{s-1-k}\delta_{ki_k} n_k (a_{s-1} - pa_s)\right) \quad \in R \\
&+ \sum_{k=1}^{s-1} \left(p^{s-k}\delta_{ki_k} n_k a_s - \delta_{ki_k} n_k \cdot p^{s-k} a_s\right) + mp^{k+1} a_{k+1} \\
&\stackrel{mod\ R}{\equiv} mp^{k+1} a_{k+1} \stackrel{mod\ R}{\equiv} mp^k a_k \stackrel{mod\ R}{\equiv} mp^{k-1} a_{k-1} \\
&\stackrel{mod\ R}{\equiv} \ldots \stackrel{mod\ R}{\equiv} mpa_1 \stackrel{mod\ R}{\equiv} 0
\end{aligned}
$$

and we have shown that $R \supseteq \ker\varphi$. Hence $F/R = F/\ker\varphi \cong G$.

6. Give an example of a quotient $R-$module M/N which is Artinian although neither the ring R nor the modules M, N are.

Reason: Artinian Modules.

Solution: An example is the p-Prüfer group considered as $\mathbb{Z}$-module:

$$\mathbb{C}_{p^\infty} = \{\, \exp(2n\pi i/p^m) \,|\, n \in \mathbb{Z}, m \in \mathbb{N}\,\} \cong_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$$

The $\mathbb{Z}$-module homomorphismus $\varphi \,:\, \mathbb{Z}[1/p] \longrightarrow \mathbb{C}_{p^\infty}$, $q \longmapsto \exp(2q\pi i)$ is surjective and has the kernel $\mathbb{Z}$.

(a) $\mathbb{Z}$ is not Artinian.

$$\ldots (q^n) \subsetneq (q^{n-1}) \subsetneq \ldots \subsetneq (q) \subsetneq \mathbb{Z}\,,\; q \text{ prime}$$

So $\mathbb{Z}$ is neither an Artinian ring nor an Artinian $\mathbb{Z}$-module.

(b) $\mathbb{Z}[1/p]$ is not Artinian.

The same descending series of $\mathbb{Z}$-modules as above is also an infinitely long descending series of $\mathbb{Z}[1/p]$-submodules, if we choose $q \neq p$, so $\mathbb{Z}[1/p]$ isn't an Artinian $\mathbb{Z}$-module.

(c) $\mathbb{C}_{p^\infty}$ is Artinian.

The chain of $\mathbb{Z}$-submodules $(1/p^{n-1}) \subsetneq (1/p^n)$ in $\mathbb{C}_{p^\infty}$ is infinitely long, so $\mathbb{C}_{p^\infty}$ is not Noetherian. Now let us consider a chain

$$\mathbb{C}_{p^\infty} \supsetneq M_1 \supsetneq M_2 \supsetneq M_3 \supsetneq \ldots$$

of descending $\mathbb{Z}$-submodules. We show that this can be written as

$$\mathbb{C}_{p^\infty} \supsetneq (1/n_1) \supsetneq (1/n_2) \supsetneq (1/n_3) \supsetneq \ldots$$

of descending $\mathbb{Z}$-submodules for some positive integers $n_j > 0$.

Let $0 \neq M \subseteq \mathbb{Z}[1/p]/\mathbb{Z}$ be a $\mathbb{Z}$-submodule. Then every element $m \in M$ can be written as

$$\mathbb{Z} \not\ni m = \sum_{k=1}^{n} \frac{a_k}{p^{r_k}} = \frac{c_m}{p^{r_m}} \quad (a_k, c_m \in \mathbb{Z},\; r_k, r_m \in \mathbb{N})$$

Now we can cancel all factors p of c_m. Since $M \neq 0 = \mathbb{Z}$ there is an element $m = \dfrac{c_m}{p^{r_m}}$ such that $r := r_m > 0$. If $(c_m, p) = 1$ then $(c_m, p^r) = 1$ and we can find $\alpha, \beta \in \mathbb{Z}$ such that $1 = \alpha c + \beta p^r$, i.e.

$$\frac{1}{p^r} = \alpha \cdot \frac{c}{p^r} + \beta \equiv \alpha \cdot \frac{c}{p^r} \in M \mod \mathbb{Z}\,.$$

Hence every $\mathbb{Z}$-submodule of $\mathbb{Z}[1/p]/\mathbb{Z}$ has the form $M = (1/n)$ with a positive integer $n > 0$.

Now $(1/n_k) \supsetneq (1/n_{k+1})$ implies $n_{k+1} \,|\, n_k$. Hence $n_1 > n_2 > n_3 > \ldots$ is a decreasing sequence of positive integers, which thus must terminate, i.e. $\mathbb{C}_{p^\infty}$ is an Artinian $\mathbb{Z}$-module.

7. (a) Let $u_1, \ldots, u_n$ be solutions of the one dimensional heat equation $\frac{du}{dt} - \frac{d^2u}{dx^2} = 0$ $(x \in \mathbb{R}, t > 0)$. Show that

$$u(x_1, \ldots, x_n, t) := \prod_{k=1}^{n} u_k(x_k, t)$$

is a solution of the n dimensional heat equation $\frac{\partial u}{\partial t} - \Delta u = 0$.

(b) Calculate a solution for

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = 0 & \text{for } x \in \mathbb{R}^3,\ t > 0 \\ u(x,0) = x_1^2 x_2^2 x_3 & \text{for } x = (x_1, x_2, x_3) \in \mathbb{R}^3 \end{cases}$$

Reason: Heat Equation.

Solution:

(a) We get by differentiating $u = u(x_1, \ldots, x_n, t)$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{k=1}^{n} \frac{\partial u_k}{\partial t}(x_k,t) \prod_{j \neq k} u_j(x_j,t)$$
$$\frac{\partial u}{\partial x_k}(x,t) = \frac{\partial u_k}{\partial x_k}(x_k,t) \prod_{j \neq k} u_j(x_j,t)$$
$$\frac{\partial^2 u}{\partial x_k^2}(x,t) = \frac{\partial^2 u_k}{\partial x_k^2}(x_k,t) \prod_{j \neq k} u_j(x_j,t)$$
$$\Delta u(x,t) = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} = \sum_{k=1}^{n} \frac{\partial^2 u_k}{\partial x_k^2}(x_k,t) \prod_{j \neq k} u_j(x_j,t)$$

and so

$$\frac{\partial u}{\partial t} - \Delta u = \sum_{k=1}^{n} \underbrace{\left(\frac{\partial u_k}{\partial t}(x_k,t) - \frac{\partial^2 u_k}{\partial x_k^2}(x_k,t) \right)}_{=0 \text{ by assumption}} \prod_{j \neq k} u_j(x_j,t) = 0$$

(b) If $u_k(x_k,t)$ with $u_k(x_k,0) = f_k(x_k)$ are solutions of the one dimensional heat equation, then $u(x,t) = u_1(x_1,t)u_2(x_2,t)u_3(x_3,t)$ solves the given problem with $u(x,0) = f_1(x_1)f_2(x_2)f_3(x_3)$. We therefore want to find solutions to

$$\begin{cases} \dfrac{du_k}{dt}(x_k,t) - \dfrac{d^2}{dx_k^2}u(x_k,t) = 0 & \text{for } x_k \in \mathbb{R},\, t > 0 \\ u_k(x_k,0) = x_k^2 & \text{for } x_k \in \mathbb{R} \end{cases}$$

$$\begin{cases} \dfrac{du_3}{dt}(x_3,t) - \dfrac{d^2}{dx_3^2}u(x_3,t) = 0 & \text{for } x_3 \in \mathbb{R},\, t > 0 \\ u_3(x_3,0) = x_3 & \text{for } x_3 \in \mathbb{R} \end{cases}$$

for $k = 1,2$. Let's set $u_k(x_k,t) = v_k(x_k) + w_k(t)$ so

$$0 = \frac{\partial}{\partial t}u_k(x_k,t) - \frac{\partial^2}{\partial x_k^2}u(x_k,t) = w_k'(t) - v_k''(x_k) \text{ for } x_k \in \mathbb{R},\, t > 0$$

i.e. $w_k'(t) = v_k''(x_k) = c_k \in \mathbb{R}$ is constant. Thus

$$w_k(t) = c_k t \text{ and } v_k(x_k) = \frac{1}{2}c_k x_k^2 + d_k x_k + e_k$$

The initial value $u_k(x_k,0) = v_k(x_k) + w_k(t) = x_k^2$ means $c_k = 2$ and $d_k = e_k =$ hence $u_k(x_k,t) = 2t + x_k^2$ for $k = 1,2$. The third equation leads in an analogue way to $u_3(x_3,t) = x_3$.

One (not necessarily all) solution to our initial value problem of the three dimensional heat equation is given by

$$u(x_1,x_2,x_3,t) = (2t + x_1^2)(2t + x_2^2)x_3$$

8. Prove and give an example of a solvable group which is not supersolvable.

   Reason: Solvable Groups.

   Solution: A_4.
   The non Abelian, alternating group $A_4 \subseteq S_4$ has $\frac{4!}{2} = 12$ elements and is solvable:

$$\{\,(1)\,\} \triangleleft \{\,(1),(12)(34)\,\} \triangleleft V_4 = \{\,(1),(12)(34),(13)(24),(14)(23)\,\} \triangleleft A_4$$

   The first two factor groups (from the left) are isomorphic to $\mathbb{Z}_2$, i.e. of index 2, hence Abelian and normal in each other. The Klein $4-$group V_4

is also normal in A_4 because it coincides with the commutator subgroup of $A_4 \,:\, V_4 = [A_4, A_4]$. This means especially, that the factor group is Abelian. It is also isomorphic to $\mathbb{Z}_3$. We calculate the examples

$$[(123),(124)] = (123)(124)(132)(142) = (12)(34) \in V_4$$
$$[(123),(12)(34)] = (123)(12)(34)(132)(12)(34) = (13)(24) \in V_4$$

Assume we have a composition series $1 = G_1 \triangleleft G_2 \triangleleft \ldots \triangleleft G_n = A_4$ where all $G_k \triangleleft A_4$ are normal in the main group, and have cyclic factor groups. Since

$$\prod_{k=1}^{n-1} |G_{k+1}/G_k| = |A_4| = 12$$

we have maximal four factors, i.e. $n \leq 4$ because we included $\{1\}$. Subgroups of A_4 with 2 or 3 elements are not normal:

$$((12)(34))\,(132)\,((12)(34)) = (132)(13)(24) = (124) \notin \{\,(1),(123),(132)\,\}$$
$$(123)\,((12)(34))\,(132) = (13)(24)(34)(12) = (14)(23) \notin \{\,(1),(12)(34)\,\}$$

A_4 can be written $A_4 \cong V_4 \rtimes_{\varphi} \mathbb{Z}_3$ with

$$\varphi \,:\, \mathbb{Z}_3 \longrightarrow \operatorname{Aut}(V_4)\,,\; \varphi(z)(v) = (243)v(234)$$

A subgroup with 6 elements would necessarily be normal as of index 2. But the isomorphism shows, there is none, because $S_3 \ntriangleleft A_4$. A_4 is herewith an example that the opposite of Lagrange's theorem does not hold: there is no normal subgroup of a finite group for any divider of the group order. As a consequence we must have $n = 4$ and $G_3 = V_4$. This leaves us with $G_2, G_3 \in \{\,\mathbb{Z}_2, \mathbb{Z}_3\,\}$ neither of which are normal in A_4 which therefore cannot be supersolvable.

9. (HS-1) For which natural numbers is $1! + \ldots + n!$ a square number? $n! = 1 \cdot 2 \cdot \ldots \cdot n$.

   Reason: Modular Arithmetic.

   Solution: $1! = 1$ and $1! + 2! + 3! = 9$ are square numbers, $1! + 2! = 3$ and $1! + 2! + 3! + 4! = 33$ are not. Now let $n \geq 5$. Then every number

$$x = 1! + 2! + 3! + 4! + 5! + 6! + \ldots + n! = 33 + m$$

   where m is divisible by 10 as 2 and 5 are included factors in each term from 5 onwards. So x divided by 10 has remainder 3. However, any square number divided by 10 must have a remainder from $\{\,0,1,4,5,6,9\,\}$. Hence $n \in \{\,1,3\,\}$ are the only solutions.

10. (HS-2) Determine $\{ (x,y) \in \mathbb{N}_0 \times \mathbb{N}_0 \,|\, x^3+8x^2-6x+8-y^3=0 \}$.

Reason: Diophantic Equation.

Solution: Assume we have an integer solution for non-negative numbers x,y to $y^3 = x^3+8x^2-6x+8$. We start with the motto: Get rid of what disturbs! These are apparently the cubes. As there are no $y-$terms of minor degree, we can consider expressions $\pm(y^3-(x+g)^3)$ for small integers g and find:

$$\begin{aligned} y^3-(x+1)^3 &= x^3+8x^2-6x+8-x^3-3x^2-3x-1 \\ &= 5x^2-9x+7 \\ &= 5\cdot\left[\left(x-\frac{9}{10}\right)^2+\frac{59}{100}\right] \\ &> 0 \\ (x+3)^3-y^3 &= x^3+9x^2+27x+27-x^3-8x^2+6x-8 \\ &= x^2+33x+19 \\ &> 0 \end{aligned}$$

Thus $(x+3)^3 > y^3 > (x+1)^3$ or $x+1 < y < x+3$ which leaves $y=x+2$ as only integer possibility. Thus we have

$$\begin{aligned} x^3+8x^2-6x+8 = (x+2)^3 &= x^3+6x^2+12x+8 \\ 2x^2-18x &= 2x(x-9) = 0 \end{aligned}$$

The only possible pairs are $(x,y)=(0,2)$ and $(x,y)=(9,11)$.
As $9^3+8\cdot 9^2-6\cdot 9+8 = 729+648-54+8 = 1331 = 11^3$ and $0^3+8\cdot 0^2-6\cdot 0+8 = 8 = 2^3$ both pairs are indeed a solution.

11. (HS-3) Given two different, coprime, positive natural numbers $a,b\in\mathbb{N}$. Then there are two natural numbers $x,y\in\mathbb{N}$ such that $ax-by=1$.

Reason: Pigeonhole Principle.

Solution: We may assume $a,b>1$, since $(x,y)=(b+1,1),(1,a-1)$ solve the equation in case $a,b=1$ respectively, and we are done. Let $a>b$ and consider the multiples

$$m_1 = a\,,\, m_2 = 2a\,,\, m_3 = 3a\,,\,\ldots\,,\, m_{b-1} = (b-1)a$$

We write $j\cdot a = m_j = q_j\cdot b + r_j$ for $1\leq j<b$ with $0\leq r_j<b\,,\, q_j>0$.

If $r_j=0$ for some j, then $a\,|\,q_j$, i.e. $q_j = aq'_j$, since a and b are coprime.

Thus $ja = q_j b = aq'_j b$ and $q'_j b = j < b$ which implies $q'_j = 0$. But this is impossible as otherwise we would have $ja = aq'_j b = 0$, hence $a = 0$, a contradiction.

Assume $r_j > 1$ for all j. Then we have

$$r_1, r_2, \ldots, r_{b-1} \in \{\, 2, 3, \ldots, b-1 \,\}$$

and two remainders have to be equal, say $r_i = r_j$. This means

$$(i-j)a = m_i - m_j = (q_i - q_j) \cdot b + (r_i - r_j) = (q_i - q_j) \cdot b$$

and by the same argument as above, all factors of a must be in $(q_i - q_j)$, $q_i - q_j = aq'$, i.e. $i - j = q'b < b$ and $q' = 0$. Then we get that either $i = j$ or $a = 0$ which is a contradiction in both cases.

We have shown that at least one remainder equals one, say $r_j = 1$. Hence $j \cdot a - q_j \cdot b = r_j = 1$ which had to be proven.

12. (HS-4) How many moves do the towers of Hanoi require to solve by an optimal strategy?

    The towers of Hanoi are three places. At the beginning there is a tower of disks on the left, the places on the right and in the middle are empty. Each disk is a bit smaller than the one below it, such that it looks like a round pyramid. The task is to move the complete tower from left to right in its original order - biggest disk at the bottom, smallest on top - where one move is the replacement of one disk at the top of a tower to the middle, to the right or to the left.

    Reason: Algorithmic Induction.

    Solution: Let n be the number of disks, i.e. the height of the tower at the beginning. We prove by induction that the solution is $2^n - 1$ moves.

    The statement is obviously true for $n = 1$. Now let us assume that the optimal strategy for n disks require $2^n - 1$ moves. What happens, if we add another biggest disk at the bottom? Since we may only move the top most disk, all towers during the game a sorted by radius, possibly upside down. The tower without the new biggest disk has to be moved twice: once to solve the problem for n disks onto the place in the middle, then to move it again to the left. The additional disk requires one additional move from left to right. Hence we get

    $$(2^n - 1) + (2^n - 1) + 1 = 2 \cdot 2^n - 2 + 1 = 2^{n+1} - 1$$

    moves total.

13. (HS-5) Among six people are always three who know each other or three who don't. Why?

    Reason: Pigeonhole Principle.

    Solution: We draw a graph of six persons and connect every knot with all others. Then we color the lines blue, if the two people representing the vertices know each other, and red if they don't. We have five edges at each vertex, so we may assume that three of them are blue at the first edge. Say we have the blue edges $\overline{AB}, \overline{AC}, \overline{AD}$. If at least one other vertex $\overline{BC}, \overline{BD}, \overline{CD}$ is blue, then we have found a blue triangle and we are done. On the other hand, if all those vertices are red, then $\triangle(BCD)$ is red and we are done again.

21 April 2020

1. Let $U \subseteq X$ be a dense subset of a normed vector space, Y a Banach space and $A \in L(U,Y)$ a linear, bounded operator. Show that there is a unique continuation $\tilde{A} \in L(X,Y)$ with $\left.\tilde{A}\right|_U = A$ and $\|\tilde{A}\| = \|A\|$.

   Reason: Operator Property.

   Solution: For an $x \in X$ we choose a sequence $(y_n)_{n\in\mathbb{N}} \subseteq U$ with $y_n \to x$. Now

   $$\|Ay_n - Ay_m\| \leq \|A\| \cdot \|y_n - y_m\| \leq \|A\|\|y_n - x\| + \|A\|\|y_m - x\| \stackrel{n,m\to\infty}{\longrightarrow} 0$$

   so $(Ay_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, which has a limit $\tilde{A}x := \lim_{n\to\infty} Ay_n$ as Y is complete. $\tilde{A}$ is linear and bounded

   $$\|\tilde{A}x\| = \|\lim_{n\to\infty} Ay_n\| \stackrel{A \text{ continuous}}{=} \lim_{n\to\infty} \|Ay_n\| \leq \|A\| \lim_{n\to\infty} \|y_n\| = \|A\|\|x\|$$

   hence continuous and $\|\tilde{A}\| \leq \|A\|$.

   Now let $\bar{A}$ be a second solution with the required properties. We choose again a sequence $(u_n)_{n\in\mathbb{N}} \subseteq U$ which converges to a given point $x \in X$. As $\bar{A}$ has to be continuous, we get

   $$\bar{A}x = \bar{A}(\lim_{n\to\infty} u_n) = \lim_{n\to\infty} \left.\bar{A}\right|_U u_n = \lim_{n\to\infty} Au_n = \tilde{A}x$$

   Finally we have

   $$\|A\| = \sup_{x\in U, \|x\|\leq 1} \|Ax\| = \sup_{x\in U, \|x\|\leq 1} \|\tilde{A}x\| \leq \sup_{x\in X, \|x\|\leq 1} \|\tilde{A}x\| = \|\tilde{A}\|$$

   where the inequality arises from the fact that we build the supremum over a larger set.

2. Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim \mathcal{N}(\lambda, \sigma^2)$ be normally distributed random variables on $\mathbb{R}$ with expectation values $\mu, \lambda \in \mathbb{R}$ and standard deviation σ. We want to test the hypothesis that $\mu = \lambda$ with n independent measurements $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$. We choose the mean distance

   $$T_n(X_1, Y_1; X_2, Y_2; \ldots; X_n, Y_n) := \frac{1}{n} \sum_{k=1}^{n} (X_k - Y_k)$$

   as estimator for the difference $\nu = \mu - \lambda$.

   (a) Does the estimator T_n have a bias and is it consistent?

(b) Let $n = 100$ and $\sigma^2 = 0.5$. We use the hypotheses $H_0 : \mu = \lambda$ and $H_1 : \mu \neq \lambda$. Determine a reasonable deterministic test φ for the error level $\alpha = 0.05$.

Reason: Normal Distribution.

Solution:

(a) Since the expectation value is linear we have

$$\begin{aligned} E(T_n) &= E\left(\frac{1}{n}\sum_{k=1}^{n}(X_k - Y_k)\right) \\ &= \frac{1}{n}\sum_{k=1}^{n}(E(X_k) - E(Y_k)) \\ &= \frac{1}{n}\sum_{k=1}^{n}(\mu - \lambda) \\ &= \mu - \lambda \end{aligned}$$

i.e. T_n is unbiased. By the strong law of large numbers

$$\frac{1}{n}\sum_{k=1}^{n}(X_k - Y_k) \xrightarrow{a.s.} E(X - Y) = \mu - \lambda$$

and T_n is consistent.

(b) Let $z := T_n(X_1, Y_1; X_2, Y_2; \ldots ; X_n, Y_n) \in \mathbb{R}$ be the result of the measurements. z is again distributed normally as the intersection of two normally distributed random variables. The expectation value is $\mu - \lambda$ and the variance

$$\begin{aligned} V(z) &= \frac{1}{n^2}\, V\left(\sum_{k=1}^{n} V(X_k - Y_k)\right) \\ &= \frac{1}{n^2}\sum_{k=1}^{n} V(X_k - Y_k) \\ &= \frac{1}{n}\, V(X_k - Y_k) \\ &= \frac{1}{n} \cdot (V(X_k) + V(Y_k)) \\ &= \frac{1}{100} \cdot \left(\frac{1}{2} + \frac{1}{2}\right) \\ &= 0.01 \end{aligned}$$

Under the null hypothesis we have $H_0 : \mu = \lambda$ and thus $z \sim \mathcal{N}(0, 0.01)$. A reasonable deterministic test rejects H_0 as soon as the measured value z differs too much from the expectation value 0. Hence we define as test function

$$\varphi : \mathbb{R} \longrightarrow \{0, 1\}$$
$$\varphi(z) := \begin{cases} 0 & , \text{ if } |z| < \varepsilon \\ 1 & , \text{ otherwise} \end{cases}$$

We must choose the critical level ε such that the first order error is at most α, i.e. $P(|z| > \varepsilon \,|\, H_0) \leq 0.05$. From $z \sim \mathcal{N}(0, 0.01)$ follows $10z \sim \mathcal{N}(0, 1)$ and thus

$$\begin{aligned} P(|z| > \varepsilon \,|\, H_0) \leq 0.05 &\Longleftrightarrow P(|z| \leq \varepsilon \,|\, H_0) \geq 0.95 \\ &\Longleftrightarrow P(-\varepsilon \leq z \leq \varepsilon \,|\, H_0) \geq 0.95 \\ &\Longleftrightarrow P(-10\varepsilon \leq 10z \leq 10\varepsilon \,|\, H_0) \geq 0.95 \\ &\Longleftrightarrow F_{0,1}(10\varepsilon) - F_{0,1}(-10\varepsilon) \geq 0.95 \\ &\Longleftrightarrow 2F_{0,1}(10\varepsilon) - 1 \geq 0.95 \\ &\Longleftrightarrow F_{0,1}(10\varepsilon) \geq 0.975 \end{aligned}$$

We find by looking up the table for standard normal distributions $\mathcal{N}(0, 1)$ that $0.975 = F_{0,1}(1.96)$, i.e. $10\varepsilon = 1.96$ and our error level is $\varepsilon = 0.196$.

If our measurement shows an average difference greater than 0.196, then we falsely reject the null hypothesis by at most a 5% chance.

3. (a) Solve $y''x^2 - 12y = 0$, $y(0) = 0$, $y(1) = 16$ and calculate

$$\sum_{n=1}^{\infty} \frac{1}{t_n},\; t_n := y(n) - \frac{1}{2}y'(n) + \frac{1}{8}y''(n) - \frac{1}{48}y'''(n) + \frac{1}{384}y^{(4)}(n).$$

(b) What do we get for the initial values $y(1) = 1$, $y(-1) = -1$ and $\sum_{n=1}^{\infty} (y'(n) + y'''(n))$?

Reason: Riemann Zeta-Function.

Solution:

(a) We have a Cauchy-Euler equation here, so we use the transformation theorem and set $y(x) = u(\log|x|)$.

$$y''(x)x^2 = \left(u' \cdot \frac{1}{x}\right)' x^2 = \left(u''\frac{1}{x^2} - u'\frac{1}{x^2}\right) x^2 = u'' - u' = 12y(x) = 12u$$

with the characteristic polynomial $\chi(\lambda) = \lambda^2 - \lambda - 12$ that has zeros $\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 12} \in \{-3, 4\}$. The fundamental system is thus $\{e^{-3u}, e^{4u}\}$ for the transformed version, and $\{x^{-3}, x^4\}$ for the original equation. Hence we have

$$y(x) = \alpha x^{-3} + \beta x^4 = 16x^4$$

if we apply $y(0) = 0$ and $y(1) = 16$. So

$$\begin{aligned} t_n &= y(n) - \frac{1}{2}y'(n) + \frac{1}{8}y''(n) - \frac{1}{48}y'''(n) + \frac{1}{384}y^{(4)}(n) \\ &= 16n^4 - 32n^3 + 24n^2 - 8n + 1 \\ &\Longrightarrow \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{t_n} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4}\right) \\ &= \frac{1}{2}\left(\zeta(4) + (1 - 2^{1-4}) \cdot \zeta(4)\right) = \frac{15}{16} \cdot \frac{\pi^4}{90} = \frac{\pi^4}{96} \approx 1.014678 \end{aligned}$$

(b) If we have the initial values $y(1) = 1$, $y(-1) = -1$ then $y(x) = \alpha x^{-3} + \beta x^4$ implies $\alpha = 1$, $\beta = 0$ and we get with $y(x) = x^{-3}$

$$\begin{aligned} \sum_{n=1}^{\infty} (y'(n) + y'''(n)) &= \sum_{n=1}^{\infty} \left(-3x^{-4} - 60x^{-6}\right) \\ &= -3 \cdot \zeta(4) - 60 \cdot \zeta(6) \\ &= -\frac{3\pi^4}{90} - \frac{60\pi^6}{945} \\ &= -\frac{\pi^4}{30} - \frac{4\pi^6}{63} \approx -64.2875534202 \end{aligned}$$

4. Solve the initial value problem $y'(x) = y(x)^2 - (2x+1)y(x) + 1 + x + x^2$ for $y(0) \in \{0, 1, 2\}$.

Reason: Differential Equation.

Solution: We are only interested in solutions, where $y(x)$ is finite and

$y'(x)$ exists. We may thus use the template $y(x) = x + \dfrac{1}{u(x)}$ and get

$$\begin{aligned} y' &= 1 - \frac{u'}{u^2} \\ &= \left(x + \frac{1}{u}\right)^2 - (2x+1)\left(x + \frac{1}{u}\right) + 1 + x + x^2 \\ &= x^2 + \frac{2x}{u} + \frac{1}{u^2} - 2x^2 - \frac{2x}{u} - x - \frac{1}{u} + 1 + x + x^2 \\ &= \frac{1}{u^2} - \frac{u}{u^2} + 1 \\ &\Leftrightarrow \\ u' &= u - 1 \\ &\Leftrightarrow \\ \frac{u'}{u-1} &= 1 \end{aligned}$$

Integrating both sides gets $\log|u-1| = x + C$ or $u(x) = 1 + e^x \cdot e^C$. Backward substitution gives us

$$y(x) = x + \frac{1}{1 + De^x}$$

Let us consider $y(0) = 0$, i.e. $y(x) = x$. It is obvious that this is a solution, too, $C = D = \infty$. For the case $y(0) = 1$ we get $y(x) = x + 1$ with $D = 0$ (or $C = -\infty$), and for $y(0) = 2$ we have $y(x) = x + \dfrac{1}{1 - \frac{1}{2}e^x}$.

5. For coprime natural numbers n, m show that

$$m^{\varphi(n)} + n^{\varphi(m)} \equiv 1 \bmod nm$$

Reason: Euler's Theorem.

Solution: Since $\gcd(n, m) = 1$ we get by Euler's theorem $m^{\varphi(n)} \equiv 1 \bmod n$ and $n^{\varphi(m)} \equiv 1 \bmod m$. Trivially true are $n^{\varphi(m)} \equiv 0 \bmod n$ and $m^{\varphi(n)} \equiv 0 \bmod m$. Thus

$$m^{\varphi(n)} + n^{\varphi(m)} \equiv 1 \bmod n \text{ and } m^{\varphi(n)} + n^{\varphi(m)} \equiv 1 \bmod m$$

Since n, m are coprime, the congruences still hold for $\operatorname{lcm}(n, m) = nm$.

6. (HS-1) Let $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ a monic, real polynomial of degree n, whose zeros are all negative.

Show that $\displaystyle\int_1^\infty \frac{dx}{P(x)}$ converges absolutely if and only if $n \geq 2$.

Reason: Integrals.

Solution: Let $n \geq 2$. Now

$$\frac{P(x)}{x^n} = 1 + a_{n-1}\frac{1}{x} + \ldots + a_1\frac{1}{x^{n-1}} + a_0\frac{1}{x^n} \longrightarrow 1$$

converges for $x \to \infty$. Multiplication by $\sqrt{x}$ yields

$$\frac{P(x)}{x^{n-\frac{1}{2}}} = \sqrt{x}\left(1 + a_{n-1}\frac{1}{x} + \ldots + a_1\frac{1}{x^{n-1}} + a_0\frac{1}{x^n}\right) \longrightarrow \infty$$

Thus there is a $x_0 > 1$, such that the numerator exceeds the denominator $P(x) > x^{n-\frac{1}{2}} \geq x_0^{n-\frac{1}{2}} > 1$ for all $x \geq x_0$ and we have

$$\begin{aligned}\int_1^\infty \left|\frac{1}{P(x)}\right| dx &= \underbrace{\int_1^{x_0} \frac{dx}{|P(x)|}}_{=:C} + \int_{x_0}^\infty \frac{dx}{|P(x)|} \\ &= C + \int_{x_0}^\infty \frac{dx}{P(x)} \\ &\leq C + \int_{x_0}^\infty x^{\frac{1}{2}-n}\,dx \\ &< \infty\end{aligned}$$

For $n = 0$ we have $P(x) = 1$ and

$$\int_1^\infty \frac{dx}{P(x)} = \int_1^\infty dx = \lim_{\zeta\to\infty}(\zeta - 1) = \infty$$

and for $n = 1$ with $P(x) = x + a_0$

$$\int_1^\infty \frac{dx}{P(x)} = \int_1^\infty \frac{1}{x + a_0}\,dx = \lim_{\zeta\to\infty}(\log|\zeta + a_0| - \log|1 + a_0|) = \infty$$

22 March 2020 - Part II

1. Let $\sum_{k=1}^{\infty} a_k$ be a given convergent series with $|a_{k+1}| \leq |a_k|$ for all k. Assume we use a computer to sum its value until the partial sum is closer than ε to the actual value of the series. Does it make sense to use $|a_n| < \varepsilon$ as a stopping criterion for the loop? Please justify your answer.

   Reason: Understanding.

   Solution: No. The smallness of the summands - even with a monotone decreasing absolute value - says nothing about the size of the remainder part, i.e. the error of the current partial sum.

   Let's consider

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} q^k \text{ and } R_n := \sum_{k=n+1}^{\infty} q^k$$

   For different values of q we list n for which $a : n = q^n < \varepsilon := 0.001$

q	0.9	0.99	0.999	0.9999
n	66	688	6905	69075
R_n	0.0086	0.0993	0.998	9.998

   This means for our remainder

$$R_n = \sum_{k=n+1}^{\infty} q^k = \sum_{k=0}^{\infty} q^{k+n+1} = q^{n+1} \sum_{k=0}^{\infty} q^k \approx \frac{0.001}{1-q} \stackrel{q \to 1-0}{\longrightarrow} +\infty$$

   Hence we can make R_n arbitrary large, although $|a_k| < 0.001 \; (k > n)$, if we choose $q < 1$ close enough to 1.

2. "Every absolutely convergent series converges." Now why is its proof so complicated, couldn't we just say: Given an absolute convergent series $\sum_{k=1}^{\infty} |a_k|$ then we have for the sequence $R_n := \sum_{k=n+1}^{\infty} a_k$

$$|R_n| = \left| \sum_{k=n+1}^{\infty} a_k \right| \leq \sum_{k+1}^{\infty} |a_k|$$

   with the remainder of a convergent series on the right, hence a null sequence. Thus R_n is a null sequence, too, and the series is convergent.

   Reason: Proof Theory.

Solution: In order to be able to estimate R_n in the described manner, we must first be sure that it exists at all, i.e. that this series converges. But that is exactly what should be shown! Thus we just noticed that if the series converges, it converges.

3. Calculate the limit (i being the imaginary unit):

$$\lim_{n\to\infty} \operatorname{Arg}\left(\sum_{k=0}^{n}\frac{1}{k+i}\right)$$

Reason: Calculus.

Solution:

$$\sum_{k=0}^{n}\frac{1}{k+i}=\sum_{k=0}^{n}\frac{k-i}{k^2+1}=\sum_{k=0}^{n}\frac{k}{k^2+1}+i\cdot\sum_{k=0}^{n}\frac{1}{k^2+1}$$

Both real and even positive sums can be estimated by known series. The real part is unbounded, because

$$\sum_{k=0}^{n}\frac{k}{k^2+1}>\sum_{k=1}^{n-1}\frac{1}{k}$$

whereas the imaginary part is bounded by

$$\sum_{k=0}^{n}\frac{1}{k^2+1}<1+\sum_{k=1}^{n}\frac{1}{k^2}<1+\frac{\pi^2}{6}$$

Since the inverse tangent is continuous, we get

$$\begin{aligned}\lim_{n\to\infty} \operatorname{Arg}\left(\sum_{k=0}^{n}\frac{1}{k+i}\right)&=\lim_{n\to\infty}\arctan\frac{\sum_{k=0}^{n}\frac{1}{k^2+1}}{\sum_{k=0}^{n}\frac{k}{k^2+1}}\\&=\arctan\lim_{n\to\infty}\frac{\sum_{k=0}^{n}\frac{1}{k^2+1}}{\sum_{k=0}^{n}\frac{k}{k^2+1}}\\&=\arctan 0\\&=0\end{aligned}$$

4. Show that there is no odd dimensional real division algebra D.

   Reason: Basic Algebra.

   Solution: Assume $\dim D = n$ is odd. Consider the left multiplication $L_a : D \longrightarrow D$, $x \longmapsto ax$ with an element $a \in D - \mathbb{R}$. It's characteristic polynomial $\chi(L_a)(x) = \det(xI_n - L_a)$ in some basis is a monic, real polynomial of odd degree and thus has a real zero z, i.e. there is an element $b \in D - \{0\}$ such that $0 = (zI_n - L_a)(b) = zb - ab = (z - a)b$. Since z is real and a is not, we have $z - a \neq 0$ and $b \neq 0$ cannot be a unit.

   Alternative Solution: A division algebra structure on R^n makes $\mathbb{R}^n \setminus \{0\}$ into a topological group, and induces a group structure on S^{n-1} via the map $x \mapsto x/|x|$. But S^m can only have a group structure for odd m by degree theory, so n must be even.

5. Let $R := \mathbb{Z}_{(5)} = \left\{ \dfrac{a}{b} \;\middle|\; 5 \nmid b \right\}$ the ring of rational numbers which don't have a factor 5 in their denominator, $M \neq \{0\}$ a finitely generated $R-$module, and $I := \left\{ \dfrac{a}{b} \in R \;\middle|\; 25 \mid a \right\}$.

   Prove that I is an ideal contained in the Jacobson radical of R and that $IM \neq M$. The Jacobson radical $J = J(R)$ is defined as the intersection of all maximal ideals.

   Reason: Nakayama's Lemma.

   Solution: We first show that $x \in J$ if and only if $1 - xy$ is a unit in R for any $y \in R$.

   If $x \in J$ and $1 - xy$ is no unit, then it belongs to some maximal ideal $K \subsetneq R$ (att.: this result uses the axiom of choice). Now $1 = k + xy \in K + JR = K + J \subseteq K$ which is absurd.
   If $x \notin J$ then $x \notin K$ for some maximal ideal $K \subsetneq R$. Hence $xR + K = R$ by the maximality of K and we can write $1 = xy + k$ for some $y \in R, k \in K$. Now $1 - xy \in K$ and thus cannot be a unit.

   R is a commutative, local ring with maximal ideal $J = \left\{ \dfrac{a}{b} \in R \;\middle|\; 5 \mid a \right\}$ which is also its Jacobson radical. It is obvious that $I \subsetneq J$ is a proper ideal. Suppose $IM = M$ and $\{ u_1, \ldots, u_n \}$ is a minimal set of generators of M. Since $M \neq \{0\}$ we have $n \geq 1$ and $u_n \neq 0$. Since $u_n \in IM$ we have an expression

$$u_n = \sum_{k=1}^{n} a_k u_k \Longrightarrow (1 - a_n) u_n = \sum_{k=1}^{n-1} a_k u_k$$

for some $a_k \in I \subseteq J$. From $a_n \in J$ we conclude that $1 - 1 \cdot a_n$ is a unit, i.e. u_n can be expressed by the other $u-k$ contradicting the minimality if the chosen system of generators.

6. Calculate
$$\lim_{n\to\infty} \frac{\sqrt{n\pi}}{2^{2n}} \cdot \binom{2n}{n}$$
   (a) without using Stirling's formula.
   (b) by using Stirling's formula, with accurate remainder terms, i.e. not simply $\sim$.

Reason: Wallis Product.

Solution:

(a) Wallis Product.
$$\prod_{k=1}^{n} \frac{4k^2-1}{4k^2} = \prod_{k=1}^{n} \frac{(2k-1)\cdot(2k+1)}{2k\cdot 2k} = \frac{\frac{(2n)!}{2^n\, n!}\cdot\frac{(2n)!}{2^n\, n!}\cdot(2n+1)}{2^n\, n!\cdot 2^n\, n!}$$
$$= (2n+1)\left[\frac{(2n)!}{2^{2n}\, n!^2}\right]^2 = (2n+1)\left[\frac{1}{2^{2n}}\binom{2n}{n}\right]^2$$
so $\lim_{n\to\infty}(2n+1)\left[\frac{1}{2^{2n}}\binom{2n}{n}\right]^2 = \prod_{k=1}^{\infty}\frac{4k^2-1}{4k^2} = \frac{2}{\pi}$ by Wallis product. Hence $\lim\limits_{n\to\infty}\frac{\sqrt{2n+1}}{2^{2n}}\binom{2n}{n} = \sqrt{\frac{2}{\pi}}$ or $\lim\limits_{n\to\infty}\frac{\sqrt{\left(n+\frac{1}{2}\right)\pi}}{2^{2n}}\binom{2n}{n} = 1$.

Because of $\lim\limits_{n\to\infty}\frac{\sqrt{n\,\pi}}{\sqrt{\left(n+\frac{1}{2}\right)\pi}} = 1$ we also have
$$\lim_{n\to\infty}\frac{\sqrt{n\,\pi}}{2^{2n}}\cdot\binom{2n}{n} = 1$$

(b) Stirling's Formula.

We use Robbins estimation for Stirling's formula:

$$\begin{aligned}
&\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}e^{\frac{1}{12n}}\\
&\Longrightarrow \sqrt[12n+1]{e} < \frac{n!}{\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}} < \sqrt[12n]{e}\\
&\Longrightarrow \frac{\sqrt[24n+1]{e}}{\sqrt[6n]{e}} < \frac{(2n)!}{\sqrt{2\pi}(2n)^{2n+\frac{1}{2}}e^{-2n}} \cdot \frac{2\pi\, n^{2n+1}e^{-2n}}{(n!)^2} < \frac{\sqrt[24n]{e}}{\sqrt[12n+1]{e^2}}\\
&\Longrightarrow e^{-\frac{18n+1}{144n^2+6n}} < \sqrt{2\pi}\cdot\sqrt{n}\cdot 2^{-2n}\cdot 2^{-\frac{1}{2}}\cdot\binom{2n}{n} < e^{-\frac{36n-1}{288n^2+24n}}\\
&\Longrightarrow 1 \leq \lim_{n\to\infty}\frac{\sqrt{\pi n}}{2^{2n}}\cdot\binom{2n}{n} \leq 1
\end{aligned}$$

7. (HS-1) Prove that if for $x \in \mathbb{R} - \{0\}$ the number $x + \dfrac{1}{x}$ is an integer, then $x^n + \dfrac{1}{x^n}$ with $n \in \mathbb{N}$ are integers, too.

   Reason: Induction.

   Solution: We have the statement for $n = 1$ by assumption. For $n = 2$ and $m := n + \frac{1}{n} \in \mathbb{Z}$ we get

$$\left(x^2 + \frac{1}{x^2}\right) = \left(x + \frac{1}{x}\right)^2 - 2 = m^2 - 2 \in \mathbb{Z}$$

   We may assume that the statement is true for every number up to n. Hence

$$\left(x^{n+1} + \frac{1}{x^{n+1}}\right) = \left(x^n + \frac{1}{x^n}\right)\cdot\left(x + \frac{1}{x}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right) \in \mathbb{Z}$$

   by induction.

8. (HS-2) We define for positive integers a, b the following sequence

$$x_n := \begin{cases} 1 & \text{if } n = 1 \\ ax_{n-1} + b & \text{if } n > 1 \end{cases}$$

   Show that the sequence contains infinitely many numbers. which are not prime, for any choice of a, b.

   Reason: Numbers.

   Solution: The sequence $(x_n)_{n\in\mathbb{N}}$ is strictly monotone increasing, since

$a,b \geq 1$. Hence it is sufficient to show, that there is a natural number $d > 1$ with the following property: There are infinitely many indices n such that $d \,|\, x_n$.

If $(a,b) > 1$ then we are done, so we may assume $(a,b) = 1$. Let $d = x_2 = ax_1 + b = a + b > 1$ and be $x_n = q_n \cdot d + r_n$, $0 \leq r_n < d$. Then we have at least two equal remainders $r_\alpha = r_{\alpha+\beta} \in \{ r_2, r_3, \ldots, r_{d+2} \}$. Now any divisor of a and $d = a + b$ would also be a divisor of a and b, which are coprime. So a and d are coprime, too. Thus

$$\begin{aligned} d \,|\, x_{\alpha+\beta} - x_\alpha = a \cdot (x_{\alpha+\beta-1} - x_{\alpha-1}) &\Longrightarrow d \,|\, x_{\alpha+\beta-1} - x_{\alpha-1} \\ &\Longrightarrow d \,|\, (q_{\alpha+\beta-1} - q_{\alpha-1})d + (r_{\alpha+\beta-1} - r_{\alpha-1}) \\ &\Longrightarrow d \,|\, r_{\alpha+\beta-1} - r_{\alpha-1} \\ &\Longrightarrow r_{\alpha+\beta-1} = r_{\alpha-1} \\ &\vdots \\ &\Longrightarrow r_2 = r_{2+\beta} \end{aligned}$$

$$\begin{aligned} d \,|\, a \cdot (x_{\alpha+\beta} - x_\alpha) = x_{\alpha+\beta+1} - x_{\alpha+1} \\ &\Longrightarrow d \,|\, r_{\alpha+\beta+1} - r_{\alpha+1} \\ &\Longrightarrow r_{\alpha+\beta+1} = r_{\alpha+1} \\ &\vdots \\ &\Longrightarrow r_{\alpha+2\beta} = r_{\alpha+\beta} = r_\alpha \end{aligned}$$

Thus we have $r_2 = r_{2+k\cdot\beta}$ for all $k \in \mathbb{N}$. But $r_2 = 0$ per construction, so for all indices $2 + k\beta$ we have that $d \,|\, x_{2+k\beta}$.

9. (HS-3) Name a convergent series $\sum_{k=1}^{\infty} a_k$ with positive a_k, where $a_{k+1}/a_k \geq 2$ holds infinitely often.

   Reason: Attention with Intuition.

   Solution: We define

$$a_k := \begin{cases} 2^{-k} & \text{if } k \text{ is even} \\ 2^{-k+2} & \text{if } k \text{ is odd} \end{cases}$$

   Now $a_{2k+1}/a_{2k} = 2^{-2k-1+2}\, 2^{-(-2k)} = 2$. The series with the first terms written out is $\sum_{k=1}^{\infty} a_k = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \frac{1}{8} + \frac{1}{64} + \frac{1}{32} + \frac{1}{256} + \frac{1}{128} + \ldots$ which converges to $2 + 1 = 3$.

10. (HS-4) For natural numbers $1 \leq k \leq 2n$ show that

$$\binom{2n+1}{k-1} + \binom{2n+1}{k+1} \geq 2 \cdot \frac{n+1}{n+2} \cdot \binom{2n+1}{k}$$

Reason: Polynomials.

Solution:

$$\binom{2n+1}{k+1} = \frac{(2n+1)!}{(k+1)!(2n-k)!} = \frac{2n+1-k}{k+1} \cdot \frac{(2n+1)!}{k!(2n+1-k)!}$$
$$= \frac{2n+1-k}{k+1} \cdot \binom{2n+1}{k}$$

$$\binom{2n+1}{k} = \frac{(2n+1)!}{k!(2n+1-k)!} = \frac{2n+2-k}{k} \cdot \frac{(2n+1)!}{(k-1)!(2n+2-k)!}$$
$$= \frac{2n+2-k}{k} \cdot \binom{2n+1}{k-1}$$

$$\binom{2n+1}{k-1} + \binom{2n+1}{k+1} = \frac{k}{2n+2-k} \cdot \binom{2n+1}{k} + \frac{2n+1-k}{k+1} \cdot \binom{2n+1}{k}$$

Hence we have to show that

$$\frac{k}{2n+2-k} + \frac{2n+1-k}{k+1} \geq 2 \cdot \frac{n+1}{n+2}$$

or with $m := n - k + \frac{1}{2}$ and $-(n-\frac{1}{2}) \leq m \leq n - \frac{1}{2}$

$$f(n,m) := \frac{2n-2m+1}{2n+3+2m} + \frac{2n+1+2m}{2n-2m+3} - 2 \cdot \frac{n+1}{n+2} \stackrel{(!)}{\geq} 0$$

$$\begin{aligned} f(n,m) &= 2 \cdot \frac{4n^2+4m^2+8n+3}{(2n+3)^2-4m^2} - 2 \cdot \frac{n+1}{n+2} \\ &= 2 \cdot \frac{8m^2n - 2n + 12m^2 - 3}{((2n+3)^2-4m^2) \cdot (n+2)} \\ &= 2 \cdot \frac{(4m^2-1)(2n+3)}{(2n+3-2m)(2n+3+2m)(n+2)} \\ &= 2 \cdot \frac{(2m+1)(2m-1)(2n+3)}{(2n+3-2m)(2n+3+2m)(n+2)} \end{aligned}$$

Since we have $-2n+1 \leq \pm 2m \leq 2n-1$ our denominator is positive. But $m \neq 0$ as k,n are integers, so $4m^2 - 1 > 2$ and the numerator is also positive, which had to be shown.

11. (HS-5) The year on the Earth-like planet Trappist-1e has 365 days divided into months of $28, 30, 31$ days. How many months does its year

have and how many months with (i) 28, (ii) 30, (iii) 31 days?

Reason: Pigeonhole Principle.

Solution: We prove the following two statements:

(a) There are 12 months on Trappist-1e.

(b) The following combinations are possible:
$(28, 30, 31) \in \{ (0, 7, 5), (1, 4, 7), (2, 1, 9) \}$.

We have to solve the equation $28a + 30b + 31c = 365$ with $a, b, c \in \mathbb{N}_0$.
$c \geq 1$, since 365 is odd.

(a) With $c' := c - 1$ we have

$$\begin{aligned} 28a + 30b + 31c' = 334 & \\ & \Longrightarrow 28(a + b + c') \leq 334 \leq 31(a + b + c') \\ & \Longrightarrow 10 < \frac{334}{31} \leq a + b + c' \leq \frac{334}{28} < 12 \\ & \Longrightarrow a + b + c' = 11 \\ & \Longrightarrow a + b + c = 12 \end{aligned}$$

since $a, b, c, c' \in \mathbb{N}_0$.

(b) The sum of days can also be written as

$$\begin{aligned} 365 &= (30 - 2)a + 30b + (30 + 1)c = 30(a + b + c) - 2a + c = 360 - 2a + c \\ &\Longrightarrow 5 = c - 2a \text{ with } 0 \leq a, 1 \leq c \text{ and } a + c \leq 12 \\ &\Longrightarrow a + c = a + (5 + 2a) = 3a + 5 \leq 12 \\ &\Longrightarrow (a, c) \in \{ (0, 5), (1, 7), (2, 9) \} \\ &\Longrightarrow (a, b, c) \in \{ (0, 7, 5), \underbrace{(1, 4, 7)}_{\text{Earth}}, (2, 1, 9) \} \end{aligned}$$

12. (HS-6) Decrypt the affine encrypted "Ara gtynd hdm hvcrsnd mthvtjph!"

Reason: Puzzle.

Solution: Non vitae sed scholae discimus! with $x \to 17(x - 1) + 14$.

23 March 2020

1. Let $\mathfrak{g} = \operatorname{lin}_{\mathbb{R}}\{ e_1, e_2, e_3, e_4 \}$ on which we define the following multiplication:

$$[e_1, e_4] = 2e_1 \ , \ [e_2, e_4] = 3e_2 - e_3 \ , \ [e_3, e_4] = e_2 + 3e_3$$

and $[e_i, e_j] = 0$ otherwise, as well as $[e_i, e_i] = 0$.

Show that

(a) $\mathfrak{g}$ is a Lie algebra.

(b) There exists an $\alpha_0 \in A(\mathfrak{g})$ where

$$A(\mathfrak{g}) := \{ \alpha : \mathfrak{g} \stackrel{\text{linear}}{\longrightarrow} \mathfrak{g} \,|\, \forall\, X, Y \in \mathfrak{g} : [\alpha(X), Y] + [X, \alpha(Y)] = 0 \}$$

such that $[\operatorname{ad} X, \alpha_0] \in \mathbb{R} \cdot \alpha_0$ for all $X \in \mathfrak{g}$.

(c) The center $Z(\mathfrak{g}) = \{ 0 \}$.

(d) $\mathfrak{g}$ has a one dimensional ideal.

Reason: Solvable Lie Algebra.

Solution:

(a) $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \subseteq \operatorname{lin}_{\mathbb{R}}\{ e_1, e_2, e_3 \}$ which all commute. Thus we only have to consider products $[e_4, [e_a, e_b]]$ with $a, b \leq 3$.

$$[e_4, [e_a, e_b]] + [e_a, [e_b, e_4]] + [e_b, [e_4, e_a]] \in [e_4, 0] + [e_a, \mathfrak{g}'] + [e_b, \mathfrak{g}'] = 0$$

(b) We have seen that $[[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}', \mathfrak{g}'] = 0$, i.e. $\mathfrak{g}$ is solvable. The setting $X.\alpha := [\operatorname{ad} X, \alpha] = \operatorname{ad} X \circ \alpha - \alpha \circ \operatorname{ad} X$ makes $A(\mathfrak{g})$ into a $\mathfrak{g}-$module. By Lie's theorem there is an invariant one dimensional submodule, spanned by α_0.

Explicitly we can choose $\alpha_0(x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4) := x_4e_1$ which is easy to verify.

(c) Let $Z = z_1e_1 + z_2e_2 + z_3e_3 + z_4e_4 \in Z(\mathfrak{g})$ be a central element. Then

$$\begin{aligned}[Z, e_4] = 0 &= 2z_1e_1 + z_2(3e_2 - e_3) + z_3(e_2 + 3e_3) \\ &\Longrightarrow z_1 = 0 \, , \, 3z_2 + z_3 = 0 \, , \, -z_2 + 3z_3 = 0 \\ &\Longrightarrow Z = z_4e_e\end{aligned}$$

which cannot be central unless $Z = 0$ since $0 = [e_1, Z] = 2z_4e_1$.

(d) It is clear from the multiplication table, that e_1 spans a one dimensional ideal.

2. Let $A, B \in \mathbb{M}(m, \mathbb{R})$ and $\|A\|, \|B\| \leq 1$, then

$$\left\| e^{A+B} - e^A \cdot e^B \right\| \leq 6e^2 \cdot \| [A,B] \|$$

Reason: Trotter's Estimation.

Solution: For a vector $\alpha \in \{0,1\}^n$ we define

$$S(\alpha) := \prod_{k=1}^{n} A^{1-\alpha_k} B^{\alpha_k}$$

Each of these vectors can be sorted in an ascending order in n^2 steps, e.g. with Bubble sort, until $S(\alpha_{Sort}) = A^{n-|\alpha|}B^{|\alpha|}$ where every commutation between A, B results in an additional term $[A,B] = AB - BA$:

$$S(BA) - S(AB) = BA - AB = -[A,B]$$

or with an example which needs more steps:

$$\begin{bmatrix} \alpha_0 = (1,0,1,0) \\ S(\alpha_0) = BABA \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha_1 = (0,1,1,0) \\ S(\alpha_1) = ABBA \end{bmatrix}$$
$$S(\alpha_0) - S(\alpha_1) = (BA - AB)BA = -[A,B]BA$$
$$\begin{bmatrix} \alpha_1 = (0,1,1,0) \\ S(\alpha_1) = ABBA \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha_2 = (0,1,0,1) \\ S(\alpha_2) = ABAB \end{bmatrix}$$
$$S(\alpha_1) - S(\alpha_2) = AB(BA - AB) = -AB[A,B]$$
$$\begin{bmatrix} \alpha_2 = (0,1,0,1) \\ S(\alpha_2) = ABAB \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha_3 = (0,0,1,1) \\ S(\alpha_3) = AABB \end{bmatrix}$$
$$S(\alpha_2) - S(\alpha_3) = A(BA - AB)B = -A[A,B]B$$

So every sorting step produces a factor $[A,B]$, which combined with the assumption $\|A\|, \|B\| \leq 1$ means $\|S(\alpha_k) - S(\alpha_{k+1})\| \leq \| [A,B] \|$ and

$$\|S(\alpha) - S(\alpha_{Sort})\| \leq \sum_{k=0}^{n^2-1} \|S(\alpha_k) - S(\alpha_{k+1})\| \leq n^2 \cdot \| [A,B] \|$$

$$
\begin{aligned}
e^{A+B} &= \sum_{n=0}^{\infty} \frac{1}{n!}(A+B)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha\in\{0,1\}^n} S(\alpha) \\
e^A \cdot e^B &= \left(\sum_{n=0}^{\infty} \frac{A^n}{n!}\right) \cdot \left(\sum_{n=0}^{\infty} \frac{B^n}{n!}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^{n-k}}{(n-k)!} \cdot \frac{B^k}{k!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} A^{n-k} B^k = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha\in\{0,1\}^n} A^{n-|\alpha|} B^{|\alpha|} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha\in\{0,1\}^n} S(\alpha_{Sort})
\end{aligned}
$$

With these equations we get

$$
\begin{aligned}
\| e^{A+B} - e^A \cdot e^B \| &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha\in\{0,1\}^n} \| S(\alpha) - S(\alpha_{Sort}) \| \\
&\leq \sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot n^2 \cdot \| [A,B] \| \\
&= 6e^2 \cdot \| [A,B] \|
\end{aligned}
$$

3. Show that for $m \times m$ matrices A, B

$$
e^{t\,(A+B)} = \lim_{n\to\infty} \left(e^{t\,\frac{A}{n}} \cdot e^{t\,\frac{B}{n}} \right)^n
$$

Reason: Trotters Formula.

Solution: We will use Trotter's estimation from the previous problem. Let $S := \exp\left(\frac{A}{n} + \frac{B}{n}\right)$ and $T := \exp\left(\frac{A}{n}\right) \cdot \exp\left(\frac{B}{n}\right)$. Then

$$
\begin{aligned}
\|S\| &\leq \exp\left(\frac{\|A\|}{n} + \frac{\|B\|}{n}\right) \\
\|T\| &\leq \exp\left(\frac{\|A\|}{n}\right) \cdot \exp\left(\frac{\|B\|}{n}\right) = \exp\left(\frac{\|A\|}{n} + \frac{\|B\|}{n}\right)
\end{aligned}
$$

$$
\begin{aligned}
S^n - T^n &= \sum_{k=0}^{n-1} S^{n-k}T^k - \sum_{k=1}^{n} S^{n-k}T^k \\
&= \sum_{k=0}^{n-1} S^{n-1-k} \cdot S \cdot T^k - \sum_{k=0}^{n-1} S^{n-k-1} \cdot T \cdot T^k \\
&= \sum_{k=0}^{n-1} S^{n-1-k} \cdot (S-T) \cdot T^k \\
\|S^n - T^n\| &\leq \sum_{k=0}^{n-1} \|S\|^{n-1-k} \cdot \|T\|^k \cdot \|S-T\| \\
&\leq n \cdot \exp\left(\frac{\|A\|}{n} + \frac{\|B\|}{n}\right)^{n-1} \cdot \|S-T\| \\
&\leq n \cdot e^{\|A\|+\|B\|} \cdot \|S-T\|
\end{aligned}
$$

Now if we choose n large enough such that $\left\|\frac{A}{n}\right\|, \left\|\frac{B}{n}\right\| \leq 1$, we know that

$$
\begin{aligned}
\|S-T\| &= \left\|\exp\left(\frac{A}{n} + \frac{B}{n}\right) - \exp\left(\frac{A}{n}\right) \cdot \exp\left(\frac{B}{n}\right)\right\| \\
&\leq 6\,e^2 \cdot \left\| \left[\frac{A}{n}, \frac{B}{n}\right] \right\| = 6\,e^2\,\frac{1}{n^2}\,\|\,[A,B]\,\|
\end{aligned}
$$

$$
\|S^n - T^n\| \leq n \cdot e^{\|A\|+\|B\|} \cdot \|S-T\| \leq \frac{6e^2}{n} \cdot e^{\|A\|+\|B\|} \cdot \|\,[A,B]\,\|
$$

With $n \to \infty$ we get $S^n = \exp\left(\frac{A}{n} + \frac{B}{n}\right)^n = \exp(A+B) \longrightarrow T^n$ or

$$
\exp(tA + tB) = \lim_{n\to\infty} \left(\exp\left(\frac{tA}{n}\right) \cdot \exp\left(\frac{tB}{n}\right)\right)^n
$$

4. (HS-1) Given two integers n, m with $nm \neq 0$. Show that there is a integer expression $1 = sn + tm$ if and only if n and m are coprime, i.e. have no proper common divisor.

   Reason: Bezout's Lemma.

   Solution: Consider all numbers $\{\, x = sn + tm > 0 \,|\, s,t \in \mathbb{Z} \,\}$. If we choose the smallest one, say d, then $\mathrm{gcd(n,m)} \,|\, \mathrm{d}$.

   Division with remainder gives us $n = qd + r$ with $0 \leq r < d$. With the expression for d we get

$$n = q(sn + tm) + r \iff r = (1 - qs)n + (-qt)m$$

We had chosen d to be minimal under these conditions, so $r = 0$ is the only possibility. But then $n = qd+0$ and $d \mid n$. With the same argument we get $d \mid m$, so as d divides both given integers, $d \mid \mathrm{gcd(n, m)}$.

With $\mathrm{gcd(n, m) \mid d \mid gcd(n, m)}$ we have $d = \mathrm{gcd(n, m)}$ and an expression $d = sn + tm$. Finally, n and m are coprime if and only if $d = 1$.

5. (HS-2) Division of an integer by a prime number p leaves us with the possible remainders $C := \{ 0, 1, 2, \ldots, p-1 \}$. We can define an addition and a multiplication on C if we wrap it around p, i.e. we identify $0 = p = 2p = 3p = \ldots, 1 = 1 + p = 1 + 2p = 1 + 3p = \ldots, \ldots$ This is called modular arithmetic (modulo p).

   Show that for any given numbers $a, b \in C$ the equations $a + x = b$ and $a \cdot x = b$ have a unique solution. Is this still true if we drop the requirement that p is prime?

   Remark: This problem is about proof techniques, so be as accurate as possible.

   Reason: Proof Techniques.

   Solution: We will make use of the laws of associativity, commutativity and distributivity on C which are all easy to check.

   If we set $x := q \cdot p - a + b$ such that $x \in C$, then

$$a + x = b \Longrightarrow a + qp - a + b = qp + b \equiv b$$

   solves the equation for addition. This does not require p to be prime. If we have $a+x = b = a+y$ then $x = qp-a+b = qp-a+a+y = qp+y \equiv y$ up to multiples of p which we all identified.

   According to Bezout's Lemma, we can find integers s, t such that $1 = sa + tp$ because all numbers in C are coprime to p, if p is prime. Hence from $ax = b$ we get

$$s \cdot b = s \cdot (a \cdot x) = (sa)x = (1 - tp)x = x - (tp)x = x - (tx)p \equiv x$$

   Now let $q \in \mathbb{Z}$ be such that $s' = s + qp \in C$. Then

$$s'b = sb + qpb \equiv sb \equiv x \in C$$

If we have $a \cdot x = b = a \cdot y$ then

$$\begin{aligned} x &= s'b = s'(ay) = (s'a)y = (sa)y + ((qp)a)y \\ &= (1 - tp)y + (qay)p = y + (qay - ty)p \\ &\equiv y \end{aligned}$$

up to multiples of p which we all identified.

The existence of a solution for $ax = b$ does not work in general, if p isn't prime. Let $p = 4$, then there is no solution for $2x = 1$.

24 February 2020

1. The function $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $\varphi(t,r) = (t(r+2), t^2 - r)$ is injective on $U := (0,1) \times (-1,1)$. Show that $\varphi : U \longrightarrow V := \varphi(U)$ is a diffeomorphism.

   Next let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be integrable over V. Write

   $$\int_V f \, d\lambda = \int_{\ldots}^{\ldots} \int_{\ldots}^{\ldots} \ldots f(\ldots, \ldots) \, dr \, dt$$

   and calculate the area of V.

   Reason: Analysis.

   Solution: We get

   $$J_{(t,r)}\varphi = \begin{bmatrix} r+2 & t \\ 2t & -1 \end{bmatrix}, \ \det(D_{(t,r)}\varphi) = \det(J_{(t,r)}\varphi) = -r - 2 - 2t^2$$

   so for $(t,r) \in U$ the determinate is $\det(D_{(t,r)}\varphi) < -1 < 0$.

   The function $\varphi : U \longrightarrow V$ is obviously continuously differentiable and surjective:

   $$\begin{aligned} (t(r+2), t^2 - r) = (a,b) &\Longrightarrow t = \frac{a}{r+2}, \ b = t^2 - r \\ &\Longrightarrow 0 = b(r+2)^2 - a^2 + r(r+2)^2 =: p(r) \end{aligned}$$

   If $a = 0$ we can choose $t = 0$ and $r = -b$ hence we may assume $a \neq 0$. Since $p(r)$ is a polynomial of degree 3 it has at least one real root r_0 which gives us $t_0 = \dfrac{a}{r_0 + 2}$ in case $r_0 \neq -2$. The case $r_0 = 2$ means $p(-2) = 0 = -a^2$ which we dealt with before.

   φ is injective, too: The differential of φ is everywhere in U an isomorphism, hence φ is a diffeomorphism by the inverse function theorem.

   $$\begin{aligned} \int_V f \, d\lambda &= \int_U |\det(D\varphi)| \cdot (f \circ \varphi) \, d\lambda \\ &= \int_0^1 \int_{-1}^1 (2t^2 + r + 2) \cdot f(t(r+2), t^2 - r) \, dr \, dt \end{aligned}$$

   with the substitution theorem and Fubini's theorem. Especially we get

   $$\operatorname{vol}(V) = \int_V 1 \, d\lambda = \int_0^1 \int_{-1}^1 2t^2 + r + 2 \, dr \, dt = \int_0^1 2(2t^2 + 2) \, dt = \frac{4}{3} + 4 = \frac{16}{3}$$

2. Calculate $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}}$

   Reason: Application of the Differential Operator.

   Solution: The Taylor series for $\arcsin^2 z$ and $-1<z<1$ is given by

$$\arcsin^2 z = \frac{1}{2}\sum_{k=1}^{\infty}\frac{(2z)^{2k}}{k^2\binom{2k}{k}}$$

   and applying the differential operator $z\dfrac{d}{dz}$ twice yields

$$\begin{aligned}\frac{2z\arcsin z}{\sqrt{1-z^2}} &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{2^{2k+1}kz^{2k}}{k^2\binom{2k}{k}}\\ \frac{2z\arcsin z}{\sqrt{1-z^2}}+\frac{2z^2}{1-z^2}+\frac{2z^3\arcsin z}{\sqrt{1-z^2}^3} &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{2^{2k+2}k^2z^{2k}}{k^2\binom{2k}{k}}\\ \frac{2z^2}{1-z^2}+\frac{2z\arcsin z}{\sqrt{1-z^2}^3} &= 2\sum_{k=1}^{\infty}\frac{(2z)^{2k}}{\binom{2k}{k}}\end{aligned}$$

   which is divided by 2 for $z=\dfrac{1}{2}$

$$\sum_{k=1}^{\infty}\frac{1}{\binom{2k}{k}} = \frac{1}{3}+\frac{\pi}{12}\cdot\frac{4}{3}\cdot\frac{2}{\sqrt{3}} = \frac{9+2\sqrt{3}\pi}{27}\approx 0.7364$$

3. Let a be an integer and p an odd prime which does not divide a. The left multiplication

$$\lambda_{a,p}\,:\,\mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p^\times\,;\, x\longmapsto ax \bmod p$$

   is then a permutation on $\{1,\ldots,p-1\}$. Prove

$$\left(\frac{a}{p}\right) = \operatorname{sgn}(\lambda_{a,p})$$

   Reason: Lemma of Zolotarev

   Solution: Let $k=\operatorname{ord} a$ in $\mathbb{Z}_p^\times$. Then $\lambda_{a,p}$ is a product of $\dfrac{p-1}{k}$ many cycles of length k. Thus

$$\operatorname{sgn}(\lambda_{a,p}) = (-1)^{(k-1)(p-1)/k}$$

If k is even, then

$$\operatorname{sgn}(\lambda_{a,p}) = (-1)^{(p-1)/k} \equiv (a^{k/2})^{(p-1)/k} \equiv a^{(p-1)/2} \bmod p$$

If k is odd, then $2k \,|\, (p-1)$ and

$$\operatorname{sgn}(\lambda_{a,p}) = 1 \equiv \left(a^k\right)^{(p-1)/2k} \equiv a^{(p-1)/2} \bmod p$$

Hence we have in both cases with Euler's criterion for the Legendre symbol of odd primes

$$\operatorname{sgn}(\lambda_{a,p}) \equiv a^{(p-1)/2} \equiv \left(\frac{a}{p}\right) \bmod p$$

4. Let $\mathcal{H}$ be a real Hilbert space and β a continuous bilinear form, $\mathcal{H}^*$ its dual space of continuous functionals on $\mathcal{H}$, and $\beta(f,f) \geq C\|f\|^2$ with $C > 0$.
   Prove that for any given continuous functional $F \in \mathcal{H}^*$ there is a unique vector $f^\dagger \in \mathcal{H}$ such that

   $$F(g) = \beta(f^\dagger, g) \quad \forall\, g \in \mathcal{H}$$

   Reason: Lemma of Babuška-Lax-Milgram.

   Solution: The statement is a generalization of Riesz' representation theorem (or theorem of Fréchet-Riesz). If we define a continuous function $B(f)(g) := \beta(f,g)$ then Riesz' representation theorem gives us an isometric isomorphism $T : \mathcal{H}^* \longrightarrow \mathcal{H}$ such that for every $B(f) \in \mathcal{H}^*$ there is a unique $T(B(f))$ such that $\|B(f)\| = \|T(B(f))\|$ and

   $$B(f)(g) = \langle T(B(f)), g\rangle_{\mathcal{H}} = \beta(f,g) \quad \forall\, g \in \mathcal{H} \quad (*)$$

   or generally $f^*(g) = \langle T(f^*), g\rangle_{\mathcal{H}}\ \forall\, g \in \mathcal{H} \quad (*)$

   The functionals $B(f)$ are bounded since β is continuous, i.e. $\|B\|$ is a finite real number. We get from our lower bound

   $$\begin{aligned} C\|f\|^2 &\leq |\beta(f,f)| = \langle T(B(f)), f\rangle_{\mathcal{H}} \\ &\leq \|T(B(f))\| \cdot \|f\| = \|B(f)\| \cdot \|f\| \leq \|B\| \cdot \|f\|^2 \end{aligned}$$

   hence $0 < \dfrac{C}{\|B\|} \leq 1$. We now define the function

   $$Q(f) := f - k \cdot (T(B(f)) - T(F))$$

on $\mathcal{H}$ with a real number $k \in \mathbb{R} - \{0\}$. A vector $f^\dagger \in \mathcal{H}$ is a fixed point of Q iff $T(B(f^\dagger)) - T(F) = 0$. In general we have for all $g \in \mathcal{H}$

$$\begin{aligned} T(B(f)) - T(F) \stackrel{(**)}{=} 0 &\Longleftrightarrow F(g) \stackrel{(**)}{=} B(f)(g) = \beta(f,g) \stackrel{(*)}{=} \langle T(B(f)), g\rangle_{\mathcal{H}} \\ &\Longleftrightarrow F(g) \stackrel{(*)}{=} \langle T(F), g\rangle_{\mathcal{H}} \stackrel{(**)}{=} \langle T(B(f)), g\rangle_{\mathcal{H}} \\ &\Longleftrightarrow \langle T(B(f)) - T(F), g\rangle_{\mathcal{H}} \stackrel{(**)}{=} 0 \end{aligned}$$

again by Riesz' representation theorem and the equations above. As $g \in \mathcal{H}$ is arbitrary, we may set $g := T(B(f^\dagger)) - T(F)$ for a fixed point of Q and get $\|T(B(f^\dagger)) - T(F)\|^2 = 0$ hence

$$B(f^\dagger) = \beta(f^\dagger, -) = F$$

which has to be shown. Thus all what's left to show is, that such a unique fixed point $f^\dagger$ of Q exists, which we will prove with Banach's fixed point theorem.

$$\begin{aligned} \|Q(f) - Q(g)\|^2 &= \|f - k(T(B(f)) - T(F)) - g + k(T(B(g)) - T(F))\|^2 \\ &= \langle (f-g) - kT(B(f-g)), (f-g) - kT(B(f-g))\rangle_{\mathcal{H}} \\ &\stackrel{(*)}{=} l, \|f-g\|^2 - 2k\,\langle T(B(f-g)), f-g\rangle_{\mathcal{H}} + k^2\,\|T(B(f-g))\|^2 \\ &\stackrel{(*)}{=} \|f-g\|^2 - 2k\,\beta(f-g, f-g) + k^2\,\|B(f-g)\|^2 \\ &\leq \|f-g\|^2 - 2k\,C\,\|f-g\|^2 + k^2\,\|B\|^2\,\|f-g\|^2 \\ &= \|f-g\|^2\left(1 - 2k\,C + k^2\,\|B\|^2\right) \\ &\stackrel{\text{set } k:=C/\|B\|^2}{=} \|f-g\|^2\left(1 - \frac{C^2}{\|B\|^2}\right) \end{aligned}$$

We have seen that $\dfrac{C}{\|B\|} \in (0,1]$ hence $q := 1 - \dfrac{C^2}{\|B\|^2} \in [0,1)$ and $\|Q(f) - Q(g)\|^2 = q\,\|f-g\|^2$ and the statement follows from Banach's fixed point theorem.

5. (HS-1)

   (a) Let $A = (-2,0)$, $B = (0,4)$ and $M = (1,3)$. What is $\alpha = \sphericalangle(AMB)$?

   (b) Let $C = (-1, 2+\sqrt{5})$, $D = (-1, 2-\sqrt{5})$ and $M = (1,3)$. What is $\beta = \sphericalangle(CMD)$?

Reason: Theorem of Thales.

Solution: We observe that A and B, as well as C and D are diametrical points on the circle $(x+1)^2+(y-2)^2=5$ and that M fulfills that equation, too. Hence by Thales' theorem, the angles at M have both to be a right angle.

6. (HS-2) Determine (with justification, but without explicit calculation) which of

   (a) 1000^{1001} and 1002^{1000}
   (b) $e^{0.000009}-e^{0.000007}+e^{0.000002}-e^{0.000001}$ and $e^{0.000008}-e^{0.000005}$

   is larger.

   Reason: Numbers.

   Solution:

   (a) We show that $1000^{1001}>1002^{1000}$:

   $$3003=1001\cdot\log_{10}(1000)>1000\cdot\frac{\log 1002}{\log 10}=1000\cdot\log_{10}(1002)$$

   This is equivalent to

   $$\log_{10}(1002)=\log_{10}\left(1000\cdot\frac{1000+2}{1000}\right)=3+\log_{10}\left(1+\frac{1}{500}\right)<3.003$$
   $$\Longleftrightarrow$$
   $$\log_{10}\left(1+\frac{1}{500}\right)<\frac{3}{1000}$$
   $$\Longleftrightarrow$$
   $$1+\frac{1}{500}<\frac{10,000}{9,961}=\frac{1,000}{\underbrace{\frac{1}{1}+\ldots+\frac{1}{1}}_{=994}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{5}+\frac{1}{5}+\frac{1}{5}}$$
   $$\stackrel{H.M.\leq G.M.}{\leq}\quad\sqrt[1000]{1\cdot\ldots\cdot 1\cdot 2^3\cdot 5^3}=\sqrt[1000]{1000}\approx 1.0069$$

   (b) With $a=e^{0.000001}$ we have $x=a^9-a^7+a^2-a$ and $y=a^8-a^5$ such that the difference is

   $$x-y=a^9-a^8-a^7+a^5+a^2-a=a(a-1)^2(a+1)(a^5-a^2-1)$$

By monotony of the exponential function and $1 < e < 4$ we get

$$1 < a < a^2 < a^5 = e^{0.000005} < 4^{0.000005} < 4^{0.5} = 2$$
$$a^5 - a^2 - 1 < 2 - 1 - 1 = 0$$
$$0 < a - 1 < a$$
$$0 < a + 1 < 2a < 4$$

and so $x - y = \underbrace{a}_{>0} \cdot \underbrace{(a-1)^2}_{>0} \cdot \underbrace{(a+1)}_{>0} \cdot \underbrace{(a^5 - a^2 - 1)}_{<0} < 0$ and $e^{0.000009} - e^{0.000007} + e^{0.000002} - e^{0.000001} < e^{0.000008} - e^{0.000005}$

7. (HS-3) Answer the following questions:

   (a) How many knights can you place on a $n \times m$ chessboard such that no two attack each other?

   (b) In how many different ways can eight queens be placed on a chessboard, such that no queen threatens another? Two solutions are not different, if they can be achieved by a rotation or by mirroring of the board.

   Reason: Puzzle. Internet Research.

   Solution:

   (a) Knights change color if they move. So the maximal possible number of knights on an $n \times m$ chessboard is the maximal number of squares of the same color $\lceil \frac{n \cdot m}{2} \rceil$. Exceptions are boards where n or m are small.

   Case $m = 1$: In this case we can place n knights on the board.

   Case $m = 2$: In this case we can place knights on blocks of four, followed by empty blocks of four. This way we get more knights on the board than we would get, if we placed all on, say black squares. This can be seen for $n = 6$, where we have eight knights by the block structure on the twelve squares, but only six black squares. If we write $n = 4k + r$ with $k \geq 0$, $r \in \{1, 2, 3, 4\}$ then the number N of possible knights is

$$N = \begin{cases} 4k + 2 \text{ if } r = 1 \\ 4k + 4 \text{ if } r > 1 \end{cases}$$

   In all cases with $m \geq 3$ we are in the general case, where the squares of one color is optimal, because these are more than those we can get by building blocks. And more than that isn't possible.

(b) There are 92 solutions total, and 12 fundamental solutions, i.e. up to reflection and rotation. For a list see

`https://en.wikipedia.org/wiki/Eight_queens_puzzle#Solutions`

25 January 2020

1. Let $f\,:\,\mathbb{R}\longrightarrow\mathbb{R}$ be a smooth, $2\pi-$periodic function with square integrable derivative, and $\displaystyle\int_0^{2\pi} f(x)\,dx=0$. Prove
$$\int_0^{2\pi}\left[f(x)\right]^2\,dx\leq\int_0^{2\pi}\left[f\,'(x)\right]^2\,dx$$
For which functions does equality hold?

   Reason: Wirtinger's Inequality.

   Solution: The function fulfills the Dirichlet conditions, so there is a real Fourier series such that
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^\infty\left(a_n\cos(nx)+b_n\sin(nx)\right)$$
See e.g. https://en.wikipedia.org/wiki/Fourier_series. The condition about the vanishing integral implies $a_0=0$. By Parseval's equation (see challenge from November 2019) for $f(x)$ and $f\,'(x)$ we get
$$\begin{aligned}\frac{1}{\pi}\int_0^{2\pi}f^2(x)\,dx&=\sum_{n=1}^\infty\left(a_n^2+b_n^2\right)\\&\leq\sum_{n=1}^\infty n^2\left(a_n^2+b_n^2\right)\\&=\frac{1}{\pi}\int_0^{2\pi}(f\,')^2(x)\,dx\end{aligned}$$
and equality holds, if $a_n=b_n=0$ for all $n>1$, i.e
$$f(x)=a_1\cos(x)+b_1\sin(x)\,.$$

2. Let M be the set of all non-negative, convex functions $f\,:\,[0,1]\longrightarrow\mathbb{R}$ with $f(0)=0$. Prove
$$\int_0^1\prod_{k=1}^n f_k(x)\,dx\geq\frac{2^n}{n+1}\prod_{k=1}^n\int_0^1 f_k(x)\,dx\quad\forall\;f_1,\ldots,f_n\in M$$

   Hint: Define and use $\hat{f}(x)=2x\displaystyle\int_0^1 f(x)\,dx$.

   Reason: Anderson's Inequality.

   Solution:

(a) Every $f \in M$ is monotone increasing.

Assume we have points $0 \leq x_1 < x_2 \leq 1$ such that $f(x_1) > f(x_2)$. Then for some value $\lambda \in [0,1]$ and because $f(0)$

$$f((\lambda)x_2) = f(x_1) \leq \lambda f(x_2) < f(x_1)$$

which is not possible. Geometrically we would get a point $(x_1, f(x_1))$ above the secant $s(x) = \dfrac{f(x_2)}{x_2} \cdot x$ between the origin and $(x_2, f(x_2))$ when it should be below.

(b) M is multiplicatively closed: $f, g \in M \Longrightarrow f \cdot g \in M$.

We have to prove convexity. Choose $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$. Since f, g are both monotone (increasing), we have

$$(f(x) - f(y))\,(g(x) - g(y)) \geq 0$$

and so

$$f(x)g(x) + f(y)g(y) \geq f(x)g(y) + f(y)g(x)$$

and thus

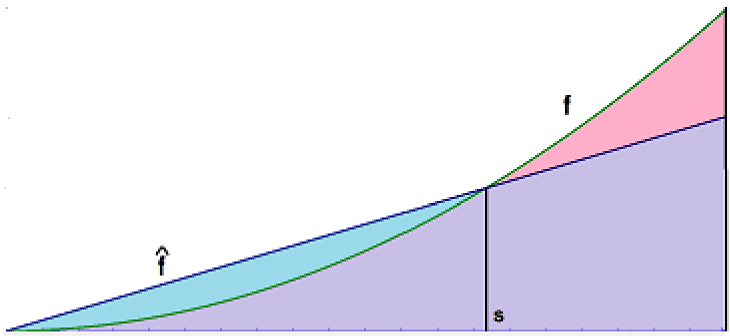
$$\begin{aligned}(fg)(\alpha x + \beta y) &= f(\alpha x + \beta y) + g(\alpha x + \beta y)\\ &\leq (\alpha f(x) + \beta f(y))\,(\alpha g(x) + \beta g(y)) \text{ by convexity}\\ &= \alpha^2 f(x)g(x) + \alpha\beta\left(f(x)g(y) + f(y)g(x) + \beta^2 f(y)g(y)\right)\\ &\leq \left(\alpha^2 + \alpha\beta\right) f(x)g(x) + \left(\alpha\beta + \beta^2\right) f(y)g(y)\\ &= \alpha f(x)g(x) + \beta f(y)g(y)\end{aligned}$$

(c) $\hat{f}(x) := 2x \displaystyle\int_0^1 f(t)\,dt \in M \;\forall\, f \in M$

since $\hat{f}(0) = 0$ and linearity in x proves convexity. Note that $\int_0^1 \hat{f}(x)\,dx = \int_0^1 f(x)\,dx$.

(d) $\displaystyle\int_0^1 g(x)f(x)\,dx \geq \int_0^1 g(x)\hat{f}(x)\,dx \;\forall\, f, g \in M$

By the equality of the areas under the function graphs of $\hat{f}(x)$ and $f(x)$ there has to be a point $0 < s < 1$ such that $f(x) \leq \hat{f}(x)$ for all $x \leq s$ and $f(x) \geq \hat{f}(x)$ for all $x \geq s$. Hence



$$\int_0^s \hat{f}(x)\,dx + \int_s^1 \hat{f}(x)\,dx = \int_0^s f(x)\,dx + \int_s^1 f(x)\,dx \text{ and}$$

$$\int_0^s \left(\hat{f}(x) - f(x)\right)\,dx = \int_s^1 \left(f(x) - \hat{f}(x)\right)\,dx$$

As g is monotone increasing we get

$$\begin{aligned}\int_0^s g(x)\left(\hat{f}(x) - f(x)\right)\,dx &\leq g(s)\int_s^1 \left(\hat{f}(x) - f(x)\right)\,dx \\ &= g(s)\int_s^1 \left(f(x) - \hat{f}(x)\right)\,dx \\ &\leq \int_s^1 g(x)\left(f(x) - \hat{f}(x)\right)\,dx \\ &= -\int_s^1 g(x)\left(\hat{f}(x) - f(x)\right)\,dx\end{aligned}$$

Thus $\displaystyle\int_0^1 g(x)\left(\hat{f}(x) - f(x)\right)\,dx \leq 0\,.$

Finally we have for any $f_1,\ldots,f_n \in M$

$$\begin{aligned}\int_0^1 \prod_{k=1}^n f_k(x)\,dx &\geq \int_0^1 \prod_{k=1}^n \hat{f}_k(x)\,dx \\ &= 2^n \prod_{k=1}^n \left(\int_0^1 f_k(x)\,dx\right)\int_0^1 x^n\,dx \\ &= \frac{2^n}{n+1}\prod_{k=1}^n \int_0^1 f_k(x)\,dx\end{aligned}$$

3. Consider the following differential operators on the space of smooth

functions $C^\infty(\mathbb{R})$

$$A = 2x \cdot \frac{d}{dx}\,,\; B = x^2 \cdot \frac{d}{dx}\,,\; C = -\frac{d}{dx}$$

Determine the eigenvectors and a (multiplicative) structure on lin $\operatorname{span}_{\mathbb{R}}\{\,A, B, C\,\}$.

Reason: Lie theory.

Solution: The eigenvectors are:

$$\begin{aligned} A.f = \lambda f &\Longleftrightarrow f = c\sqrt{x}^{\lambda} \\ B.g = \lambda g &\Longleftrightarrow g = ce^{-\lambda/x} \\ C.h = \lambda h &\Longleftrightarrow h = ce^{-\lambda x} \end{aligned}$$

and the structure is: $\operatorname{span}_{\mathbb{R}}\{\,A, B.C\,\} = \mathfrak{sl}(2,\mathbb{R})$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\,,\; B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\,,\; C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

4. Prove

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k}$$

Reason: Useful Series Identity.

Solution: We define $a_n := \dfrac{2^{n+1}}{n+1} \sum_{k=0}^{n} \dfrac{1}{\binom{n}{k}}$. So

$$\begin{aligned} a_n - a_{n-1} &= \frac{2^{n+1}}{n+1}\left(1 + \sum_{k=0}^{n-1}\left(\frac{(n-k)!k!}{n!} - \frac{(n+1)(n-1-k)!k!}{2n(n-1)!}\right)\right) \\ &= \frac{2^n}{n+1}\left(2 + \sum_{k=0}^{n-1}\frac{(n-k)!k!}{n!}\left(2 - \frac{n+1}{n-k}\right)\right) \\ &= \frac{2^n}{n+1}\left(2 + \sum_{k=0}^{n-1}\frac{(n-k)!k!}{n!}\cdot\frac{n-2k-1}{n-k}\right) \\ &= \frac{2^{n+1}}{n+1} + \frac{2^n}{(n+1)!}\sum_{k=0}^{n-1}\underbrace{((n-1)-k)!\,k!\,((n-1)-2k)}_{=:b_k} \end{aligned}$$

Now we get

$$\begin{aligned} b_{(n-1)-k} &= k!\,((n-1)-k)!\,((n-1)-2((n-1)-k)) \\ &= k!\,((n-1)-k)!\,(-1)\,((n-1)-2k) = -b_k \end{aligned}$$

which implies, that $\sum_{k=0}^{n-1} b_k = 0$ and

$$\begin{aligned} a_{n-1} &= a_{n-1} - a_{n-2} + a_{n-2} - a_{n-3} \pm \ldots + a_1 - a_0 + a_0 - \underbrace{a_{-1}}_{=0} \\ &= \sum_{k=0}^{n-1}(a_k - a_{k-1}) = \sum_{k=0}^{n-1} \frac{2^{k+1}}{k+1} + \frac{2^k}{(k+1)!}\sum_{k=0}^{n-1} b_k \\ &= \sum_{k=1}^{n} \frac{2^k}{k} \quad \text{and} \quad \sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}} \cdot a_n = \frac{n+1}{2^{n+1}} \sum_{k=1}^{n+1} \frac{2^k}{k} \end{aligned}$$

5. Calculate $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{2}}$

   Reason: Easy series.

   Solution:

$$\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{2}} = 2\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} = 2\sum_{k=1}^{\infty}\left(\frac{1}{2k-1} - \frac{1}{2k}\right) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

   For the last equality, we use the Taylor series for natural logarithm and plug $x = 1$ in it:

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$$

   such that

$$\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{2}} = 2\log 2$$

6. Prove for $b > 0$

$$\int_{-\infty}^{\infty} f\left(x - \frac{b}{x}\right)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

   Reason: Integration Methods. Translation Invariance.

   Solution:

$$\begin{aligned} I := \int_{-\infty}^{\infty} f\left(x - \frac{b}{x}\right)\, dx &= \int_{-\infty}^{0} f\left(x - \frac{b}{x}\right)\, dx + \int_{0}^{\infty} f\left(x - \frac{b}{x}\right)\, dx \\ &\overset{u=-b/x}{=} \int_{0}^{\infty} f\left(-\frac{b}{u} + u\right) \frac{b}{u^2}\, du + \int_{-\infty}^{0} f\left(-\frac{b}{u} + u\right) \frac{b}{u^2}\, du \\ &= \int_{-\infty}^{\infty} f\left(u - \frac{b}{u}\right) \frac{b}{u^2}\, du = \int_{-\infty}^{\infty} f\left(x - \frac{b}{x}\right) \frac{b}{x^2}\, dx \end{aligned}$$

Therefore

$$\begin{aligned} 2I &= \int_{-\infty}^{\infty} f\left(x - \frac{b}{x}\right)\left(1 + \frac{b}{x^2}\right) dx \\ &= \int_{-\infty}^{0} f\left(x - \frac{b}{x}\right)\left(1 + \frac{b}{x^2}\right) dx + \int_{0}^{\infty} f\left(x - \frac{b}{x}\right)\left(1 + \frac{b}{x^2}\right) dx \\ &\stackrel{v=x-(b/x)}{=} \int_{-\infty}^{\infty} f(v)\,dv + \int_{-\infty}^{\infty} f(v)\,dv \\ &= 2\int_{-\infty}^{\infty} f(v)\,dv \end{aligned}$$

which is what has to be proven.

7. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a $\pi-$periodic function. Prove that

$$\int_{-\infty}^{\infty} f(x)\,\frac{\sin x}{x}\,dx = \int_0^{\pi} f(x)\,dx \text{ and } \int_{-\infty}^{\infty} f(x)\,\frac{\tan x}{x}\,dx = \int_0^{\pi} f(x)\,dx$$

so the integrals exist. See **(***)** at the end of the proof.

Reason: Integration Methods. Lobachevski's Formulas.

Solution: We omit will the epsilontic around the poles, and start with the series expansion of $\csc(x)$

$$\begin{aligned} \csc(x) &= \frac{1}{x} + 2x\sum_{k\in\mathbb{N}} \frac{(-1)^k}{x^2 - k^2\pi^2} \\ &= \frac{1}{x} + \sum_{k\in\mathbb{N}} (-1)^k \frac{(x+k\pi)+(x-k\pi)}{(x-k\pi)(x+k\pi)} \\ &= \frac{1}{x} + \sum_{k\in\mathbb{N}} \left(\frac{(-1)^k}{x+k\pi} + \frac{(-1)^k}{x-k\pi}\right) \\ &\stackrel{(*)}{=} \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{x+k\pi} = \frac{1}{\sin x} \end{aligned}$$

and for the second part with the series expansion of $\cot(x)$

$$\begin{aligned} \cot(x) &= \frac{1}{x} + \sum_{k\in\mathbb{N}} \left(\frac{1}{x+k\pi} + \frac{1}{x-k\pi}\right) \\ &\stackrel{(**)}{=} \sum_{k\in\mathbb{Z}} \frac{1}{x+k\pi} = \frac{1}{\tan x} \end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,\frac{\sin x}{x}\,dx &= \sum_{k\in\mathbb{Z}}\int_{k\pi}^{(k+1)\pi} f(x)\,\frac{\sin x}{x}\,dx\\ &\stackrel{y=x-k\pi}{=} \sum_{k\in\mathbb{Z}}\int_0^{\pi} f(y)\,\frac{\sin(y+k\pi)}{y+k\pi}\,dy\\ &= \sum_{k\in\mathbb{Z}}\int_0^{\pi} f(y)\,\frac{(-1)^k\sin(y)}{y+k\pi}\,dy\\ &= \int_0^{\pi} f(y)\underbrace{\sum_{k\in\mathbb{Z}}\frac{(-1)^k}{y+k\pi}}_{\stackrel{(*)}{=}\csc(y)}\sin y\,dy\\ &= \int_0^{\pi} f(y)\,dy = \int_0^{\pi} f(x)\,dx\end{aligned}$$

Note that similar can be done with the weight function $\dfrac{\sin^2 x}{x^2}$.
For the second part we get

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,\frac{\tan x}{x}\,dx &= \sum_{k\in\mathbb{Z}}\int_{(k-1/2)\pi}^{(k+1/2)\pi} f(x)\,\frac{\tan x}{x}\,dx\\ &\stackrel{y=x-k\pi}{=} \sum_{k\in\mathbb{Z}}\int_{-\pi/2}^{\pi/2} f(y)\,\frac{\tan y}{y+k\pi}\,dy\\ &= \int_{-\pi/2}^{\pi/2} f(y)\underbrace{\sum_{k\in\mathbb{Z}}\frac{1}{y+k\pi}}_{\stackrel{(**)}{=}\cot y}\tan y\,dy\\ &\stackrel{z=y+\pi/2}{=} \int_0^{\pi} f(z)\,dz = \int_0^{\pi} f(x)\,dx\end{aligned}$$

(***) The second equation is wrong in general, e.g. choose $f(x)=1$. We used exchangeability of integral, series and implicitly limits (at the poles). Hence this example shows that conditions such as in Fubini's theorem have to be carefully checked!

8. (a) If $\varphi : G \longrightarrow H$ is a homomorphism of finite groups, then $\operatorname{ord}(\varphi(g)) \mid \operatorname{ord}(g)$ for all elements $g \in G$.
   (b) Determine all group homomorphisms $\varphi : \mathbb{Z}_4 \longrightarrow \operatorname{Sym}(3)$ and $\psi : \operatorname{Sym}(3) \longrightarrow \mathbb{Z}_4$.

Reason: Basic Group Theory.

Solution:

(a) Let $n := \operatorname{ord}(g)$ and $k := \operatorname{ord}(g)$. Since $\varphi(g)^n = \varphi(g^n) = \varphi(e) = e$ so $k \leq n$. We can write $n = s \cdot k + r$ with non-negative integers $s, 0 \leq r < k$ by division with remainder. Now

$$\varphi(g)^r = \varphi(g)^{n-sk} = \varphi(g)^n \cdot \left(\varphi(g)^k\right)^{-s} = e \cdot e^{-s} = s$$

By minimality of $k > r$ this is only possible, if $r = 0$, i.e. $n = s \cdot k$ and $k \,|\, n$.

(b) Every homomorphism $\varphi \,:\, \mathbb{Z}_4 \longrightarrow \operatorname{Sym}(3)$ is uniquely determined by its image $\varphi([1])$, since $[1] = 1 + 4\mathbb{Z}$ generates $\mathbb{Z}_4$. As $\varphi([1]) \mid \operatorname{ord}([1]) = 4$ the only possible elements of Sym(3) are $X = \{\,(1), (12), (13), (23)\,\}$. For every element $x \in X$ we define $\varphi_x([0]) = (1)$ and $\varphi_x([1]) = x$. Given that φ_x needs to be a homomorphism, we get

$$\begin{aligned}\varphi_x([2]) &= \varphi_x([1]) + \varphi_x([1]) = x^2 = (1)\\ \varphi_x([3]) &= \varphi_x([1]) + \varphi_x([1]) + \varphi_x([1]) = x^3 = x\end{aligned}$$

so every element of X defines a group homomorphism φ_x .

Let on the other hand be $\psi \,:\, \operatorname{Sym}(3) \longrightarrow \mathbb{Z}_4$ be a group homomorphism. Then $\operatorname{ord}(123) = 3$ and $\mathbb{Z}_4$ doesn't have an element of order 3. So $\psi(123) = \psi(132) = [0]$.
All transpositions are conjugates of each other:

$$(12)(23)(12) = (13)\,,\ (13)(12)(13) = (23)\,,\ (23)(13)(23) = (12)$$

and $\mathbb{Z}_4$ is Abelian, hence $\psi(12) = \psi(13) = \psi(23)$. Furthermore $\operatorname{ord}(\psi(\tau)) \,|\, \operatorname{ord}(\tau) = 2$ so $\psi(\tau) \in \{\,[0], [2]\,\}$ for all transpositions τ . There are therefore two possibilities for ψ, the homomorphism which transforms every element onto [0], or ψ given by $\psi(1) = \psi(123) = \psi(132) = [0]\,,\ \psi(12) = \psi(13) = \psi(23) = [2]$ which is induced by the signum-function, the sign of a permutation.

9. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}_{\geq 0}$ be non-negative real numbers. The elementary symmetric polynomials are

$$\sigma_k(\mathbf{a}) = \sum_{1 \leq j_1 < \ldots < j_k \leq n} a_{j_1} a_{j_2} \ldots a_{j_k}$$

and

$$S_k(\mathbf{a}) = \frac{1}{\binom{n}{k}} \cdot \sigma_k(\mathbf{a})$$

the corresponding elementary symmetric mean value. Prove

(a) $S_1(\mathbf{a}) \geq \sqrt{S_2(\mathbf{a})} \geq \sqrt[3]{S_3(\mathbf{a})} \geq \ldots \geq \sqrt[n]{S_n(\mathbf{a})}$

(b) $S_m(\mathbf{a})^2 \geq S_{m+1}(\mathbf{a}) \cdot S_{m-1}(\mathbf{a})$ for $m = 1, \ldots, n-1$

Reason: MacLaurin's and Newton's Inequality.

Solution:

$$F(x) := (x + a_1) \cdot \ldots \cdot (x + a_n) = \sum_{k=0}^{n} \sigma_k(\mathbf{a}) x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} S_k(\mathbf{a}) x^{n-k}$$

$$F'(x) = \sum_{k=0}^{n} (n-k) \binom{n}{k} S_k(\mathbf{a}) x^{n-k-1} = n \sum_{k=0}^{n-1} \binom{n-1}{k} S_k(\mathbf{a}) x^{n-k-1}$$

Say $a_k < a_{k+1}$. Since $F(-a_k) = 0 = F(-a_{k+1})$ we have by Rolle's theorem a $-b_k \in]-a_{k+1}, -a_k[$ such that $F'(-b_k) = 0$. The same is true in case $a_k = a_{k+1}$. Hence we can write

$$F'(x) = n(x - (-b_1)) \cdot \ldots \cdot (x - (b_n)) = (x + b_1) \cdot \ldots \cdot (x + b_{n-1})$$

with also non-negative numbers $\mathbf{b} = (b_1, \ldots, b_{n-1})$ and thus

$$F'(x) = n \sum_{k=0}^{n-1} \sigma_k(\mathbf{b}) x^{n-1-k} = n \sum_{k=0}^{n-1} \binom{n-1}{k} S_k(\mathbf{b}) x^{n-1-k}$$

By comparison of the two polynomials $F'(x)$ we get $S_k(\mathbf{a}) = S_k(\mathbf{b})$ for all $k \leq n-1$. Every further derivative has also all nonpositive zeros by induction and

$$\begin{aligned} F^{(n-m)}(x) &= \frac{n!}{m!} (X + r_1^{(m_1)}) \cdot \ldots \cdot (X + r_m^{(m_m)}) \\ &= \frac{n!}{m!} \sum_{k=0}^{m} \binom{m}{k} S_k(\mathbf{r}^{(m)}) x^{m-k} \\ &= \left(\frac{d}{dx}\right)^{n-m} \sum_{k=0}^{n} \binom{n}{k} S_k(\mathbf{a}) x^{n-k} \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} S_k(\mathbf{a}) \frac{(n-k)!}{(m-k)!} x^{m-k} \\ &= \frac{n!}{m!} \sum_{k=0}^{m} \binom{m}{k} S_k(\mathbf{a}) x^{m-k} \end{aligned}$$

which again shows by comparison that $S_k(\mathbf{a}) = S_k(\mathbf{r}^{(m)})$ for all $k \leq m$. Especially we get

$$S_m(\mathbf{r}^{(m)}) = S_m(\mathbf{a}) = r_1^{(m)} \cdot \ldots \cdot r_m^{(m)}$$

and ($\widehat{a}$ here means "without a")

$$\begin{aligned}
\widehat{r_1^{(m)}}r_2^{(m)}\cdots r_m^{(m)} + r_1^{(m)}\widehat{r_2^{(m)}}\cdots r_m^{(m)} + \ldots + r_1^{(m)}r_2^{(m)}\cdots\widehat{r_m^{(m)}}&\\
= r_1^{(m)}r_2^{(m)}\cdots r_m^{(m)}\cdot\left(\frac{1}{r_1^{(m)}}+\ldots+\frac{1}{r_m^{(m)}}\right)&\\
= m\cdot S_{m-1}(\mathbf{r}^{(m)})&\\
= m\cdot S_{m-1}(\mathbf{a})&
\end{aligned}$$

$$r_1^{(m)}\cdots r_m^{(m)}\sum_{1\leq i<j\leq m}\frac{1}{r_i^{(m)}r_j^{(m)}} = \frac{m(m-1)}{2}\cdot S_{m-2}(\mathbf{a})$$

(a) McLaurin's inequality. Since the arithmetic mean is greater than the geometric mean, we have

$$\frac{\widehat{r_1^{(m)}}r_2^{(m)}\cdots r_m^{(m)} + r_1^{(m)}\widehat{r_2^{(m)}}\cdots r_m^{(m)} + \ldots + r_1^{(m)}r_2^{(m)}\cdots\widehat{r_m^{(m)}}}{m} \geq \sqrt[m]{\left(r_1^{(m)}\cdot\ldots\cdot r_m^{(m)}\right)^{m-1}}$$

hence for all $m\leq n$

$$S_{m-1}(\mathbf{a})\geq\sqrt[m]{S_m(\mathbf{a})^{m-1}}\Longrightarrow\sqrt[m-1]{S_{m-1}(\mathbf{a})}\geq\sqrt[m]{S_m(\mathbf{a})}$$

(b) Newton's inequality.

$$S_{m-1}(\mathbf{a})^2 \geq S_m(\mathbf{a}) \cdot S_{m-2}(\mathbf{a})$$
$$\Longleftrightarrow$$
$$\left(r_1^{(m)} \cdots r_m^{(m)}\right)^2 \cdot \frac{1}{m^2} \cdot \left(\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2 \geq$$
$$\left(r_1^{(m)} \cdots r_m^{(m)}\right)^2 \cdot \frac{2}{m(m-1)} \cdot \sum_{1\leq i<j\leq m} \frac{1}{r_i^{(m)} r_j^{(m)}}$$
$$\Longleftrightarrow$$
$$(m-1)\left(\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2 \geq 2m \sum_{1\leq i<j\leq m} \frac{1}{r_i^{(m)} r_j^{(m)}}$$
$$\Longleftrightarrow$$
$$(m-1)\left(\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2 \geq m \cdot \left(\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2 - m \cdot \sum_{k=1}^{m} \frac{1}{(r_k^{(m)})^2}$$
$$\Longleftrightarrow$$
$$m \cdot \sum_{k=1}^{m} \frac{1}{(r_k^{(m)})^2} \geq \left(\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2$$
$$\Longleftrightarrow$$
$$\frac{1}{m}\sum_{k=1}^{m} \frac{1}{(r_k^{(m)})^2} \geq \left(\frac{1}{m}\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}\right)^2$$
$$\Longleftrightarrow$$
$$\sqrt{\frac{1}{m}\sum_{k=1}^{m} \frac{1}{(r_k^{(m)})^2}} \geq \frac{1}{m}\sum_{k=1}^{m} \frac{1}{r_k^{(m)}}$$
$$\Longleftrightarrow$$
$$M^1_{1/m} \leq M^2_{1/m}$$

since the arithmetic mean is less or equal than the quadratic mean by the generalized or weighted Hölder mean inequality (see problem 1 above).

10. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers, not all zero.

Prove

$$\left(\sum_{n\in\mathbb{N}} a_n\right)^4 < \pi^2 \sum_{n\in\mathbb{N}} a_n^2 \cdot \sum_{n\in\mathbb{N}} n^2 a_n^2$$

Reason: Carlson's Inequality.

Solution: We quote two different proofs by Hardy.

(a) Consider the Fourier series $f(x) = \sum_{k=1}^{\infty} a_k \cos kx$ and its derivative $f'(x) = -\sum_{k=1}^{\infty} a_k \sin kx$. By Parseval's equation (see no. 1. in the November challenge) we get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \frac{2}{\pi}\int_0^{\pi} |f(x)|^2\,dx = \sum_{k=1}^{\infty} a_k^2 =: S$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f'(x)|^2\,dx = \frac{2}{\pi}\int_0^{\pi} |f'(x)|^2\,dx = \sum_{k=1}^{\infty} k^2 a_k^2 =: T$$

Now $\int_0^{\pi} f(x)\,dx = \sum_{k=1}^{\infty} a_k \left[\frac{\sin kx}{k}\right]_0^{\pi} = 0$ so there is a $0 < \xi < \pi$ with $f(\xi) = 0$ and

$$\begin{aligned}\left(\sum_{k=1}^{\infty} a_k\right)^2 &= f^2(0) - f^2(\xi) = 2\int_{\xi}^{0} f(x)f'(x)\,dx \\ &\stackrel{\text{Cauchy-Schwarz}}{<} 2\sqrt{\int_0^{\pi} f^2(x)\,dx} \cdot \sqrt{\int_0^{\pi} f'^2(x)\,dx} \\ &= 2\cdot\sqrt{\frac{\pi}{2}S}\cdot\sqrt{\frac{\pi}{2}T} = \pi\sqrt{ST}\end{aligned}$$

and squaring completes the proof.

(b) Let $\alpha, \beta > 0$ and $S = \sum_{k=1}^{\infty} a_k^2$, $T = \sum_{k=1}^{\infty} k^2 a_k^2$. By the Cauchy-

Schwarz inequality we get

$$\begin{aligned}\left(\sum_{k=1}^{\infty} a_k\right)^2 &= \left(\sum_{k=1}^{\infty} a_k \cdot \sqrt{\alpha+\beta k^2} \cdot \frac{1}{\sqrt{\alpha+\beta k^2}}\right)^2 \\ &\leq \sum_{k=1}^{\infty} a_k^2 \cdot (\alpha+\beta k^2) \cdot \sum_{k=1}^{\infty} \frac{1}{\alpha+\beta k^2} \\ &= (\alpha S+\beta T)\sum_{k=1}^{\infty}\frac{1}{\alpha+\beta k^2} < (\alpha S+\beta T)\int_0^{\infty}\frac{dx}{\alpha+\beta x^2} \\ &= (\alpha S+\beta T)\cdot\frac{\pi}{2}\cdot\frac{1}{\sqrt{\alpha\beta}} = \frac{\pi}{2}\left(\sqrt{\frac{\alpha}{\beta}}\cdot S+\sqrt{\frac{\beta}{\alpha}}\cdot T\right)\end{aligned}$$

With $\alpha := T$, $\beta := S$ we have

$$\left(\sum_{k=1}^{\infty} a_k\right)^2 < \frac{\pi}{2}(\sqrt{ST}+\sqrt{ST}) = \pi\sqrt{ST}$$

and squaring completes the proof.

11. (HS-1) On how many ways can 2020 be written as a sum of consecutive natural numbers (greater than zero)?

Reason: Prime numbers.

Solution:

$$\begin{aligned}2020 &= 2^3\cdot 5\cdot 101 \\ &= n+(n+1)+\ldots+(n+k) \\ &= n(k+1)+\frac{k}{2}\,(k+1) \\ 4040 &= (k+1)(2n+k)\end{aligned}$$

If $101\,|\,(k+1)$ then $k\geq 100$ and $2n+k\geq 102$ but 4040 doesn't have two such great divisors. Hence $(k+1, 2n+k)$ can only be one of the pairs

$$\{\,(2,2020),(4,1010),(8,505),(5,808),(10,404),(20,202),(40,101)\,\}$$

But if both components were even, then n wouldn't be a natural number, so we are left with 3 possibilities:

$$\begin{aligned}2020 &= 249+250+\ldots+256 \\ 2020 &= 402+403+\ldots+406 \\ 2020 &= 31+32+\ldots+70\end{aligned}$$

12. (HS-2) A binary operation on a set S is a mapping, which maps a pair from $S \times S$ to S. E.g. addition is a binary operation on integers. Find two different binary operations for $S = \{ A, B, C, D \}$ which have a neutral element, $A \circ X = X$, and can be inverted: for all $X \in S$ there is a $Y \in S$ with $X \circ Y = A$, and are associative: $X \circ (Y \circ Z) = (X \circ Y) \circ Z$.

    Reason: Group Theory.

    Solution: There are two groups of order 4: $\mathbb{Z}_4$ and $V_4 = \mathbb{Z}_2^2$. Their Abelian multiplications are given by

$\circ$	A	B	C	D
A	A	B	C	D
B	B	C	D	A
C	C	D	A	B
D	D	A	B	C

$\circ$	A	B	C	D
A	A	B	C	D
B	B	A	D	C
C	C	D	A	B
D	D	C	B	A

13. (HS-3) Find all six digit numbers with the following property: If we move the first (highest) digit at the end, we will get three times the original number.

    Reason: Puzzle.

    Solution: Set $n = 100,000\, a + b$. Then $3n = 10\, b + a$ and so

$$300,000\, a + 3\, b = 10\, b + a \Longleftrightarrow 299,999\, a = 7\, b \Longleftrightarrow b = 42,857\, a$$

    Therefore $n = 142,857\, a$ and $3n = 428,571\, a$ which means $a \in \{ 1, 2 \}$ and the only six digit numbers are $142,857$ and $285,714$.

14. (HS-4) The Pell sequence named after the English mathematician John Pell is defined by

$$P(n) = \begin{cases} 0\,,\; n = 0 \\ 1\,,\; n = 1 \\ P(n-2) + 2P(n-1)\,,\; n > 1 \end{cases}$$

    Calculate the limit $\delta_s := \lim_{n\to\infty} \dfrac{P(n)}{P(n-1)}$.

    Reason: Silver Ratio.

Solution:

$$\begin{aligned}\delta_s &= \lim_{n\to\infty} \frac{P(n-2)+2P(n-1)}{P(n-1)} \\ &= \lim_{n\to\infty} \frac{P(n-2)}{P(n-1)} + 2 \\ &= 2 + \delta_s^{-1}\end{aligned}$$

and thus $0 = \delta_s^2 - 2\delta_s - 1$ which has the solutions $1 \pm \sqrt{2}$. Since $P(n) > P(n-1)$ only $\delta_s = 1 + \sqrt{2}$ is a possible solution.

Two quantities are in the **silver ratio** if the ratio of the sum of the smaller and twice the larger of those quantities, to the larger quantity, is the same as the ratio of the larger one to the smaller one. This leads to

$$\frac{2a+b}{a} = \frac{a}{b} = \delta_s = [2; 2, 2, 2, \ldots]$$

where the last representation is the continued fraction of the silver ratio.

15. (HS-5) Consider the graph of $f(x) = 1/x$ with $x \geq 1$ and let it rotate around the $x-$axis. This solid of revolution looks like an infinitely long trumpet. Calculate its volume V and its surface A.
If we fill it with paint, pour it out again, then we have painted it from inside. Explain this apparent contradiction to the surface you computed.

Reason: Gabriel's Horn (Wikipedia).

Solution:

$$\begin{aligned}V &= \pi \int_1^\infty \frac{1}{x^2}\,dx = \pi \\ A &= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > 2\pi \int_1^\infty \frac{1}{x}\,dx = \infty\end{aligned}$$

We filled in paint, poured it out again, and thus have painted an infinitely large inner surface with a finite amount of paint!
The solution is, that in reality paint has a certain thickness, so we didn't need to "fill" the entire horn, only a finite part of it.

26 December 2019

1. Let (X, d) be a metric space. The open ball with center $z \in X$ of radius $r > 0$ is defined as

$$B_r(z) := \{ x \in X \mid d(x,z) < r \}$$

(a) Give an example for

$$\overline{B_r(z)} \neq K_r(z) := \{ x \in X \mid d(x,z) \leq r \}$$

Does at least one of the inclusions $\subseteq$ or $\supseteq$ always hold?

(b) What are the answers in the previous case, if we additionally assume that (X, d) has an inner metric?

An inner metric d_0 associated to d is defined as the infimum of all lengths of rectified curves between two points:

Let $\sigma : [0,1] \longrightarrow X$ with $\sigma(0) = x$, $\sigma(1) = y$ a rectified curve with length

$$L(\sigma) = \sup \left\{ \sum_{k=1}^{n} d(\sigma(t_{k-1}), \sigma(t_k)) \,\middle|\, 0 = t_0 < t_1 < \cdots < t_n = 1\,,\, n \in \mathbb{N} \right\}$$

Then $d_0(x, y) = \inf L(\sigma)$.

Reason: Exceptions in Metric Spaces.

Solution: The function $d_z := d(z, .) : X \longrightarrow \mathbb{R}$ is Lipschitz continuous with constant 1 and thus continuous: w.l.o.g. we may assume $d_z(x) \geq d_z(y)$ so

$$\begin{aligned} |d_z(x) - d_z(y)| &= d_z(x) - d_z(y) = d(z,x) - d(z,y) \\ &\leq d(z,y) + d(y,x) - d(z,y) = d(y,x) = d(x,y) \end{aligned}$$

Hence $d^{-1}([0,r]) = K_r(z)$ is closed. As $B_r(z) \subseteq K_r(z)$ we have

$$\overline{B_r(z)} \subseteq K_r(z)$$

in any case. Now we define a metric space $(\mathbb{R}^n, d)$ by

$$d(x,y) := \begin{cases} ||x-y|| & , \ ||x-y|| \leq 1 \\ 1 & , \ ||x-y|| > 1 \end{cases}$$

As $||.||$ is the ordinary Euclidean norm, we have

$$\overline{B_1(z)} = \{\, x \in \mathbb{R}^n \mid ||x-z|| \leq 1 \,\} \subsetneq \mathbb{R}^n = \{\, x \in \mathbb{R}^n \mid d(x,z) \leq 1 \,\} = K_r(z)$$

Let (X,d) now be an inner metric space and $x \in K_r(z)$.
Then $\overline{B_r(z)} = K_r(z)$ if there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq B_r(z)$ such that $\lim_{n\to\infty} x_n = x$.

If $x \in B_r(z)$ we can simply choose the constant sequence, hence we may assume $d(x,z) = r$. We choose a monotone decreasing sequence $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ of positive real numbers which converges to 0. By assumption there are rectified curves σ_n with length $L(\sigma_n) = r + \varepsilon_n$ for every $n \in \mathbb{N}$ with $\sigma_n(0) = z$, $\sigma_n(1) = x$. Let's assume the curves are parameterized by their paths so we can choose $x_n := \sigma_n(r - \varepsilon_n)$. Then

$$d(z,x_n) = d(z,\sigma_n(r-\varepsilon_n)) = L(\sigma_n) = r - \varepsilon_n < r$$

which means that all $x_n \in B_r(z)$ and $x_n \stackrel{n\to\infty}{\longrightarrow} x$ follows from

$$d(x,x_n) = d(x,z) - d(z,x_n) = r + \varepsilon_n - (r - \varepsilon_n) \leq 2\varepsilon_n \stackrel{n\to\infty}{\longrightarrow} 0$$

2. Let $f(z) = \dfrac{7z-51}{z^2-12z+27}$ be a complex function.

   (a) Determine the Laurent series of $f(z)$ and their radius of convergences around $z=3$ in the cases where 0 is in the area of convergence, and 10 is in the area of convergence.

   (b) Determine $\lim_{z\to 3} f(z)$, $\operatorname{Res}(f,3)$ and the kind of singularity in $z=3$.

Reason: Laurent Series.

Solution: We write

$$\begin{aligned} f(z) &= \frac{7z-51}{z^2-12z+27} = \frac{5}{z-3} + \frac{2}{z-9} = \frac{5}{z-3} + \frac{2}{(z-3)-6} \\ &= \frac{5}{z-3} - \frac{2}{6}\cdot\frac{1}{1-\left(\dfrac{z-3}{6}\right)} = \frac{5}{z-3} - \frac{2}{6}\cdot\sum_{n=0}^{\infty}\left(\frac{z-3}{6}\right)^n \\ &= \frac{5}{z-3} - \frac{1}{3} - \frac{1}{18}(z-3) - \frac{1}{108}(z-3)^2 - \frac{1}{648}(z-3)^3 - \frac{1}{3888}(z-3)^4 - \ldots \end{aligned}$$

which converges for $\left|\dfrac{z-3}{6}\right| < 1 \Longleftrightarrow 0 < |z-3| < 6$ and includes $z=0$.

In the other case we write

$$\begin{aligned} f(z) &= \frac{7z-51}{z^2-12z+27} = \frac{5}{z-3} + \frac{2}{(z-3)-6} \\ &= \frac{5}{z-3} + \frac{2}{z-3} \cdot \frac{1}{1-\left(\dfrac{6}{z-3}\right)} \\ &= \frac{5}{z-3} + \frac{2}{z-3} \cdot \sum_{n=0}^{\infty} \left(\frac{6}{z-3}\right)^n \\ &= \frac{7}{z-3} + \frac{12}{(z-3)^2} + + \frac{72}{(z-3)^3} + \frac{432}{(z-3)^4} + \frac{2592}{(z-3)^5} + \ldots \end{aligned}$$

which converges for $\left|\dfrac{6}{z-3}\right| < 1 \Longleftrightarrow 6 < |z-3|$ and includes $z = 10$.

To determine the singularity at $z = 3$ we can only use the first expansion due to the area of convergence. The Laurent series

$$f(z) = \frac{5}{z-3} - \frac{1}{3} - \frac{1}{18}(z-3) - \frac{1}{108}(z-3)^2 - \frac{1}{648}(z-3)^3 - \frac{1}{3888}(z-3)^4 - \ldots$$

has only one power -1 and all others are higher. Hence $f(z)$ has a first order singularity at $z = 3$. It also implies $\lim_{z\to 3} f(z) = \infty$ and $\operatorname{Res}(f,3) = c_{-1} = 5$.

3. Write the following groups as amalgamated products of cyclic groups:

   (a) $G = \langle x, y \,|\, x^3y^{-3}, y^6 \rangle$

   (b) $H = \langle x, y \,|\, x^{30}, y^{70}, x^3y^{-5} \rangle$

   Reason: Amalgamations.

   Solution: By definition we have groups $F(x,y)/N$ where $F(x,y)$ is the free group generated by two elements and N the normal subgroup generated by the given relations. We prove

   (a) $G \cong \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$

   Since $N \trianglelefteq G$ is normal, we get from $x^3y^{-3}, y^6 \in N$

   $$x^6 = (x^3y^{-3}) \cdot y^3x^3 = (x^3y^{-3}) \cdot \left(y^3(x^3y^{-3})y^{-3} \cdot y^6\right) \in N$$

   and so

   $$G = \langle x, y \,|\, x^3y^{-3}, y^6 \rangle = \langle x, y \,|\, x^3y^{-3}, y^6, x^6 \rangle$$

Hence it is sufficient to show that the amalgamated product

$$\mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$$

has this presentation, too.
We have the free product

$$\mathbb{Z}/6\mathbb{Z} * \mathbb{Z}/6\mathbb{Z} \cong \langle x \,|\, x^6 \rangle * \langle y \,|\, y^6 \rangle = \langle x, y \,|\, x^6, y^6 \rangle$$

With $\mathbb{Z}/2\mathbb{Z} = \langle t \,|\, t^2 \rangle$ we have two inclusions

$$\iota_1 : t \longmapsto x^3 \,,\ \iota_2 : t \longmapsto y^3$$

of $\mathbb{Z}/2\mathbb{Z}$ into the two factors $\mathbb{Z}/6\mathbb{Z}$. By definition of the amalgamated product we thus have

$$\begin{aligned}\mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} &= (\mathbb{Z}/6\mathbb{Z} * \mathbb{Z}/6\mathbb{Z}) \,/ \langle \iota_1(u)\iota_2^{-1}(u) \,|\, u \in \mathbb{Z}/2\mathbb{Z} \rangle \\ &\cong \langle x, y \,|\, x^6, y^6 \rangle / \langle \iota_1(t)\iota_2^{-1}(t) \rangle \\ &= \langle x, y \,|\, x^6, y^6 \rangle / \langle x^3 y^{-3} \rangle \\ &= \langle x, y \,|\, x^6, y^6, x^3 y^{-3} \rangle = G\end{aligned}$$

(b) $H \cong \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/10\mathbb{Z}$

As before with $\mathbb{Z}/2\mathbb{Z} = \langle t \,|\, t^2 \rangle$ we have two inclusions

$$\iota_1 : t \longmapsto a^3 \,,\ \iota_2 : t \longmapsto b^5$$

of $\mathbb{Z}/2\mathbb{Z}$ into the two groups $\mathbb{Z}/6\mathbb{Z}$, $\mathbb{Z}/10\mathbb{Z}$. By definition of the amalgamated product we thus have

$$\begin{aligned}\mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/10\mathbb{Z} &= (\mathbb{Z}/6\mathbb{Z} * \mathbb{Z}/10\mathbb{Z}) \,/ \langle \iota_1(u)\iota_2^{-1}(u) \,|\, u \in \mathbb{Z}/2\mathbb{Z} \rangle \\ &\cong \langle a, b \,|\, a^6, b^{10} \rangle / \langle \iota_1(t)\iota_2^{-1}(t) \rangle \\ &= \langle a, b \,|\, a^6, b^{10} \rangle / \langle a^3 b^{-5} \rangle \\ &= \langle a, b \,|\, a^6, b^{10}, a^3 b^{-5} \rangle =: H'\end{aligned}$$

We now have to find an isomorphism $\varphi : H \longrightarrow H'$.

The mapping $x \longmapsto a$, $y \longmapsto b$ induces a homomorphism $\tilde{\varphi} : F(x,y) \longrightarrow H'$ by the universal property of the free group. Now

$$\begin{aligned}\tilde{\varphi}(x^{30}) &= \tilde{\varphi}(x)^{30} = (a^6)^5 = 1^5 = 1 \\ \tilde{\varphi}(y^{70}) &= \tilde{\varphi}(y)^{70} = (b^{10})^7 = 1^7 = 1 \\ \tilde{\varphi}(x^3 y^{-5}) &= \tilde{\varphi}(x)^3 \tilde{\varphi}(y)^{-5} = a^3 b^{-5} = 1\end{aligned}$$

Since all relations of H are mapped onto 1, there is a homomorphism $\varphi : H \longrightarrow H'$ by the universal property of the presentation of $H = \langle x, y \,|\, x^{30}, y^{70}, x^3y^{-5} \rangle$ such that $\varphi \circ \pi = \tilde{\varphi}$ with the canonical projection $\pi : F(x, y) \longrightarrow H$. In order to show that φ is actually an isomorphism, we construct a homomorphism $\psi : H' \longrightarrow H$ which is inverse to φ .

As before, this time by mapping $a \longmapsto x$, $b \longmapsto y$, we get a homomorphic mapping $\psi : H' \longrightarrow H$. In order that this mapping is well-defined, we must ensure that the relations in H' are mapped onto $1 \in H$, i.e. we must show $x^6 = y^{10} = 1$.

From $x^3y^{-5} = 1$ we get $H \ni 1 = x^{30} = y^{50}$, hence

$$y^{10} = y^{-200}y^{210} = (y^{50})^{-4} \cdot (y^{70})^3 = 1^{-4} \cdot 1^3 = 1$$

and

$$x^6 = (x^3)^2 = (y^5)^2 = y^{10} = 1$$

So ψ is a well-defined homomorphism. By their mappings of the generators, it is obvious that they are inverse to one another and

$$H \cong H' = \mathbb{Z}/6\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/10\mathbb{Z}$$

4. Prove that there are uncountably many groups, which are generated by two elements, and not finitely presented.

   Hint: There are uncountably many non-isomorphic groups with two generators [Bernhard Neumann, 1937].

   Reason: Group Presentations.

   Solution: Assume there are countably many groups, which are generated by two elements, and not finitely presented. We show that the set X of groups, which are generated by two elements, and are finitely presented, is countable. If both those sets are countable, then so is their union, contradicting the given hint.

   Let $G \in X$ with generators a, b . The set of possibly not reduced words of length k over the alphabet $\{ a, b, a^{-1}, b - 1 \}$ has $4^k < \infty$ elements. Thus the set of those words of length not greater than k is also finite. Hence there are only finitely many possibilities for m many relations r of length $l(r) \leq k$. So there are only finitely many possible groups

$$G(m, k, R) = \langle a, b \,|\, R\, ;\, |R| \leq m \text{ and } \max_{r \in R} l(r) \leq k \rangle$$

and the set of those groups $X_{m,k} := \{\, G(m,k,R) \,|\, R \subseteq F(a,b) \,\}$ is finite, and so is the countably infinite union

$$X = \bigcup_{m,k\in\mathbb{N}} X_{m,k}$$

5. Let $f : [1,\infty) \longrightarrow [0,\infty)$ be a continuously differentiable function. Write S for the solid of revolution of the graph $y = f(x)$ about the $x-$axis. If the surface area of S is finite, then so is the volume.

   Reason: Gabriel's Horn.

   Solution: Since the surface area is finite we get

$$\begin{aligned}
\limsup_{t\to\infty} \sup_{x\geq t} f(x)^2 - f(1) &= \limsup_{t\to\infty} \int_1^t \left(f(x)^2\right)' \,dx \\
&\leq \int_1^\infty \left|\left(f(x)^2\right)'\right| \,dx \\
&= 2\int_1^\infty f(x)|f'(x)|\,dx \\
&\leq 2\int_1^\infty f(x)\sqrt{1+f'(x)^2}\,dx \\
&= \frac{A}{\pi} < \infty
\end{aligned}$$

   Hence there is a $t_0 \geq 1$ such that $\sup_{x\geq t_0} f(x) < \infty$ and so is $L := \sup_{x\geq 1} f(x) < \infty$ because $f(x)$ is continuous with values in $[0,\infty)$, i.e. bounded on $[1,\infty)$. For the volume we have

$$\begin{aligned}
V &= \int_1^\infty f(x)\cdot \pi f(x)\,dx \\
&\leq \int_1^\infty \frac{L}{2}\cdot 2\pi f(x)\,dx \\
&\leq \frac{L}{2}\int_1^\infty 2\pi f(x)\sqrt{1+f'(x)^2}\,dx \\
&= \frac{L}{2}\cdot A \\
&< \infty
\end{aligned}$$

6. Calculate $\sum_{k,j=1}^{\infty} \dfrac{1}{kj(k+j)^2}$

   Reason: Harmonic Series.

Solution: We will make use of the Taylor expansion at $x = 0$

$$\frac{1}{1-x}\log\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} H_n x^n \text{ where } H_n = \sum_{k=0}^{n}\frac{1}{k} \text{ and } |x| < 1 \quad (*)$$

Let $S := \sum_{k,j=1}^{\infty} \dfrac{1}{kj(k+j)^2}$ which is

$$\begin{aligned}
S &= \sum_{k,j=1}^{\infty}\frac{1}{kj}\int_0^1 x^{k+j}\,\frac{dx}{x}\int_0^1 y^{k+j}\,\frac{dy}{y} = \sum_{k,j=1}^{\infty}\frac{1}{kj}\int_0^1\int_0^1 (xy)^{k+j}\frac{dx\,dy}{xy}\\
&= \int_0^1\int_0^1\sum_{k,j=1}^{\infty}\frac{(xy)^k}{k}\,\frac{(xy)^j}{j}\,\frac{dx\,dy}{xy} = \int_0^1\int_0^1\frac{\log^2(1-xy)}{xy}\,dx\,dy\\
&\overset{u=xy}{=} \int_0^1\int_0^y\frac{\log^2(1-u)}{u}\,\frac{du}{y}\,dy = \int_0^1\int_u^1\frac{\log^2(1-u)}{u}\,\frac{dy}{y}\,du\\
&= \int_0^1\frac{\log^2(1-u)}{u}\cdot(-\log(u))\,du \overset{v=1-u}{=} \int_1^0\frac{\log^2 v}{1-v}\log(1-v)\,dv\\
&= \int_0^1\left(\frac{1}{1-v}\right)\log\left(\frac{1}{1-v}\right)(\log^2 v)\,dv = \int_0^1\sum_{n=0}^{\infty}H_n v^n\,\log^2 v\,dv\\
&= \sum_{n=0}^{\infty}H_n\int_0^1 v^n\,\log^2 v\,dv\\
&= \sum_{n=0}^{\infty}H_n\left[\frac{v^{n+1}\left((n+1)^2\log^2 v - 2(n+1)\log v + 2\right)}{(n+1)^3}\right]_0^1\\
&= 2\sum_{n=1}^{\infty}H_{n-1}\frac{1}{n^3} = 2\left(\sum_{n=1}^{\infty}\frac{H_n}{n^3} - \sum_{n=1}^{\infty}\frac{1}{n^4}\right)\\
&= 2\left(\frac{\pi^4}{72} - \zeta(4)\right) = \frac{\pi^4}{36} - \frac{\pi^4}{45} = \frac{\pi^4}{180}
\end{aligned}$$

7. Calculate $S := \displaystyle\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{3^k(2n-2k)!(2k)!}{2^k 8^n[(n-k)!]^2[k!]^2(2n(1+2k)+(1-4k^2))}$

   Reason: Cauchy Product.

   Solution: We first clean up the various parts of the quotient. The faculties are $\binom{2n-2k}{n-k}\binom{2k}{k}$, the powers are $\dfrac{3^k}{16^k}\cdot\dfrac{1}{8^{n-k}}$, and the polynomial part is $(2n+1-2k)(1+2k)$. Thus we can write the series as a Cauchy product

$$\begin{aligned} S &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2k}{k}\frac{3^k}{16^k(2k+1)}\cdot\binom{2n-2k}{n-k}\frac{1}{8^{n-k}(2n-2k+1)} \\ &= \left(\sum_{n=0}^{\infty}\binom{2n}{n}\frac{3^n}{16^n(2n+1)}\right)\left(\sum_{m=0}^{\infty}\binom{2m}{m}\frac{1}{8^m(2m+1)}\right) \\ &= \left(\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(\frac{\sqrt{3}}{2})^{2n}}{4^n(2n+1)}\right)\left(\sum_{m=0}^{\infty}\binom{2m}{m}\frac{(\frac{1}{\sqrt{2}})^{2m}}{4^m(2m+1)}\right) \\ &= \frac{2}{\sqrt{3}}\arcsin\left(\frac{\sqrt{3}}{2}\right)\cdot\sqrt{2}\arcsin\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{2}{\sqrt{3}}\cdot\frac{\pi}{3}\cdot\sqrt{2}\cdot\frac{\pi}{4} = \frac{1}{3\sqrt{6}}\pi^2 \end{aligned}$$

8. Solve $y'x - y = \sqrt{x^2 - y^2}$

Reason: Jacobian Differential Equation.

Solution: $y'x - y = \sqrt{x^2 - y^2}$ can be transformed into a Jacobian differential equation. First we divide x and substitute $z = \dfrac{y}{x}$ so we get

$$y' = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2} = z + \sqrt{1 - z^2}$$

with

$$z' = \frac{y'x - y}{x^2} = \frac{\sqrt{x^2 - y^2}}{x^2} = \frac{1}{x}\sqrt{1 - z^2} = \frac{1}{x}\left[\underbrace{z + \sqrt{1 - z^2}}_{:=g(z)=y'} - z\right]$$

Now we have

$$\int \frac{dz}{g(z) - z} = \int \frac{dz}{\sqrt{1 - z^2}} = \arcsin(z) + C = \int \frac{dx}{x} = \log|x| + C'$$

hence

$$y = x \cdot \sin\left(\log|x| + C\right)$$

We cannot rule out that $g(z_0) = z_0$ for some value z_0. This means we have $\sqrt{1 - z_0^2} = 0$ or $z_0 = \pm 1$, i.e. $y = \pm x$ are also solutions.

9. Let $(a_n)_{n\in\mathbb{N}}\,,\,(b_n)_{n\in\mathbb{N}}\subseteq\mathbb{R}_{\geq 0}$ be two sequences of non-negative numbers, where not all sequence elements vanish, and be $p,q\in\mathbb{R}$ with $1<p,q<\infty\,,\,\frac{1}{p}+\frac{1}{q}=1$. Prove

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_nb_m}{n+m}<\frac{\pi}{\sin(\pi/p)}\cdot\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\cdot\left(\sum_{m=1}^{\infty}b_m^q\right)^{\frac{1}{q}}$$

Reason: Hilbert's Inequality.

Solution: $f(x)=\frac{1}{(1+x)x^\alpha}$ is for $0<\alpha<1$ strictly monotone decreasing, hence

$$\sum_{m=1}^{\infty}\frac{1}{(1+\frac{m}{n})\cdot(\frac{m}{n})^\alpha}\cdot\frac{1}{n}\overset{\text{Riemann sum}}{<}\int_0^\infty\frac{dx}{(1+x)x^\alpha}\overset{(*)}{=}\frac{\pi}{\sin\pi\alpha}$$

Proof of $(*)$:

$$\begin{aligned}\int_0^\infty\frac{dx}{(1+x)x^\alpha}&=\int_0^\infty x^{-\alpha}\int_0^\infty e^{-(1+x)t}\,dt\,dx\\&=\int_0^\infty\int_0^\infty x^{-\alpha}e^{-t}e^{-xt}\,dt\,dx\\&\overset{u=tx}{=}\int_0^\infty\int_0^\infty\left(\frac{u}{t}\right)^{-\alpha}e^{-u}e^{-t}t^{-1}\,du\,dx\\&=\int_0^\infty u^{-\alpha}e^{-u}\,du\int_0^\infty t^{\alpha-1}e^{-t}\,dt\\&=\Gamma(1-\alpha)\Gamma(\alpha)\overset{\alpha\notin\mathbb{Z}}{=}\frac{\pi}{\sin\pi\alpha}\end{aligned}$$

Now we have

$$\sum_{n,m=1}^{\infty}\frac{a_nb_m}{n+m}=\sum_{n,m=1}^{\infty}\frac{a_n}{(n+m)^{1/p}\left(\frac{m}{n}\right)^{1/(pq)}}\cdot\frac{b_m}{(n+m)^{1/q}\left(\frac{n}{m}\right)^{1/(pq)}}$$

$$\overset{\text{Hölder}}{\leq}\left(\sum_{n,m=1}^{\infty}\frac{a_n^p}{(n+m)\left(\frac{m}{n}\right)^{\frac{1}{q}}}\right)^{\frac{1}{p}}\cdot\left(\sum_{n,m=1}^{\infty}\frac{b_m^q}{(n+m)\left(\frac{n}{m}\right)^{\frac{1}{p}}}\right)^{\frac{1}{q}}$$

$$=\left(\sum_{n=1}^{\infty}a_n^p\cdot\sum_{m=1}^{\infty}\frac{1}{(1+\frac{m}{n})\left(\frac{m}{n}\right)^{\frac{1}{q}}}\cdot\frac{1}{n}\right)^{\frac{1}{p}}$$

$$\cdot\left(\sum_{m=1}^{\infty}b_m^q\cdot\sum_{n=1}^{\infty}\frac{1}{(\frac{n}{m}+1)\left(\frac{n}{m}\right)^{\frac{1}{p}}}\cdot\frac{1}{m}\right)^{\frac{1}{q}}$$

$$<\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\cdot\left(\sum_{m=1}^{\infty}b_m^q\right)^{\frac{1}{q}}\cdot\left(\frac{\pi}{\sin\left(\frac{\pi}{q}\right)}\right)^{\frac{1}{p}}\cdot\left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\right)^{\frac{1}{q}}$$

$$\overset{(**)}{=}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\cdot\left(\sum_{m=1}^{\infty}b_m^q\right)^{\frac{1}{q}}\cdot\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}$$

Proof of $(**)$:

$$\sin\frac{\pi}{q}=\sin\left(\pi\left(1-\frac{1}{p}\right)\right)=\sin\pi\cos\left(\frac{\pi}{p}\right)-\cos\pi\sin\left(\frac{\pi}{p}\right)=\sin\frac{\pi}{p}$$

10. Let $f\,:\,\mathbb{R}_{\geq 0}\longrightarrow\mathbb{R}_{\geq 0}$ be an integrable function and $p>1$. Prove

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx\leq\left(\frac{p}{p-1}\right)^p\int_0^{\infty}(f(x))^p\,dx$$

Hint: Substitute $t=xu^{p/r}$ and at the end $r=p-1$.

Reason: Hardy's Inequality for Integrals.

Solution: Let $F(x):=\int_0^x f(t)\,dt$ which becomes by the substitution $t=xu^{p/r}$, $dt=x\frac{p}{r}u^{-1+p/r}du$, $u=t^{r/p}x^{-r/p}$, $du=\frac{r}{p}t^{-1+r/p}x^{-r/p}$

$$F(x)^p=\left(\int_0^x f(t)\,dt\right)^p=x^p\left(\frac{p}{r}\right)^p\left(\int_0^1 f\left(xu^{\frac{p}{r}}\right)u^{\frac{p}{r}-1}\,du\right)^p$$

Since $u \longmapsto u^p$ is convex, we can apply Jensen's theorem for convex functions (see October 2019 / 4b) and get

$$\begin{aligned}\left(\int_0^1 f\left(xu^{\frac{p}{r}}\right) u^{\frac{p}{r}-1}\,du\right)^p &\leq \int_0^1 \left[f\left(xu^{\frac{p}{r}}\right)\right]^p \cdot u^{p(\frac{p}{r}-1)}\,du \\ &= \int_0^x f(t)^p t^{p-r} x^{r-p} \left(\frac{r}{p}\right) t^{\frac{r}{p}-1} x^{-\frac{r}{p}}\,dt \\ &= \frac{x^{r-\frac{r}{p}}}{x^p} \cdot \frac{r}{p} \cdot \int_0^x f(t)^p\, t^{p-r+\frac{r}{p}-1}\,dt\end{aligned}$$

Hence

$$F(x)^p \leq x^{r-\frac{r}{p}} \left(\frac{p}{r}\right)^{p-1} \int_0^x f(t)^p\, t^{p-r+\frac{r}{p}-1}\,dt$$

$$\begin{aligned}\int_0^\infty F(x)^p x^{-r-1}\,dx &\leq \int_0^\infty x^{r-\frac{r}{p}} \left(\frac{p}{r}\right)^{p-1} \int_0^x f(t)^p\, t^{p-r+\frac{r}{p}-1}\,dt\,x^{-r-1}\,dx \\ &= \left(\frac{p}{r}\right)^{p-1} \int_0^\infty \int_0^x f(t)^p\, t^{p-r+\frac{r}{p}-1}\, x^{-\frac{r}{p}-1}\,dt\,dx \\ &= \left(\frac{p}{r}\right)^{p-1} \int_0^\infty f(t)^p\, t^{p-r+\frac{r}{p}-1} \int_t^\infty x^{-\frac{r}{p}-1}\,dx\,dt \\ &= \left(\frac{p}{r}\right)^{p} \int_0^\infty f(t)^p\, t^{p-r-1}\,dt\end{aligned}$$

With $r = p-1$ we get

$$\int_0^\infty F(x)^p x^{-p}\,dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(t)^p\,dt$$

which had to be shown.

11. (HS-1) Choose any odd prime, square it and subtract one. Show that the result is always divisible by twenty-four except for three. What can be said, if we take the prime up to the power four, and subtract one?

 Reason: Divisibility.

 Solution: Let p be the chosen prime. Then the number we get is $n(p) = p^2-1 = (p-1)(p+1)$. $n(3) = 8$ is obviously not divisible by 24, but e.g. $n(5) = 24$, $n(7) = 48$ are. Now let us assume $p > 3$. Since p is odd, $p \pm 1$ are both even, hence $4 = 2 \cdot 2 \,|\, n(p)$. But from to consecutive even numbers, one has to be divisible by 4, which means $8 = 2 \cdot 4 \,|\, n(p)$. Finally we have three consecutive numbers $p-1, p, p+1$ and one of them has to be divisible by three. As it cannot be p by assumption,

$3 \,|\, n(p)$. Because 3 and 8 are coprime, we even get $3 \cdot 8 = 24 \,|\, n(p)$.

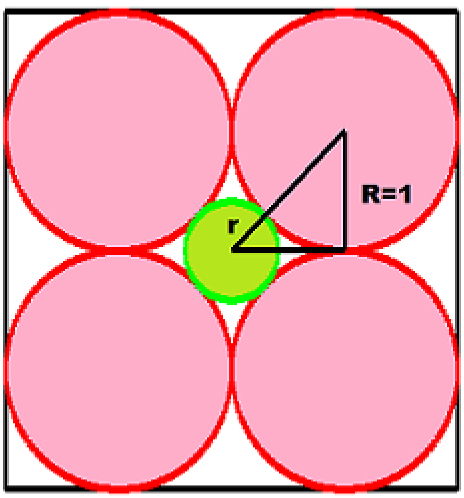
In this case we have $m(p) = p^4 - 1 = (p^2 - 1)(p^2 + 1) = (p - 1)(p + 1)(p^2 + 1)$. For small primes we get $m(3) = 80$, $m(5) = 624$, $m(7) = 2401$, $m(11) = 14,460$. Let us assume $p > 5$. As before we get $24 \,|\, m(p)$. Since $p^2 + 1$ is always even for odd values of p, we have another factor 2 and $48 \,|\, m(p)$. Now primes greater than 5 can only have one of the digits $\{1, 3, 7, 9\}$ as their last one.
If $p \equiv 1 \mod 10$ then $5 \,|\, (p - 1)$.
If $p \equiv 3, 7 \mod 10$ then $5 \,|\, (p^2 + 1)$ since $3^{+}1 = 10$, $7^2 + 1 = 50$ and for $p = a \cdot 10 + r > 11$ we get for

$$p^2 + 1 = (a \cdot 10 + r)^2 = 100a^2 + 20ar + r^2 \equiv r^2 \mod 10$$

and again a zero at the end for $r = 3, 7$.
If $p \equiv 9 \mod 10$ then $5 \,|\, (p + 1)$.
Thus we have in total $5 \cdot 48 = 240 \,|\, m(p)$ again because 5 and 48 are coprime.

12. (HS-2) In a square of side length 4, there is a circle of radius 1 in each corner. In the center of the square is another circle that touches the other four. Analogously, in the three-dimensional case, in the center of a cube of edge length 4, there would be a sphere which would touch eight spheres of radius 1 placed in the corners of the cube. In which dimension does the central hypersphere become so large that it touches all sides of the hypercube?

Reason: Abstract Geometry.



Solution: Pythagoras gives us $(r+R)^2 = R^2 + R^2$ and for $R = 1$ we have $r = \sqrt{2} - 1$. Pythagoras applied once more for the next dimension results in

$$(r+R)^2 = (R^2 + R^2) + R^2 = 3R^2 \Longrightarrow r = \sqrt{3} - 1$$

which continues with every new dimension. So the radius of the sphere inside equals $r = \sqrt{n} - 1$ in dimension n. In dimension 4 the inner sphere is as big as the outer ones. In order to touch the boundary of the hypercube, we need $r \geq 2$, i.e. $n = 9$.
The nine-dimensional central hypersphere touches all 18 bounding sides, eight-dimensional hypercubes, of the nine-dimensional hypercube. In the ten-dimensional space, parts of the central hypersphere are even outside the ten-dimensional hypercube. The higher the dimensions get, the more this effect intensifies.

13. (HS-3) There is only one rule at Christmas at the world's richest family: The gifts have to be expensive, heavy and glamorous. So they all present statues of pure gold. It may be large figure, a tiger sculpture or an opulent candlestick. The eldest son who doesn't live at home anymore receives gifts of nine tons total, but none of which is heavier than a ton. He wants to bring home all of them, but only could rent trucks which can load three tons maximal. How many trucks are needed to at least be able to transport all gifts of gold at the same time?

    Reason: Gold Transport.

    Solution: If the gifts were ten statues of 900 kg each, then three trucks wouldn't be sufficient. Now we load the first truck until it carries at least two tons, which is possible, since all gifts weigh less than a ton. We do the same for truck number two and three. Hence we are left with less than three tons and a fourth truck will be sufficient.

14. (HS-4) Prove $\dfrac{a}{b+c} + \dfrac{b}{c+a} + \dfrac{c}{a+b} \geq \dfrac{3}{2}$ for $a, b, c > 0$

    Reason: Nesbitt's Inequality.

    Solution: As it is so important we recall the order of various mean values: Let $x_1,\ldots,x_n \in \mathbb{R}$. Then

$$\begin{aligned}
\overline{x}_{min} &= \min\{\,x_1,\ldots,x_n\,\} && \text{minimum}\\
\overline{x}_{harm} &= \frac{n}{\dfrac{1}{x_1}+\cdots+\dfrac{1}{x_n}} && \text{harmonic mean}\\
\overline{x}_{geom} &= \sqrt[n]{x_1\cdot\cdots\cdot x_n}\;,\; x_k>0 && \text{geometric mean}\\
\overline{x}_{arithm} &= \frac{x_1+\cdots+x_n}{n} && \text{arithmetic mean}\\
\overline{x}_{quadr} &= \sqrt{\frac{1}{n}\left(x_1^2+\ldots+x_n^2\right)} && \text{quadratic}\\
\overline{x}_{cubic} &= \sqrt[3]{\frac{1}{n}\left(x_1^3+\ldots+x_n^3\right)} && \text{cubic}\\
\overline{x}_{max} &= \max\{\,x_1,\ldots,x_n\,\} && \text{maximum}
\end{aligned}$$

$$\overline{x}_{min} \leq \overline{x}_{harm} \leq \overline{x}_{geom} \leq \overline{x}_{arithm} \leq \overline{x}_{quadr} \leq \overline{x}_{cubic} \leq \overline{x}_{max}$$

As a mnemonic we can think of $x_1=3, x_2=5$ where we have

$$3<\frac{15}{4}=3.75<\sqrt{15}\approx 3.87<4<\sqrt{17}\approx 4.12<\sqrt[3]{76}\approx 4.24<5$$

This means in our situation

$$\begin{aligned}
\frac{(a+b)+(b+c)+(c+a)}{3} &\geq \frac{3}{\dfrac{1}{a+b}+\dfrac{1}{b+c}+\dfrac{1}{c+a}}\\
2(a+b+c)\cdot\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right) &\geq 9\\
1+\frac{c}{a+b}+1+\frac{a}{b+c}+1+\frac{b}{c+a} &\geq \frac{9}{2}\\
\frac{c}{a+b}+\frac{a}{b+c}+\frac{b}{c+a} &\geq \frac{3}{2}
\end{aligned}$$

15. (HS-5) Let $\mathbf{x}=(x_1,\ldots,x_n)\,,\,\mathbf{y}=(y_1,\ldots,y_n)$ be tuples of positive numbers. Prove

$$\prod_{k=1}^{n}(x_k+y_k)^{1/n} \geq \prod_{k=1}^{n}x_k^{1/n}+\prod_{k=1}^{n}y_k^{1/n}$$

Reason: Mahler's Inequality.

Solution: By the arithmetic-geometric mean inequality we have

$$\prod_{k=1}^{n}\left(\frac{x_k}{x_k+y_k}\right)^{1/n} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{x_k}{x_k+y_k}\,,\;\prod_{k=1}^{n}\left(\frac{y_k}{x_k+y_k}\right)^{1/n} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{y_k}{x_k+y_k}$$

Hence

$$\prod_{k=1}^{n}\left(\frac{x_k}{x_k+y_k}\right)^{1/n} + \prod_{k=1}^{n}\left(\frac{y_k}{x_k+y_k}\right)^{1/n} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{x_k+y_k}{x_k+y_k} = 1$$

and multiplying with the denominator

$$\prod_{k=1}^{n} x_k^{1/n} + \prod_{k=1}^{n} y_k^{1/n} \leq \prod_{k=1}^{n}(x_k+y_k)^{1/n}$$

27 November 2019

1. If f has the real Fourier representation

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)$$

prove

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx=\frac{a_0^2}{2}+\sum_{k=1}^{\infty}\left(a_k^2+b_k^2\right)$$

Reason: Parseval Equation.

Solution: We can write f as complex Fourier series

$$f(x)=\sum_{k=-\infty}^{\infty}c_ke^{ikx}\text{ with }2c_k=\begin{cases}a_k-ib_k & ,\,k>0\\ a_0 & ,\,k=0\\ a_k+ib_k & ,\,k<0\end{cases}$$

Then $|f(x)|^2=f(x)\overline{f(x)}=\sum_{k,l=-\infty}^{\infty}c_k\overline{c_l}e^{i(k-l)x}$ and

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx&=\frac{1}{\pi}\sum_{k,l=-\infty}^{\infty}c_k\overline{c_l}\underbrace{\int_{-\pi}^{\pi}e^{i(k-l)x}\,dx}_{=2\pi\delta_{k,l}}\\&=2\sum_{k=-\infty}^{\infty}|c_k|^2=\frac{1}{2}\sum_{k=-\infty}^{\infty}|2c_k|^2\\&=\frac{1}{2}\left(\sum_{k=-\infty}^{-1}|a_k+ib_k|^2+|a_0|^2+\sum_{k=1}^{\infty}|a_k-ib_k|^2\right)\\&=\frac{a_0^2}{2}+\sum_{k=1}^{\infty}(a_k^2+b_k^2)\end{aligned}$$

2. We define the weighted Hölder-mean as

$$M_w^p:=\left(\sum_{k=1}^{n}w_kx_k^p\right)^{\frac{1}{p}}\,,\;M_w^0:=\lim_{p\to 0}M_w^p=\prod_{k=1}^{n}x_k^{w_k}$$

for positive, real numbers $x_1,\ldots,x_n>0$ and a weight $w=(w_1,\ldots,w_n)$ with $w_1+\ldots+w_n=1$, $w_k>0$ and a $p\in\mathbb{R}-\{\,0\,\}$.

Prove $M_w^r \leq M_w^s$ whenever $r < s$.

Hint: Use Jensen's theorem for convex functions (see October 2019 / 4a).

Reason: Inequality of Weighted Hölder Means.

Solution: By the rule of L'Hôpital we get

$$\begin{aligned}\log \prod_{k=1}^{n} x_k^{w_k} &= \sum_{k=1}^{n} w_k \log x_k \\ &= \lim_{p\to 0} \frac{\sum_{k=1}^{n} w_k x_k^p \log x_k}{\sum_{k=1}^{n} w_k x_k^p} \\ &= \lim_{p\to 0} \frac{(\log \sum_{k=1}^{n} w_k x_k^p)'}{(p)'} \\ &= \lim_{p\to 0} \frac{\log \sum_{k=1}^{n} w_k x_k^p}{p} \\ &= \lim_{p\to 0} \log (M_w^p)\end{aligned}$$

We now apply Jensen's inequality (see see October 2019 / 4a) for functions $x \longmapsto x^q$ which are convex for $q \geq 1$, $x > 0$.

(a) $0 < r < s$.
In this case $q = \frac{s}{r} > 1$ and

$$\left(\sum_{k=1}^{n} w_k x_k^r\right)^{\frac{s}{r}} \leq \sum_{k=1}^{n} w_k \left(x_k^r\right)^{\frac{s}{r}} \Longrightarrow \left(\sum_{k=1}^{n} w_k x_k^r\right)^{\frac{1}{r}} \leq \left(\sum_{k=1}^{n} w_k x_k^s\right)^{\frac{1}{s}}$$

(b) $r < s < 0$.
In this case $0 < -s < -r$ and by the previous case we have

$$\left(\sum_{k=1}^{n} w_k x_k^{-s}\right)^{\frac{1}{-s}} \leq \left(\sum_{k=1}^{n} w_k x_k^{-r}\right)^{\frac{1}{-r}} \Longrightarrow \left(\sum_{k=1}^{n} w_k x_k^{-r}\right)^{\frac{1}{r}} \leq \left(\sum_{k=1}^{n} w_k x_k^{-s}\right)^{\frac{1}{s}}$$

As we have proven it for any $x_k > 0$ we have proven it for $\frac{1}{x_k}$ as well, which is what had to be shown.

(c) $r = 0$ or $s = 0$.
Since $\lim_{r\to 0} M_w^r = M_w^0 = \lim_{s\to 0} M_w^s$ the inequality $M_w^r \leq M_w^s$ holds true for $0 \leq r < s$ and $r < s \leq 0$, too.

(d) $r < 0 < s$.
This case follows from the transitive ordering $M_w^r \leq M_w^0 \leq M_w^s$.

3. (HS-1) Mr. Smith on a full up flight with 50 passengers on a CRJ100 had lost his boarding pass. The flight attendant tells him to sit anywhere. All other passengers sit on their booked seats, unless it is already occupied, in which case they randomly choose another seat just like Mr. Smith did. What are the chances that the last passenger gets the seat printed on his boarding pass?

   Reason: Combinatorics.

   Solution: Let's say passengers in the boarding queue and seats are numbered $1, \ldots, 50$ and Mr.Smith is passenger 1. He could choose seat number one and seat number fifty with the same probability. Either case determines whether passenger 50 gets his correct seat or not. Now if he chooses, say seat number 25, passengers $1, \ldots, 24$ can seat correctly and passenger 25 is now in the same situation Mr. Smith had been at the beginning, i.e. seat with the same probability on seat 1 or seat 50 which again determines the last passenger's fate by the same probability. If passenger 25 chooses another seat, then our situation loops.

   These considerations show that ultimately only occupancies of places 1 and 50 are important. Once a passenger has chosen one of these two places at random, the outcome of the story is decided. If it's number one, passenger 50 will sit right. If it is number 50, it will not work anymore. How often passengers sit on seats during boarding that are not theirs, does not matter - as long as neither number 1 nor number 50 is affected. So the probability is exactly 0.5.

4. (HS-2) On the first flight day of a little island hopper there was no wind during the return flight. How does the total flight duration from outward and return flight change if, instead, a strong headwind blows on the way to the neighboring island - and on the way back, an equally strong tailwind?

   Reason: Wind and Flight Duration.

   Solution: Let us assume the flight path is of length 1 one way, at speed v and wind w. Then we need a time of $F_0 = \frac{2}{v}$ without wind. With wind, we need a time $F_1 = \frac{1}{v-w} + \frac{1}{v+w} = \frac{2v}{v^2 - w^2}$. Hence

$$v^2 - w^2 < v^2 \Longleftrightarrow F_0 = \frac{2}{v} < \frac{2v}{v^2 - w^2} = F_1$$

   and the complete flight is longer with wind.

28 October 2019

1. Let $A=\sum_{k=0}^\infty a_k$, $B=\sum_{k=0}^\infty b_k$ be two convergent series one of which absolutely. The Cauchy-product $C=\sum_{k=0}^\infty c_k$ with $c_k=\sum_{j=0}^k a_jb_{k-j}$ converges then to AB. Give an example that absolute convergence of one factor is necessary.

   Reason: Mertens' Theorem.

   Solution: W.l.o.g. we assume that A converges absolutely. We note the partial sums $A_n=\sum_{k=0}^n a_k$, $B_n=\sum_{k=0}^n b_k$.

$$
\begin{aligned}
AB &= (A-A_n)B+\sum_{k=0}^n a_kB\\
S_n &= \sum_{k=0}^n c_k=\sum_{k=0}^n\sum_{j=0}^k a_jb_{k-j}=\sum_{k=0}^n a_kB_{n-k}\\
AB-S_n &= (A-A_n)B+\sum_{k=0}^n a_k(B-B_{n-k})
\end{aligned}
$$

   The first term converges to 0 and with $N:=\lfloor\frac{n}{2}\rfloor$ we can write the second term

$$
\sum_{k=0}^N(B-B_{n-k})=\underbrace{\sum_{k=0}^n a_k(B-B_{n-k})}_{=P_n}+\underbrace{\sum_{k=N+1}^n a_k(B-B_{n-k})}_{=Q_n}
$$

   For P_n we have

$$
|P_n|\leq\sum_{k=0}^N|a_k|\cdot|B-B_{n-k}|\leq\max_{N\leq k\leq n}|B-B_k|\cdot\sum_{k=0}^N|a_k|\longrightarrow 0
$$

   because A converges absolutely and $(B-B_k)_k$ is a bounded sequence converging to 0, i.e. there is a constant c such that $|B-B_k|<c$ for all $k\in\mathbb{N}_0$. Therefore we get

$$
|Q_n|\leq\sum_{k=N+1}^n|a_k|\cdot|B-B_{n-k}|\leq c\sum_{k=N+1}^n|a_k|\longrightarrow 0
$$

   by the Cauchy criterion. Hence $AB-S_n\longrightarrow 0$ or $S_n\longrightarrow AB$.

   An example for the necessity of absolute convergence for at least one factor is $A = B = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k+1}}$ where $A = B = -(\sqrt{2}-1)\zeta\left(\frac{1}{2}\right) \approx 0.605$. For a proof that the factors actually converge and the Cauchy product diverges, see the problems from September.

2. Prove that a T_0 topological group (Kolmogorov space) is already T_2 (Hausdorff space). Show that an infinite linear algebraic group with the Zariski topology is always T_0 but never T_2. Why the discrepancy?

   Reason: Topological Groups.

   Solution: Let G be a T_0 topological group. We first show that the singleton $\{e\}$ is a closed subset, where $e \in G$ is the neutral element.

   Given any element $x \in G$, there is either an open neighborhood U_x of x with $e \notin U_x$ or an open neighborhood V of e with $x \notin V$. In the latter case, we may assume that $V = V^{-1}$. If not then we replace V with $V \cap V^{-1}$. The homeomorphism $f : G \longrightarrow G\,,\, y \longmapsto xy$ maps $e \longmapsto x$. Let $U_x := f(V)$, which is an open neighborhood of $x = f(e)$ with $e \notin U_x$ since $e \in U_x$ would imply $e = f(y) = xy$ and $y = x^{-1} \in V$ contradicting $x \notin V = V^{-1}$. In any case we find for all $x \neq e$ and open neighborhood that doesn't contain e. We now take the union

$$U := \bigcup_{x \neq e} U_x$$

   over all these neighborhoods. By construction, this is an open set with $U = G - \{e\}$, so $\{e\}$ is indeed closed.

   The map $G \times G \longrightarrow G\,,\,(g,h) \longrightarrow gh^{-1}$ is continuous as G is a topological group. Its preimage of the closed subset $\{e\}$ is the diagonal $\Delta G = \{\,(g,g) \mid g \in G\,\}$ which is therefore closed. Any topological space X is T_2 if and only if the diagonal ΔX is a closed subset of $X \times X$.

   The statements about linear algebraic groups follow from general properties of the Zariski topology: points of affine varieties are closed because they correspond to maximal ideals; hence varieties are T_0 (and even T_1 but note that schemes have more points and are only T_0 in general). Since any two non-empty open subsets of an irreducible variety meet, varieties are never T_2. Infinity is crucial here.

   Explanation of the discrepancy: The above shows that while linear algebraic groups are groups with a topology, they are not topological groups! The reason is that the topology on $G \times G$ for a variety is not the product topology!

3. Let $\mathcal{O}(n)$ be the group of orthogonal real $n \times n$ matrices. For $f \in L^p = L^p(\mathbb{R}^n)$ we set
$$A.f(x) = f(A^{-1}x)$$
Show that $\mathcal{O}_f = \{\, A.f \mid A \in \mathcal{O}(n) \,\} \subseteq L^p(\mathbb{R}^n)$ is compact.

Reason: Orthogonal Groups and Hilbert Spaces.

Solution: We use the theorem of Fréchet-Riesz-Kolmogorov. The set $\mathcal{O}_f$ is closed and bounded:
$$||A.f||_p^p = \int_{\mathbb{R}^n} |A.f(x)|^p\,dx = \int_{\mathbb{R}^n} |f(A^{-1}x)|^p\,dx = \int_{\mathbb{R}^n} |f(x)|^p\,dx = ||f||_p^p$$
due to the transformation theorem of integration ($|\det A| = 1$).

Let $(A_n.f)$ be an L^p convergent sequence with limit $h \in L^p$. Since $\mathcal{O}(n)$ is compact, the sequence of A_n has a convergent subsequence with limit $A \in \mathcal{O}(n)$.
$$||A_n.f - A.f||_p^p = \int_{\mathbb{R}^n} |f(A_n^{-1}Ax) - f(x)|^p\,dx$$
and this converges to 0 for $n \to \infty$. Hence $h = A.f$ and the sequence $(A_n.f)$ converges to $A.f$, i.e. $\mathcal{O}_f \subseteq L^p$ is closed.

To obtain compactness by Fréchet-Riesz-Kolmogorov, we have to check two conditions.

(a) to be shown:
$$\lim_{t\to 0} \int_{\mathbb{R}^n} |A.f(x+t) - A.f(x)|^p\,dx \stackrel{\text{uniformly in } A}{\longrightarrow} 0$$
For $\varepsilon > 0$ there is a $\delta > 0$ with
$$||t|| < \delta \Longrightarrow \int_{\mathbb{R}^n} |f(x+t) - f(x)|^p\,dx < \varepsilon$$
and
$$\int_{\mathbb{R}^n} |A.f(x+t) - A.f(x)|^p\,dx = \int_{\mathbb{R}^n} |f(x + A^{-1}t) - f(x)|^p\,dx$$
and $||A^{-1}t|| = ||t||$. Hence we obtain $\int_{\mathbb{R}^n} |A.f(x+t) - A.f(x)|^p\,dx < \varepsilon$ for $||t|| < \delta$ so the first condition is fulfilled.

(b) to be shown: For every $\varepsilon > 0$ there is an $M > 0$ such that

$$\int_{\mathbb{R}^n - B_M(0)} |A.f(x)|^p \, dx < \varepsilon \; \forall \; A \in \mathcal{O}(n)$$

where $B_M(0)$ is the closed ball of radius M and center 0.

Since $f \in L^p$, we have for every $\varepsilon > 0$ an $M > 0$ with

$$\int_{\mathbb{R}^n - B_M(0)} |f(x)|^p \, dx < \varepsilon$$

But the ball $B_M(0)$ is invariant under $A \in \mathcal{O}(n)$, so we get $A(B_M(0)) = B_M(0)$ and with the transformation theorem of integration

$$\int_{\mathbb{R}^n - B_M(0)} |A.f(x)|^p \, dx = \int_{\mathbb{R}^n - B_M(0)} |f(x)|^p \, dx < \varepsilon$$

and the second condition of the theorem is fulfilled, i.e. $\mathcal{O}_f \subseteq L^p$ is compact.

4. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $\lambda_1, \ldots, \lambda_n$ positive weights, i.e. $\sum_{i=1}^n \lambda_i = 1$. Show that

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

(b) Let $g : [0,1] \longrightarrow \mathbb{R}$ be an integrable function such that the continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is convex on the image of g. Prove

$$f\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \leq \frac{1}{b-a}\int_a^b f(g(x))\,dx$$

(c) Prove without differentiation that the cylinder with the least surface area among the ones with given volume V is the cylinder whose height equals the diameter of its base.

(d) Prove that for any sequence $a_n \geq \ldots \geq a_1 > 0$ of positive real numbers

$$\frac{1}{\frac{1}{a_1}} + \frac{2}{\frac{1}{a_1}+\frac{1}{a_2}} + \ldots + \frac{n}{\frac{1}{a_1}+\ldots+\frac{1}{a_n}} < 2(a_1 + \ldots + a_n)$$

Reason: Jensen's Inequality.

Solution:

(a) The definition of convexity is

$$f(\lambda x - (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

which is our induction base. The step then is

$$\begin{aligned}
f\left(\sum_{i=1}^{n} \lambda_i x_i\right) &= f\left(\sum_{i=1}^{n-1} \lambda_i x_i + \lambda_n x_n\right) \\
&= f\left((1-\lambda_n)\underbrace{\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i}_{=:y} + \lambda_n x_n\right) \\
&\leq (1-\lambda_n)f(y) + \lambda_n f(x_n) \\
&= (1-\lambda_n) f\left(\sum_{i=1}^{n-1} \frac{\lambda_i}{1-\lambda_n} x_i\right) + \lambda_n f(x_n) \\
&\leq \sum_{i=1}^{n-1} \lambda_i f(x_i) + \lambda_n f(x_n) \\
&= \sum_{i=1}^{n} \lambda_i f(x_i)
\end{aligned}$$

(b) By the previous part we have for an integrable function $\varphi : [0,1] \longrightarrow \mathbb{R}$ such that f is convex on its image, $\lambda_k = \frac{1}{n}$ and $x_k = \varphi\left(\frac{k}{n}\right)$

$$f\left(\sum_{k=1}^{n} \varphi\left(\frac{k}{n}\right) \cdot \frac{1}{n}\right) \leq \sum_{k=1}^{n} f\left(\varphi\left(\frac{k}{n}\right)\right) \cdot \frac{1}{n}$$

which becomes by the limit $n \to \infty$

$$f\left(\int_0^1 \varphi(u)\,du\right) \leq \int_0^1 f(\varphi(u))\,du$$

Now we substitute $u = \dfrac{x-a}{b-a}$, $du = \dfrac{dx}{b-a}$, hence

$$f\left(\int_a^b \varphi\left(\frac{x-a}{b-a}\right)\frac{dx}{b-a}\right) \leq \int_0^1 f\left(\varphi\left(\frac{x-a}{b-a}\right)\right)\frac{dx}{b-a}$$

where we set $g(x)=\varphi\left(\dfrac{x-a}{b-a}\right)$ and get

$$f\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\leq \frac{1}{b-a}\int_a^b f(g(x))\,dx$$

(c) Let r,h,A,V be radius, height, surface and volume of the cylinder, resp. Then

$$\frac{A}{3\pi}=\frac{2r^2+rh+rh}{3}\stackrel{AM\geq GM}{\geq}\sqrt[3]{2r^2\cdot rh\cdot rh}=\sqrt[3]{\frac{2V^2}{\pi^2}}=: \text{ const. } >0$$

and equality holds for $h=2r$.

(d) From $a_n\geq\ldots\geq a_1>0$ we get

$$\frac{1}{a_1}+\ldots+\frac{1}{a_n}\geq\frac{n}{a_n}\Longrightarrow\frac{n}{\dfrac{1}{a_1}+\ldots+\dfrac{1}{a_n}}\leq a_n<2a_n$$

The inequality of our statement is clearly true for $n=1$. By induction we have

$$\begin{aligned}\sum_{k=1}^{n}\frac{k}{\dfrac{1}{a_1}+\ldots+\dfrac{1}{a_k}}&=\frac{n}{\dfrac{1}{a_1}+\ldots+\dfrac{1}{a_n}}+\sum_{k=1}^{n-1}\frac{k}{\dfrac{1}{a_1}+\ldots+\dfrac{1}{a_k}}\\&<\frac{n}{\dfrac{1}{a_1}+\ldots+\dfrac{1}{a_n}}+2(a_1+\ldots+a_{n-1})\\&<2a_n+2(a_1+\ldots+a_{n-1})\\&=2(a_1+\ldots+a_n)\end{aligned}$$

5. Let $p(x)=x^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ be a nonlinear polynomial with $a_n=1$ and suppose $(x-1)^{k+1}\,|\,p(x)$ for some positive integer k. Prove that

$$\sum_{j=0}^{n-1}|a_j|>1+\frac{2k^2}{n}$$

Hint: At some stage of the proof you will need Chebyshev polynomials.

Reason: Tricky polynomial inequality.

Solution: We first prove the following statement:

For any polynomial $q(y)$ with degree at most k, we have

$$\sum_{j=0}^{n} a_j q(j) = 0 \qquad (*)$$

We define for $0 \leq \nu \leq k$ the polynomials

$$\varphi_0(x) = 1\,,\; \varphi_\nu(x) = x(x-1)(x-2)\cdot\ldots\cdot(x-\nu+1)$$

and prove

$$\sum_{j=0}^{n} a_j \varphi_\nu(j) = p^{(\nu)}(1)$$

by induction on ν. For $v = 0$ we have $a_0 + \ldots + a_n = p(1)$ and for $v = 1$ it's

$$\begin{aligned} a_0\varphi_1(0) &+ a_1\varphi_1(1) + a_2\varphi_1(2) + \ldots + a_{n-1}\varphi_1(n-1) + a_n\varphi(n) \\ &= a_0 \cdot 0 + a_1 \cdot 1 + a_2 \cdot 2 + \ldots + a_{n-1}\cdot(n-1) + 1 \cdot n \\ &= p'(1) \end{aligned}$$

$$p^{(\nu)}(x) = \sum_{j=0}^{n} a_j \left(x^j\right)^{(\nu)} = \sum_{j=v}^{n} a_j \left(x^j\right)^{(\nu)} = \sum_{j=v}^{n} a_j \varphi_\nu(j) x^{j-\nu}$$

The induction step is now

$$\begin{aligned} p^{(v+1)}(1) &= \left(\sum_{j=v}^{n} a_j\varphi_\nu(j)x^{j-\nu}\right)'(1) = \left(\sum_{j=\nu+1}^{n} a_j\varphi_\nu(j)(j-\nu)x^{j-\nu-1}\right)(1) \\ &= \left(\sum_{j=\nu+1}^{n} a_j\varphi_{\nu+1}(j)x^{j-\nu-1}\right)(1) = \sum_{j=\nu+1}^{n} a_j\varphi_{\nu+1}(j) = \sum_{j=0}^{n} a_j\varphi_{\nu+1}(j) \end{aligned}$$

Since $\{\,\varphi_0, \varphi_1, \ldots, \varphi_k\,\}$ is a basis of the vector space of all polynomials up to degree k we may write $q(x) = \sum_{\nu=0}^{k} q_\nu \varphi_\nu(x)$ which gives us

$$\sum_{j=0}^{n} a_j q(j) = \sum_{j=0}^{n} a_j \sum_{\nu=0}^{k} q_\nu\varphi_\nu(j) = \sum_{\nu=0}^{k} q_\nu \sum_{j=0}^{n} a_j\varphi_\nu(j) = \sum_{\nu=0}^{k} q_\nu p^{(\nu)}(1) = 0$$

as $(x-1)^{k+1}\,|\,p(x)$, so $(*)$ is proven.

To prove the original statement now let

$$q(x) = T_k\left(\frac{2}{n-1}\,x - 1\right)$$

with the $k-$th Chebyshev polynomial.

`https://en.wikipedia.org/wiki/Chebyshev_polynomials`

Then $q(0),\ldots,q(n-1)\in T_k([-1,1])\subseteq[-1,1]$ and

$$\begin{aligned} q(n)=T_k\left(\frac{n+1}{n-1}\right)&=\cosh\left(k\cdot\operatorname{arcosh}\left(\frac{n+1}{n-1}\right)\right)\\ &=\cosh\left(k\cdot\log\left(\frac{n+1}{n-1}+\sqrt{\left(\frac{n+1}{n-1}\right)^2-1}\right)\right)\\ &=\cosh\left(k\cdot\log\left(\frac{(\sqrt{n}+1)^2}{n-1}\right)\right)=\cosh\left(k\cdot\log\left(\frac{\sqrt{n}+1}{\sqrt{n}-1}\right)\right)\\ &=\cosh\left(k\cdot\log\left(\frac{1+\frac{1}{\sqrt{n}}}{1-\frac{1}{\sqrt{n}}}\right)\right)>\cosh\left(k\cdot\frac{2}{\sqrt{n}}\right) \end{aligned}$$

where we have used that $n>1$, and that cosh is strictly monotone increasing for positive arguments, and $\log\left(\frac{1+x}{1-x}\right)>2x$ for $x<1$.

$$\frac{1+x}{1-x}=1+2\sum_{n=1}^{\infty}x^n>1+2x+2\cdot\frac{2}{2!}x^2+2\cdot\frac{2^{3-1}}{3!}x^3+2\cdot\frac{2^3}{4!}x^4+\ldots=e^{2x}$$

Note that by definition of $q(x)$ we have $q(0),\ldots,q(n-1)\in[-1,1]$ and we have shown

$$\sum_{j=0}^{n-1}|a_j|\geq\sum_{j=0}^{n-1}a_j(-q(j))\overset{(*)}{=}a_nq(n)=q(n)>\cosh\left(k\cdot\frac{2}{\sqrt{n}}\right)>1+\frac{2k^2}{n}$$

6. Consider the triangle $A=(0,0)$, $B=(2\sqrt{3},0)$, $C=(3-\sqrt{3},\,-3+3\sqrt{3})$. Now choose on each side a point, M_a,M_b,M_c, such that the new triangle built by those points is of minimal perimeter.
What is the area of the $\triangle(M_a,M_b,M_c)$?

Reason: Fagnano Triangle.

Solution: The solution to the optimization problem is the Fagnano or orthic triangle, which is built by the base points of all heights. In our triangle these are

$$\begin{aligned} M_a &= (\sqrt{3}\,,\,\sqrt{3}) & a &= \frac{1}{2}+\frac{1}{6}\sqrt{3} \\ M_b &= (\frac{1}{2}\sqrt{3}\,,\,\frac{3}{2}) & b &= \frac{1}{4}+\frac{1}{4}\sqrt{3} \\ M_c &= (3-\sqrt{3}\,,\,0) & c &= -\frac{1}{2}+\frac{1}{2}\sqrt{3} \end{aligned}$$

We take M_aM_c as baseline, which results in the straight line, height and base point equations

$$\begin{aligned} g \ &: \ \vec{x}_g = (3-2\sqrt{3},-\sqrt{3})^\tau \cdot g + (\sqrt{3},\sqrt{3})^\tau & 0 \leq g \leq 1 \\ h_g \ &: \ \vec{x} = (-\sqrt{3},-3+2\sqrt{3})^\tau \cdot h_q + (\frac{1}{2}\sqrt{3},\frac{3}{2})^\tau & 0 \geq h_q \geq -\frac{1}{4}\sqrt{3} \\ H \ &: \ (\frac{3}{4}+\frac{1}{2}\sqrt{3}\,,\,\frac{3}{4}\sqrt{3}) & g = \frac{1}{4} \end{aligned}$$

For the area of $\triangle(M_a,M_b,M_c)$ we get

$$\begin{aligned} A &= \frac{1}{2}\cdot|g|\cdot|h_g| \\ &= \frac{1}{2}\cdot ||(3-2\sqrt{3}\,,\,-\sqrt{3})||\cdot 1\cdot ||(-\sqrt{3}\,,\,-3+2\sqrt{3})||\cdot\left|-\frac{1}{4}\sqrt{3}\right| \\ &= \frac{1}{8}\sqrt{3}\cdot\sqrt{3+(3-2\sqrt{3})^2}\cdot\sqrt{(-3+2\sqrt{3})^2+3} \\ &= -\frac{9}{2}+3\sqrt{3}\approx 0.696 \end{aligned}$$

The three straights of the triangle and their heights are

$$\begin{aligned} a \ &: \ \vec{x}_a = (3-3\sqrt{3},-3+3\sqrt{3})^\tau\cdot a + (2\sqrt{3},0)^\tau = \dot{\vec{x}}_a\cdot a + \vec{s} & 0 \leq a \leq 1 \\ b \ &: \ \vec{x}_b = (3-\sqrt{3},-3+3\sqrt{3})^\tau\cdot b = \dot{\vec{x}}_b\cdot b & 0 \leq b \leq 1 \\ c \ &: \ \vec{x}_c = (2\sqrt{3},0)^\tau\cdot c = \dot{\vec{x}}_c\cdot c & 0 \leq c \leq 1 \\ h_a \ &: \ \vec{x} = (-3+3\sqrt{3},-3+3\sqrt{3})^\tau\cdot h_a & 0 \leq h_a \leq \frac{1}{2}+\frac{1}{6}\sqrt{3} \\ h_b \ &: \ \vec{x} = (-3+3\sqrt{3},-3+\sqrt{3})^\tau\cdot h_b + (2\sqrt{3},0)^\tau & 0 \geq h_b \geq -\frac{3}{4}-\frac{1}{4}\sqrt{3} \\ h_c \ &: \ \vec{x} = (0,2\sqrt{3})^\tau\cdot h_c + (3-\sqrt{3},-3+3\sqrt{3})^\tau & 0 \geq h_c \geq -\frac{3}{2}+\frac{1}{2}\sqrt{3} \end{aligned}$$

Solution

The orthic triangle, with vertices at the base points of the altitudes of the given triangle, has the smallest perimeter of all triangles inscribed into an acute triangle, hence it is the solution of Fagnano's problem. Fagnano's original proof used calculus methods and an intermediate result given by his father Giulio Carlo de'Toschi di Fagnano. Later however several geometric proofs were discovered as well, amongst others by Hermann Schwarz and Lipót Fejér. These proofs use the geometrical properties of reflections to determine some minimal path representing the perimeter.

Physical principles

A solution from physics is found by imagining putting a rubber band that follows Hooke's Law around the three sides of a triangular frame ABC, such that it could slide around smoothly. Then the rubber band would end up in a position that minimizes its elastic energy, and therefore minimize its total length. This position gives the minimal perimeter triangle. The tension inside the rubber band is the same everywhere in the rubber band, so in its resting position, we have, by Lami's theorem, $\angle bcA = \angle acB, \angle caB = \angle baC, \angle abC = \angle cbA$

Therefore, this minimal triangle is the orthic triangle.

`https://en.wikipedia.org/wiki/Fagnano%27s_problem`

Proof by geometry:

`https://azimpremjiuniversity.edu.in/`
`SitePages/pdf/05-shailesh_fagnanosproblemaddendum_classroom.pdf`

Proof by analytical geometry:

`http://forumgeom.fau.edu/FG2007volume7/FG200728.pdf`

7. (HS-1) Calculate the masses of Sun, Earth and Jupiter. You may assume circular orbits. We further calculate with the following data:

   the gravitational constant $G = 6.67 \cdot 10^{-11} \,\dfrac{m^3}{kg \cdot s^2}$

   the Kepler constant for our solar system $C = \dfrac{T^2}{R^3} = 0.29 \cdot 10^{-18} \,\dfrac{s^2}{m^3}$

   the acceleration by gravity on earth $\gamma = 9.81 \,\dfrac{m}{s^2}$

the earth's radius $R = 6,370\,km$

Io's orbital radius $R_I = 4.22 \cdot 10^8\,m$

Io's orbital period $T_I = 1.77\,d$

Reason: Some Basic Astronomy.

Solution:

(a) Sun. The gravitational force of the sun on a planet is given by $F_G = G\,\dfrac{m_s \cdot m_p}{R^2}$ which is the radial force of the planet, i.e. $F_R = \dfrac{4\pi^2 R m_p}{T^2}$. Both forces are equal so we get

$$m_s = \frac{4\pi^2}{G}\cdot\frac{R^3}{T^2} = \frac{4\pi^2}{G \cdot C} \approx \frac{4\pi^2}{1.9343}\cdot 10^{29}\cdot\frac{kg \cdot s^2}{m^3}\cdot\frac{m^3}{s^2} \approx 2.041\cdot 10^{30}\,kg$$

(b) Earth. In case of our own planet, we have again $F_G = G\,\dfrac{m_e \cdot m}{R^2}$ as gravitational force of a mass m on the planet. It equals its weight $F_w = m \cdot \gamma$, hence we have

$$\begin{aligned} m_e &= \frac{\gamma \cdot R^2}{G} \approx \frac{9.81 \cdot 6,370,000^2}{6.67 \cdot 10^{-11}} \cdot \frac{m \cdot m^2 \cdot kg \cdot s^2}{s^2 \cdot m^2} \\ &\approx \frac{9.81 \cdot 6.37^2}{6.67} \cdot 10^{23}\,kg \approx 5.968 \cdot 10^{24}\,kg \end{aligned}$$

(c) Jupiter. The calculations for Jupiter (and Io) is analog to that of the Sun (and Jupiter). Hence we get

$$\begin{aligned} m_j &= \frac{4\pi^2}{G} \cdot \frac{R_I^3}{T_I^2} \\ &\approx \frac{4\pi^2}{6.67} \cdot 10^{11} \cdot \frac{4.22^3}{1.77^2} \cdot 10^{24} \cdot \frac{1}{8.64^2} \cdot 10^{-8} \cdot \frac{kg \cdot s^2 \cdot m^3 \cdot}{m^3 \cdot s^2} \\ &\approx 1.9 \cdot 10^{27}\,kg \end{aligned}$$

8. (HS-2) A car drives at 72 km/h directly past a resting observer when the driver presses its horn. By what interval does the pitch of the horn change as the car passes the observer? (Speed of sound $s = 340$ m/s.)

Reason: Doppler Effect.

Solution: The car's speed is 72 km/h= 20 m/s. Let the horn's pitch be ν. If the car approaches the observer, he will hear a frequence

$$\nu' = \frac{\nu}{1 - \dfrac{v}{s}}$$

If the car evicts the observer, he will hear a frequence

$$\nu'' = \frac{\nu}{1 + \dfrac{v}{s}}$$

For the interval we get

$$\frac{\nu'}{\nu''} = \frac{s+v}{s-v} = \frac{360 \text{ m/s}}{320 \text{ m/s}} = \frac{9}{8}$$

which is a full musical tone.

9. (HS-3) Consider the sphere $\mathbb{S}^2 = \{\,(x,y,z)\,|\,x^2+y^2+z^2 = r^2\,\}$ and a point $P \in \mathbb{S}^2$. Determine the set of all center points of all chords starting in P.

Reason: Geometry.
Vector calculations are easier than coordinate calculations.

Solution: The variable endpoint X of the chord is on the sphere, so for its position vector we have $\vec{x}^2 = r^2$. The position vector of the center of the chord $\overline{PX}$ is thus

$$\vec{c} = \frac{\vec{p}+\vec{x}}{2} \iff \vec{x} = 2\vec{c} - \vec{p}$$

hence $r^2 = (2\vec{c}-\vec{p})^2$ or $\left(\vec{c} - \dfrac{\vec{p}}{2}\right)^2 = \dfrac{r^2}{4}$. So the set of points we were looking for are all on a sphere with center $\overline{OP}/2 = \vec{p}/2$ and radius $r/2$. All points of this sphere are on the other hand a center of some chord of the original sphere with endpoint P, since we can go back. The point P itself is the center of the chord $\overline{PP}$.

10. (HS-4) At the monthly meeting of former mathematics students, six members choose a real number a, which has to be guessed by a seventh mathematician who had left the room before. He gets the following information after he returned:

(1) a is rational.
(2) a is an integer divisible by 14.
(3) a is real and its square equals 13.
(4) a is an integer divisible by 7.

(5) a is real and the inequality $0 < a^3 + a < 8,000$ holds.
(6) a is even.

He is told, that all pairs $(1,2),(3,4),(5,6)$ always consist of a true and a false statement. What is a?

Reason: Puzzle.

Solution: Assume $a \notin \mathbb{Z}$. Then (2),(4),(6) are false and thus (1),(3),(5) true, which cannot be since $\sqrt{13} \notin \mathbb{Q}$. This means that $a \in \mathbb{Z}$ and statement (4) is true. As $\mathbb{Z} \subseteq \mathbb{Q}$ statement (2) is false and a is not divisible by 14, hence odd. So we have additionally that $0 < a^3 + a = a(a^2+1) < 8,000$. This implies $a > 0$. On the other end it implies $a < 20$. But only $a = 7$ is odd and divisible by 7 in this range. So (1),(4),(5) are true and (2),(3),(6) are false.

29 September 2019

1. We all know that the geometric mean is less than the arithmetic mean. I memorize it with $3{\cdot}5 < 4{\cdot}4$. Now we consider the arithmetic-geometric mean $M(a,b)$ between the two others. Let a,b be two non-negative real numbers. We set $a_0 = a$, $b_0 = b$ and define the sequences (a_k) , (b_k) by

$$a_{k+1} := \frac{a_k + b_k}{2}\,,\; b_{k+1} = \sqrt{a_k b_k} \quad k = 0, 1, \ldots$$

Then the arithmetic-geometric mean $M(a,b)$ is the common limit

$$\lim_{n\to\infty} a_n = M(a,b) = \lim_{n\to\infty} b_n$$

It is not hard to show that both sequences converge and that their limit is the same by using the known inequality and the monotony of the sequences.

Prove that for positive $a, b \in \mathbb{R}$ holds

$$T(a,b) := \frac{2}{\pi}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}} = \frac{1}{M(a,b)}$$

Reason: The arithmetic-geometric mean.

Solution: We show $T(a,b) = T\left(\frac{1}{2}(a+b), \sqrt{ab}\right)$, the Laden transformation. By repetition and the limiting process we get

$$T(a,b) = T(M(a,b), M(a,b)) = \frac{2}{\pi}\int_0^{\pi/2} \frac{d\varphi}{M(a,b)} = \frac{1}{M(a,b)}$$

With $t := b\tan\varphi$ we get

$$\cos^2\varphi = \frac{b^2}{b^2+t^2}\,,\; \sin^2\varphi = \frac{t^2}{b^2+t^2}\,,\; d\varphi = \frac{b}{b^2+t^2}\,dt$$

and

$$\begin{aligned} T(a,b) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}} \\ &= \frac{1}{\pi}\int_{-\infty}^{0} \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}} + \frac{1}{\pi}\int_{0}^{+\infty} \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}} \end{aligned}$$

In the first integral we substitute $t = x - C(x)$, in the second $t = x + C(x)$ where we set $C(x) = \sqrt{ab + x^2}$ for short. Then

$$
\begin{aligned}
T(a,b) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \left\{ \frac{1 - x/C(x)}{\sqrt{[a^2 + (x - C(x))^2] \cdot [b^2 + (x - C(x))^2]}} \right. \\
&\qquad \left. + \frac{1 + x/C(x)}{\sqrt{[a^2 + (x + C(x))^2] \cdot [b^2 + (x + C(x))^2]}} \right\} dx \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dx}{C(x)} \left\{ \frac{C(x) - x}{\sqrt{[a^2 + (C(x) - x)^2] \cdot [b^2 + (C(x) - x)^2]}} \right. \\
&\qquad \left. + \frac{C(x) + x}{\sqrt{[a^2 + (C(x) + x)^2] \cdot [b^2 + (C(x) + x)^2]}} \right\} \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dx}{C(x)} \left\{ \frac{C(x)^2 - x^2}{\sqrt{(C(x) + x)^2 \cdot [a^2 + (C(x) - x)^2] \cdot [b^2 + (C(x) - x)^2]}} \right. \\
&\qquad \left. + \frac{C(x)^2 - x^2}{\sqrt{(C(x) - x)^2\, [a^2 + (C(x) + x)^2] \cdot [b^2 + (C(x) + x)^2]}} \right\} \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dx}{C(x)} \left\{ \frac{ab}{\sqrt{(C(x) + x)^2 \cdot [a^2b^2 + (C(x) - x)^4 + (a^2 + b^2) \cdot (C(x) - x)^2]}} \right. \\
&\qquad \left. + \frac{ab}{\sqrt{(C(x) - x)^2 \cdot [a^2b^2 + (C(x) + x)^4 + (a^2 + b^2) \cdot (C(x) + x)^2]}} \right\} \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dx}{C(x)} \left\{ \frac{1}{\sqrt{(C(x) + x)^2 + (C(x) - x)^2 + (a^2 + b^2)}} \right. \\
&\qquad \left. + \frac{1}{\sqrt{(C(x) - x)^2 + (C(x) + x)^2 + (a^2 + b^2)}} \right\} \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{2}{C(x)} \cdot \frac{dx}{\sqrt{2C(x)^2 + 2x^2 + a^2 + b^2}} \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{2\,dx}{\sqrt{2(ab + x^2)^2 + 2x^2(ab + x^2) + (a^2 + b^2)(ab + x^2)}} \\
&= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{dx}{\sqrt{[((a + b)/2)^2 + x^2] \cdot \left[\sqrt{ab}^2 + x^2\right]}} \\
&= T\left(\frac{a + b}{2}, \sqrt{ab}\right)
\end{aligned}
$$

2. If A, B, C, D are four points in the plane, show that

$$\det \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & |AB|^2 & |AC|^2 & |AD|^2 \\ 1 & |AB|^2 & 0 & |BC|^2 & |BD|^2 \\ 1 & |AC|^2 & |BC|^2 & 0 & |CD|^2 \\ 1 & |AD|^2 & |BD|^2 & |CD|^2 & 0 \end{bmatrix} = 0$$

Reason: Cayley Menger Determinant.

Solution: If we view A, B, C, D as vectors in $\mathbb{R}^2$, then we have the usual cosine rule $|AB|^2 = |A|^2 + |B|^2 - 2A \cdot B$, and similarly for all the other distances. The matrix can then be written as $M + M^\tau - 2G$, where

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & |A|^2 & |B|^2 & |C|^2 & |D|^2 \\ 1 & |A|^2 & |B|^2 & |C|^2 & |D|^2 \\ 1 & |A|^2 & |B|^2 & |C|^2 & |D|^2 \\ 1 & |A|^2 & |B|^2 & |C|^2 & |D|^2 \end{bmatrix}, \; G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & A \cdot A & A \cdot B & A \cdot C & A \cdot D \\ 0 & B \cdot A & B \cdot B & B \cdot C & B \cdot D \\ 0 & C \cdot A & C \cdot B & C \cdot C & C \cdot D \\ 0 & D \cdot A & D \cdot B & D \cdot C & D \cdot D \end{bmatrix}$$

$\operatorname{rk} M = \operatorname{rk} M^\tau = 1$ and $G = S \cdot S^\tau$ with the 5×2 matrix with rows $0, A, B, C, D$, i.e. $\operatorname{rk} G \leq 2$. Hence $\operatorname{rk}(M + M^\tau - 2G) \leq 1 + 1 + 2 = 4 < 5$ and its determinant vanishes.

3. Let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ a linear, continuous (= bounded) operator on Hilbert spaces. Prove that the following are equivalent:

   (a) T is invertible.

   (b) There exists a constant $\alpha > 0$, such that $T^*T \geq \alpha I_{\mathcal{H}_1}$ and $TT^* \geq \alpha I_{\mathcal{H}_2}$. $A \geq B$ means $\langle (A-B)\xi \,,\, \xi \rangle \geq 0$ for all ξ.

Reason: Linear Operators.

Solution:
$(a) \Longrightarrow (b)$ If T is invertible so is $T^* : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ with inverse $(T^*)^{-1} = (T^{-1})^*$. Define $\alpha := ||T^{-1}||^{-2} = ||(T^*)^{-1}||^{-2}$. Note that for $\xi \in \mathcal{H}_1$ we have

$$||\xi|| = |T^{-1}(T(\xi))|| \leq ||T^{-1}|| \cdot ||T(\xi)||$$

and therefore

$$\langle T^*T\xi \,,\, \xi \rangle_{\mathcal{H}_1} = \langle T\xi \,,\, T\xi \rangle_{\mathcal{H}_2} = ||T\xi||^2 \geq ||T^{-1}||^{-2}||\xi||^2 = \alpha \langle \xi \,,\, \xi \rangle_{\mathcal{H}_1}$$

This shows $\langle (T^*T-\alpha I_{\mathcal{H}_1})\,\xi\,,\,\xi\rangle \geq 0$ for all $\xi\in\mathcal{H}_1$, so $T^*T-\alpha I_{\mathcal{H}_1}\in\mathcal{B}(\mathcal{H}_1)$ is positive. The positivity of $TT^*-\alpha I_{\mathcal{H}_2}\in\mathcal{B}(\mathcal{H}_2)$ follows accordingly.

$(b) \Longrightarrow (a)$ Assume there is an $\alpha>0$ such that $T^*T\geq \alpha I_{\mathcal{H}_1}$ and $TT^*\geq\alpha I_{\mathcal{H}_2}$. Thus

$$||T\xi||^2=\langle T\xi\,,\,T\xi\rangle=\langle T^*T\xi\,,\,\xi\rangle\geq\alpha\langle\xi\,,\,\xi\rangle=||\xi||^2$$

and we get

$$||T\xi||\geq\sqrt{\alpha}||\xi|| \quad \forall\,\xi\in\mathcal{H}_1$$

On the one hand this shows that T is injective, and similar that T^* is injective, too. Therefore

$$\overline{R(T)}=(\ker(T^*))^\perp=\{\,0_{\mathcal{H}_2}\,\}^\perp=\mathcal{H}_2$$

and we only have to show that $R(T)$ is closed, hence T is also surjective. Now since T is bounded, we have

$$C\cdot||\xi||\geq||T||\cdot||\xi||\geq||T\xi||\geq\sqrt{\alpha}\cdot||\xi||$$

which makes the norm on $\mathcal{H}_1$ equivalent to the by $\mathcal{H}_2$ induced norm on $R(T)$. Thus $R(T)$ is again a Banach space and therefore closed.

4. Let $a,b\in\mathbb{F}$ be non-zero elements in a field of characteristic not two. Let A be the four dimensional $\mathbb{F}-$space with basis $\{\,1,\mathbf{i},\mathbf{j},\mathbf{k}\,\}$ and the bilinear and associative multiplication defined by the conditions that 1 is a unity element and

$$\mathbf{i}^2=a\,,\,\mathbf{j}^2=b\,,\,\mathbf{ij}=-\mathbf{ji}=\mathbf{k}\,.$$

Then $A=\left(\dfrac{a,b}{\mathbb{F}}\right)$ is called a (generalized) quaternion algebra over $\mathbb{F}$. Show that A is a simple algebra whose center is $\mathbb{F}$.

Reason: Associative Algebras.

Solution: For convenience we use the Lie bracket for $[x,y]=xy-yx$. If $x=c_0+c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$ then

$$\begin{aligned}[\mathbf{i},x]&=(2ac_3)\mathbf{j}+(2c_2)\mathbf{k}\\ [\mathbf{j},x]&=(-2bc_3)\mathbf{i}+(-2c_1)\mathbf{k}\\ [\mathbf{k},x]&=(2bc_2)\mathbf{i}+(-2ac_1)\mathbf{j}\end{aligned}$$

In case $x \in Z(A)$ we get $c_3 = c_2 = c_1 = 0$ and vice versa, hence $Z(A) = \mathbb{F}$. Now let $\{ 0 \} \neq I \trianglelefteq A$ be a non-zero ideal of A and $0 \neq x \in I$. As a two sided ideal we have

$$\begin{aligned}[\mathbf{j},[\mathbf{i},x]] &= (-4bc_2)\mathbf{i} \in I\\ [\mathbf{k},[\mathbf{j},x]] &= (4abc_3)\mathbf{j} \in I\\ [\mathbf{i},[\mathbf{k},x]] &= (-4ac_1)\mathbf{k} \in I\end{aligned}$$

If one of the coefficients c_1, c_2, c_3 is unequal zero, then I contains a unit of A and thus $I = A$. If $c_1 = c_2 = c_3 = 0$ then $x \neq 0$ implies $c_0 \neq 0$ and I again contains a unit. In all cases we have $A = I$.

5. Prove that the quaternion algebra $\left(\dfrac{a,1}{\mathbb{F}}\right) \cong \mathbb{M}(2,\mathbb{F})$ is isomorphic to the matrix algebra of 2×2 matrices for every $a \in \mathbb{F} - \{ 0 \}$.

   Reason: Quaternions.

   Solution: Direct calculation of the multiplication tables by setting

$$\begin{aligned}e_{11} &= \frac{1}{2}(1-\mathbf{j})\\ e_{22} &= \frac{1}{2}(1+\mathbf{j})\\ e_{12} &= \frac{1}{2a}(\mathbf{i}-\mathbf{k})\\ e_{21} &= \frac{1}{2}(\mathbf{i}+\mathbf{k})\end{aligned}$$

6. Show that there are infinitely many primes of the form $4k+3$, $k \in \mathbb{N}_0$.

   Reason: Number Theory.

   Solution: Assume there are only finitely many primes of the form $4k+3$: $p_1, \ldots, p_n$. Set $z := 4p_1 \cdot \ldots \cdot p_n - 1$. Then $z = 4k+3$ with $k = p_1 \cdot \ldots \cdot p_n - 1 \in \mathbb{N}_0$. Let $z = q_1 \cdot \ldots \cdot q_m$ be the prime decomposition of z. Since z is odd, $q_i \neq 2$ for all $i \in \{ 1, \ldots, m \}$. This all q_i are either of the form $q_i = 4r_i + 1$ or of the form $q_i = 4s_i + 3$ with $r_i, s_i \in \mathbb{N}_0$. Assume all q_i were of the form $4r_i + 1$, then $3 \equiv z \equiv 1 \mod 4$ which is impossible. Therefore at least one q_i has the form $4s_i + 3$. But for this prime we have $q_i \in \{ p_1, \ldots, p_n \}$, say $q_i = p_j$. Now

$$p_j = q_i \,|\, q_1 \cdot \ldots \cdot q_m = z = 4p_1 \cdot \ldots \cdot p_n - 1$$

which is impossible. Hence there are infinitely many primes of the form $4k+3$.

7. Do $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ and $\displaystyle\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}\right)^2$ converge or diverge?

Reason: Product of converging series can diverge.

Solution: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ converges according to the Leibniz criterion, because $a_n := \dfrac{1}{\sqrt{n+1}}$ is strictly monotone decreasing. For the Cauchy product $\displaystyle\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)^2 = \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}\right)^2$ we get

$$\begin{aligned} c_n &= \sum_{k=0}^{n} a_{n-k}a_k \\ &= \sum_{k=0}^{n}\frac{(-1)^{n-k}}{\sqrt{n-k+1}}\cdot\frac{(-1)^k}{\sqrt{k+1}} \\ &= (-1)^n\sum_{k=0}^{n}\frac{1}{\sqrt{n-k+1}\cdot\sqrt{k+1}} \end{aligned}$$

From $0 \leq (\sqrt{a}-\sqrt{b})^2$ we get $\sqrt{a}\sqrt{b} \leq \frac{1}{2}(a+b)$ for $a,b>0$ and so

$$\begin{aligned} |c_n| &= \sum_{k=0}^{n}\frac{1}{\sqrt{n-k+1}\cdot\sqrt{k+1}} \\ &\geq \sum_{k=0}^{n}\frac{1}{\frac{1}{2}(n-k+1+k+1)} \\ &= \sum_{k=0}^{n}\frac{2}{\sqrt{n+2}} \\ &= \frac{2(n+1)}{n+2} \\ &= \frac{2+\frac{2}{n}}{1+\frac{2}{n}} \to 2 \quad (n\to\infty) \end{aligned}$$

Hence (c_n) isn't a null sequence and $\displaystyle\sum_{n=0}^{\infty} c_n$ diverges.

8. Consider the curve $\gamma\,:\,\mathbb{R}\longmapsto\mathbb{C}$, $\gamma(t)=\cos(\pi t)\cdot e^{\pi i t}$. Find the minimal period of γ (a), prove that $\gamma(\mathbb{R})\equiv\{\,(x,y)\in\mathbb{R}^2\,|\,x^2+y^2-x=0\,\}$ (b), show that $\gamma(\mathbb{R})$ is symmetric to the $x-$axis (c), and parameterize γ with respect to its arc length (d).

   Reason: Differential Geometry.

   Solution: We have a natural parameterization by

$$\gamma(t)=(\cos^2(\pi t),\sin(\pi t)\cos(\pi t))$$

   and thus $\gamma(t+2)=\gamma(t)$. Since $\sin(\pi t)\cos(\pi t)=\frac{1}{2}\sin(2\pi t)$ we even get $\gamma(t)=\gamma(t+1)$, and thus a minimal period of 1.

   Now let $(x,y)\in\gamma(\mathbb{R})\subseteq\mathbb{R}^2$. Then

$$\begin{aligned}x^2+y^2-x&=\cos^4(\pi t)+\sin^2(\pi t)\cos^2(\pi t)-\cos^2(\pi t)\\&=\cos^2(\pi t)\cdot\big(\cos^2(\pi t)+\sin^2(\pi t)-1\big)\\&=0\end{aligned}$$

   If $(x,y)\in\mathbb{R}^2$ with $x^2+y^2-x=0$, then we have to find a $t\in\mathbb{R}$ with $\gamma(t)=(x,y)$.

$$x^2+y^2-x=0\Longleftrightarrow x=\frac{1}{2}\pm\sqrt{\frac{1}{4}-y^2}$$

   Since (x,y) exists by assumption, $\dfrac{1}{4}-y^2\geq 0$ and $x\in[0,1]$ which allows us to chose $x=\cos^2(\pi t)$.

$$y^2=x-x^2=\cos^2(\pi t)-\cos^4(\pi t)=\cos^2(\pi t)(1-\cos^2(\pi t))=\cos^2(\pi t)\sin^2(\pi t)$$

   For the positive solution we are done. If $y=-\cos(\pi t)\sin(\pi t)$ we observe that $y=-\cos(\pi t)\sin(\pi t)=\cos(-\pi t)\sin(-\pi t)$. As $x=\cos^2(\pi t)=\cos^2(-\pi t)$, we also have shown the existence of t. In combination we have $\gamma(\mathbb{R})\equiv\{\,(x,y)\in\mathbb{R}^2\,|\,x^2+y^2-x=0\,\}$. As $y,-y$ yield the same point, the symmetry with respect to the $x-$axis is obvious. By

$$x^2+y^2-x=0\Longleftrightarrow\left(x-\frac{1}{2}\right)^2+y^2=\frac{1}{4}$$

we see, that γ is the circle with center $(\frac{1}{2},0)$ and radius $\frac{1}{2}$. The arc length is given by

$$\begin{aligned}s(t) &= \int_0^t ||\dot\gamma(t)||\,dt\\ &= \int_0^t \sqrt{(-\pi\sin(\pi t)e^{\pi i t}+i\pi\cos(\pi t)e^{\pi i t})}\,\cdot\\ &\quad\cdot\sqrt{(-\pi\sin(\pi t)e^{-\pi i t}-i\pi\cos(\pi t)e^{-\pi i t})}\,dt\\ &= \int_0^t \sqrt{\pi^2\sin^2(\pi t)+\pi^2\cos^2(\pi t)}\,dt\\ &= \int_0^t \pi\,dt\\ &= \pi t\end{aligned}$$

and the inverse is $\Phi(s)=\frac{s}{\pi}$. Then we have $p:=\gamma\circ\Phi\,:\,\mathbb{R}\longrightarrow\mathbb{C}$ with

$$p(s)=\gamma(\Phi(s))=\gamma\left(\frac{s}{\pi}\right)=\cos(s)\cdot e^{is}$$

9. Let $\gamma\,:\,I\longrightarrow\mathbb{R}^n$ be a regular curve with unit tangential vector $T=\dfrac{d}{dt}\gamma$. A (orthonormal) **frame** is a (smooth) $C^\infty-$ transformation $F\,:\,I\longrightarrow \mathrm{SO}(n)$ with $F(t)e_1=T(t)$ where $\{\,e_i\,\}$ is the standard basis of $\mathbb{R}^n$. The pair (γ,F) is called a **framed curve**, and the matrix A given by $\dfrac{d}{dt}F=F'=FA$ is called **derivation matrix** of F.

   Let $F_0\,:\,\mathbb{R}\longrightarrow\mathrm{SO}(n)$ be a frame of a regular curve $\gamma\,:\,\mathbb{R}\longrightarrow\mathbb{R}^n$. Show that

   (a) If $F\,:\,\mathbb{R}\longrightarrow\mathrm{SO}(n)$ is another frame of γ, then there exists a transformation $\Phi\,:\,\mathbb{R}\longrightarrow\mathrm{SO}(n)$ with $\Phi(t)e_1=e_1$ for all $t\in\mathbb{R}$ and $F=F_0\Phi$.

   (b) If on the other hand $\Phi\,:\,\mathbb{R}\longrightarrow\mathrm{SO}(n)$ is a smooth transformation with $\Phi(t)e_1=e_1$, then $F:=F_0\cdot\Phi$ defines a new frame of γ.

   (c) If A_0 is the derivation matrix of F_0, and A the derivation matrix of the transformed frame $F:=F_0\Phi$ with Φ as above, then

   $$A=\Phi^{-1}A_0\Phi+\Phi^{-1}\Phi'$$

   Reason: Gauge Transformation.

   Solution:

(a) We define Φ by $\Phi(t) := F_0(t)^{-1} \cdot F(t)$ which is a transformation $\mathbb{R} \longrightarrow \mathrm{SO}(n)$. Since both are frames of the same curve γ we get for all $t \in \mathbb{R}$

$$F(t)e_1 = F_0(t)e_1 = T(t) \Longrightarrow \Phi(t)e_1 = F_0(t)^{-1} \cdot F(t)e_1 = e_1$$

(b) F is obviously a transformation from $\mathbb{R}$ to $\mathrm{SO}(n)$ with

$$F(t)e_1 = F_0(t) \cdot \Phi(t)e_1 = F_0(t)e_1 = T(t)$$

and F is a frame of γ.

(c) We calculate

$$A = F^{-1}F' = (F_0\Phi)^{-1}(F_0\Phi)' = \Phi^{-1}F_0^{-1}\left(F_0'\Phi + F_0\Phi'\right) = \Phi^{-1}A_0\Phi + \Phi^{-1}\Phi'$$

10. (HS-1) Show that the number of ways to express a positive integer n as the sum of consecutive positive integers is equal to the number of odd factors of n.

Reason: Partitions.

Solution: From $n = r + (r+1) + \ldots + (r+k) = \frac{1}{2}k(k+1) + (k+1)r$ we get $2n = (k+1)(2r+k)$. Now either $k+1$ or $2k+r$ is odd, so every odd factor of n results in a partition of n as sum of consecutive positive integers. If we have two decompositions $2n = (k+1)(2r+k) = (l+1)(2s+l)$ and $k+1 = l+1$ or $2r+k = 2s+l$ are the same odd numbers, then $k = l$ in both cases. If we have $k+1 = 2s+l$ then $l+1 = k+2-2s = \dfrac{2n}{2s+l} = 2r+k$ and $r = 1-s$ or $r, s \in \{0, 1\}$. For $(r, s) = (0, 1)$ we get $2n = (k+1)k = (l+1)(k+1)$ or $k = l+1$, and for $(r, s) = (1, 0)$ we have $2n = (l+1)l = l(2+k)$ or $l = k+1$ which is again the same odd factor as $2n = (l+2)(l+1) = (k+1)(k+2)$ is the same decomposition.

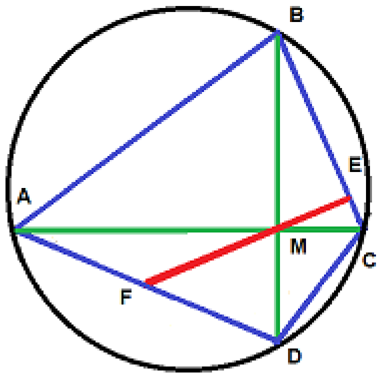
11. (HS-2) How many solutions in non-negative integers are there to the equation:

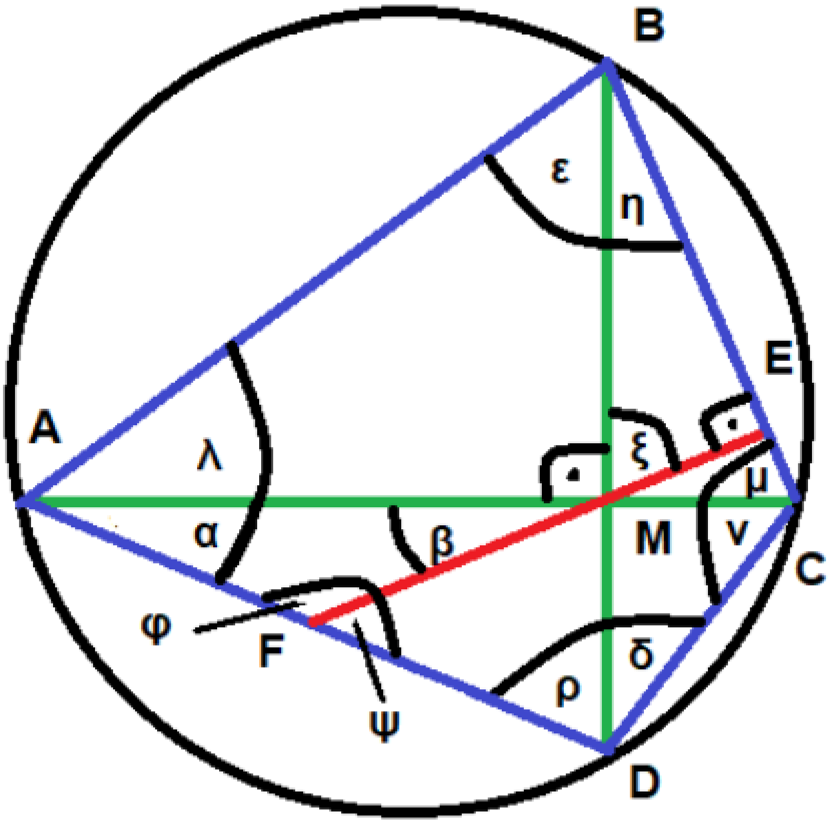
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 32$$

Reason: Partitions.

Solution: Assume we have 37 boxes in which we place 32 pebbles and 5 partition sticks. Then we have $\binom{37}{5} = 435,897$ possibilities, if we allow zero (stick in the first box) as a summand.

12. (HS-3) Let A, B, C and D be four points on a circle such that the lines AC and BD are perpendicular. Denote the intersection of AC and BD by M. Drop the perpendicular from M to the line BC, calling the intersection E. Let F be the intersection of the line EM and the edge AD. Then F is the midpoint of AD.





Reason: Brahmagupta Theorem.

Solution: We need to prove that $AF = FD$. We will prove that both AF and FD are in fact equal to FM. By the first theorem of chords in an inscribed quadrilateral: $AM \cdot CM = BM \cdot DM$ we see that the slopes of AD and BC are reciprocal. Thus a rotary reflection maps BC on AD, i.e. the triangles $\triangle AMD$ and $\triangle BMC$ are similar. Hence $\alpha = \eta$, $\rho = \mu$. Since $\eta + \xi = \beta + \xi = \pi/2$ we get $\beta = \eta = \alpha$ and $\triangle AMF$ is an isosceles triangle, i.e. $AF = FM$.

$$\begin{aligned}\varphi &= \pi - \alpha - \beta \\ &= \pi - 2\beta \\ &= \psi + \varphi - 2\beta\end{aligned}$$

and $\psi = 2\beta$. Hence $\rho = \pi - 2\beta - \xi = \pi/2 - \beta = \xi$ so $\triangle FDM$ is also an isosceles triangle, i.e. $FM = FD$.

13. (HS-4) Prove that every non-negative natural number $n \in \mathbb{N}_0$ can be written as
$$n = \frac{(x+y)^2 + 3x + y}{2}$$
with uniquely determined non-negative natural numbers $x, y \in \mathbb{N}_0$.

Reason: Olympiad Problem 581244 Bund 12b 18/19.

Solution: We set $s = x + y$, so $s \geq x \geq 0$ and for a given s we get as possible values for n the numbers
$$n = \frac{s^2+s}{2} + x \in \left\{ \frac{s^2+s}{2}, \frac{s^2+s}{2} + 1, \ldots, \frac{s^2+s}{2} + s \right\} \subseteq \mathbb{N}_0$$
If we define $I_s := \left[\frac{s^2+s}{2}, \frac{s^2+s}{2} + s \right] \cap \mathbb{N}_0$ we observe that
$$\left(\frac{s^2+s}{2} + s \right) + 1 = \frac{(s+1)^2 + (s+1)}{2}$$
and the I_s are a disjoint coverage of $\mathbb{N}_0$. Thus all n belong to some I_s and it cannot belong to two.

14. (HS-5) Calculate
$$S = \int_{\frac{1}{2}}^{3} \frac{1}{\sqrt{x^2+1}} \frac{\log(x)}{\sqrt{x}}\,dx + \int_{\frac{1}{3}}^{2} \frac{1}{\sqrt{x^2+1}} \frac{\log(x)}{\sqrt{x}}\,dx$$

Reason: Multiplicative Integration Symmetry.

Solution: The functions are continuous in the area of integration. Assume that the anti-derivative is $F(x)$. Then we have to calculate $S = F(3) - F\left(\frac{1}{2}\right) + F(2) - F\left(\frac{1}{3}\right) = F(3) - F\left(\frac{1}{3}\right) + F(2) - F\left(\frac{1}{2}\right)$ and we can calculate the integrals
$$\mathcal{I}_n = \int_{\frac{1}{n}}^{n} \frac{1}{\sqrt{x^2+1}} \frac{\log(x)}{\sqrt{x}}\,dx = \int_{\frac{1}{n}}^{n} \frac{1}{\sqrt{x + \frac{1}{x}}} \frac{\log(x)}{x}\,dx$$
Set $y = \frac{1}{x}$. This means $\dfrac{dy}{dx} = -\frac{1}{x^2} = -y^2$. We also have to switch the

integration bounds $x = n$ to $y = \frac{1}{n}$ and $x = \frac{1}{n}$ to $y = n$. Thus

$$\begin{aligned}
\mathcal{I}_n &= \int_n^{\frac{1}{n}} \frac{y}{\sqrt{\frac{1}{y}+y}} \log\left(\frac{1}{y}\right) \frac{-1}{y^2}\,dy \\
&= \int_n^{\frac{1}{n}} \frac{1}{\sqrt{\frac{1}{y}+y}} \log(y) \frac{1}{y}\,dy \\
&= -\int_{\frac{1}{n}}^{n} \frac{1}{\sqrt{\frac{1}{y}+y}} \frac{\log(y)}{y}\,dy \\
&= -\int_{\frac{1}{n}}^{n} \frac{1}{\sqrt{y\left(\frac{1}{y}+y\right)}} \frac{\log(y)}{\sqrt{y}}\,dy \\
&= -\int_{\frac{1}{n}}^{n} \frac{1}{\sqrt{1+y^2}} \frac{\log(y)}{\sqrt{y}}\,dy \\
&= -\int_{\frac{1}{n}}^{n} \frac{1}{\sqrt{x^2+1}} \frac{\log(x)}{\sqrt{x}}\,dx \\
&= -\mathcal{I}_n
\end{aligned}$$

If the integral equals its negative, then it has to be zero for any positive n. Hence $S = 0$.

30 August 2019

1. Three identical airplanes start at the same time at the vertices of an equilateral triangle with side length L. Let's say the origin of our coordinate system is the center of the triangle. The planes fly at a constant speed v above ground in the direction of the clockwise next airplane. How long will it take for the planes to reach the same point, and which are the flight paths?

   Reason: Mechanics.

   Solution: The side length of the triangle at $t = 0$ is $L(0) = L$. For the position $\vec{r}(t)$ of the first airplane we have $|\vec{r}(0)| = r(0) = \dfrac{2}{3}L\cos\dfrac{\pi}{6} = \dfrac{L}{\sqrt{3}}$. The distance between the airplanes are the same at any point in time, because of the symmetry, i.e. the airplanes will always mark the vertices of an equilateral triangle with its center at the origin. Thus the angle between the velocity $\vec{v}(t)$ and the position $\vec{r}(t)$ is always

$$\sphericalangle(\vec{v}(t), \vec{r}(t)) = \psi(t) = \psi(0) = \psi = \pi - \frac{\pi}{6}$$

   Thus we have

$$\begin{aligned}
\vec{v}(t) &= \dot{\vec{r}}(t) \\
\vec{r}(t))\vec{v}(t) &= \vec{r}(t)\dot{\vec{r}}(t) \\
r \cdot v \cdot \cos\psi &= \frac{1}{2}\frac{d}{dt}(\vec{r}(t)\vec{r}(t) \\
r \cdot v \cdot \cos\psi &= \frac{1}{2}\frac{dr^2}{dt} \\
r \cdot v \cdot \cos\psi &= r\frac{dr}{dt} \\
\frac{dr}{dt} &= -v\frac{\sqrt{3}}{2} \\
r(t) &= \frac{L}{\sqrt{3}} - v\frac{\sqrt{3}}{2}t
\end{aligned}$$

   Hence $r(t_f) = 0$ implies $t_f = \dfrac{2L}{3v}$.

   To get the flight path we decompose $\vec{v}(t)$ in components parallel and perpendicular to $\vec{r}(t)$. The perpendicular component is $|v_\perp| = v \cdot \sin\psi = \dfrac{v}{2}$ so we have the angular velocity $\dot{\omega}(t) = \dfrac{v_\perp(t)}{r(t)}$. We parameterize the motion by cylindric coordinates $\vec{r}(t) = (r(t)\cos\varphi(t)\,,\,-r(t)\sin\varphi(t)\,,\,0)^\tau$ and receive the momentary rotation angle by the integration

$$\begin{aligned}\varphi(t) &= \varphi(0) + \int_0^t \omega(t')\,dt' \\ &= \varphi(0) + \frac{v}{2}\int_0^t \frac{1}{r(t')}\,dt' \\ &= \varphi(0) + \frac{v}{2}\int_0^t \frac{1}{\frac{L}{\sqrt{3}} - v\frac{\sqrt{3}}{2}t'}\,dt' \\ &= \varphi(0) + \int_0^t \frac{1}{\dfrac{2L}{\sqrt{3}v} - \sqrt{3}t'}\,dt' \\ &= \varphi(0) + \frac{1}{\sqrt{3}}\int_0^t \frac{1}{\frac{2L}{3v} - t'}\,dt' \\ &= \varphi(0) + \frac{1}{\sqrt{3}}\log\left(\frac{\frac{2L}{3v}}{\frac{2L}{3v} - t}\right) \\ &= \varphi(0) + \frac{1}{\sqrt{3}}\log\left(\frac{r(0)}{r(t)}\right)\end{aligned}$$

so the flight path is the logarithmic spiral with

$$r(t) = r(0)\cdot e^{-\sqrt{3}(\varphi(t)-\varphi(0))}$$

The distance towards the center decreases by a factor of $e^{-2\pi\sqrt{3}} \approx 1.88\cdot 10^{-5}$ with every complete turn.

2. The Schwarzian derivative of a holomorphic function f is given by

$$S_f(z) = \{\,f,z\,\} := \frac{d}{dz}\left(\frac{f''(z)}{f'(z)}\right) - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

Prove a chain rule for the Schwarzian derivative and show that

$$\{\,f,z\,\} < 0 \wedge \{\,h,z\,\} < 0 \Longrightarrow \{\,f\circ h,z\,\} < 0$$

Reason: Dynamical Systems.

Solution: The formula we want to prove is

$$S_{f\circ h}(z) = S_f(h(z))\cdot(h'(z))^2 + S_h(z)$$

$$\begin{aligned}
(f\circ h)'(z) &= f'(h(z))h'(z)\\
(f\circ h)''(z) &= f''(h(z))(h'(z))^2+f'(h(z))h''(z)\\
(f\circ h)'''(z) &= f'''(h(z))(h'(z))^3+3f''(h(z))h'(z)h''(z)+f'(h(z))h'''(z)\\
S_{fh}(z) &= \frac{(fh)'''(z)}{(fh)'(z)}-\frac{3}{2}\left(\frac{(fh)''(z)}{(fh)'(z)}\right)^2\\
&= \frac{f'''(h(z))(h'(z))^3+3f''(h(z))h'(z)h''(z)+f'(h(z))h'''(z)}{f'(h(z))h'(z)}\\
&\quad -\frac{3}{2}\left(\frac{f''(h(z))(h'(z))^2+f'(h(z))h''(z)}{f'(h(z))h'(z)}\right)^2\\
&= \frac{f'''(h(z))}{f'(h(z))}\cdot(h'(z))^2+3\,\frac{f''(h(z))}{f'(h(z))}\cdot h''(z)+\frac{h'''(z)}{h'(z)}\\
&\quad -\frac{3}{2}\left(\frac{f''(h(z))}{f'(h(z))}\cdot h'(z)+\frac{h''(z)}{h'(z)}\right)^2\\
&= S_f(h(z))\cdot(h'(z))^2+S_h(z)\\
&\quad +3\,\frac{f''(h(z))}{f'(h(z))}\cdot h''(z)-\frac{3}{2}\cdot 2\cdot\frac{f''(h(z))}{f'(h(z))}\cdot h''(z)\\
&= S_f(h(z))\cdot(h'(z))^2+S_h(z)
\end{aligned}$$

and from $S_f(z)<0$ and $S_h(z)<0$ we thus have $S_{fh}(z)<0$.
Schwarzian derivatives are used in dynamical systems to investigate attractors, in flows of surfaces, or in the theory of Schwarz-Christoffel mappings.

3. (HS-1) David drives to work every working day by car. Outside towns he drives at an average speed of 180 km/h. On the 10 km in town, he drives at an average speed of 40 km/h. As a result, he is often too fast and gets a ticket. Meanwhile he has realized that things can not go on like this and he decides to reduce his average speed by 20 km/h in town as well as outside. How long is his way to work, if this reduces his average speed by 40 km/h on total?

   Reason: Olympiad Problem.

   Solution: Let y be the length of his path in town, and x outside of town, each measured in km. We will later set $y = 10$. Originally he needed $\dfrac{x}{180}+\dfrac{y}{40}$ hours, and now he needs $\dfrac{x}{160}+\dfrac{y}{20}$ hours. In order that the average speed decreases by exactly 40 km/h the following equation has to hold:

$$\frac{x+y}{\frac{x}{180}+\frac{y}{40}} - \frac{x+y}{\frac{x}{160}+\frac{y}{20}} = 40$$

$$\begin{aligned}
40 &= (x+y)\cdot\left(\frac{180\cdot 40}{40x+180y} - \frac{160\cdot 20}{20x+160y}\right)\\
1 &= (x+y)\cdot\left(\frac{9}{2x+9y} - \frac{4}{x+8y}\right)\\
(2x+9y)\cdot(x+8y) &= (x+y)\cdot(9x+72y-8x-36y)\\
2x^2+72y^2+25xy &= x^2+36y^2+37xy\\
x^2+36y^2-12xy &= 0\\
(x-6y)^2 &= 0
\end{aligned}$$

So a necessary and sufficient condition is $x = 6y = 60$ km and his total way is 70 km long.

4. (HS-2) Show that $2x^6+3y^6=z^3$ has no other rational solutions than $x=y=z=0$.

   Reason: Olympiad Problem. **Solution:** For an integer p the cube has only possible remainders $\{0,1,6\}$ from division by 7 so the remainder of p^6 will be either one or zero.

$$p = 2^n\cdot(2k+1) \Longrightarrow p^3 = 8^n\cdot(2k+1)^3 \equiv r^3 \bmod 7$$

   where r is odd, i.e. $r^3 \in \{1^3,3^3,5^3,7^3\} \equiv \{0,1,6\} \bmod 7$. Hence for any integer solution

$$\{0,1,6\} \ni z^3 = 2x^6+3y^6 \in \{0,2,3,5\} \bmod 7$$

   and z is divisible by 7 and then all are: $x,y,z \equiv 0 \bmod 7$.

   Let $q = \max\{p\in\mathbb{N}\,|\,7^p|x \text{ and } 7^p|y\}$. Then $7^{6q}|z^3$, i.e. $7^{2q}|z$ and $\left(\frac{x}{7^q},\frac{y}{7^q},\frac{z}{7^{2q}}\right)$ is again an integer solution, so 7 divides all of them, which is impossible by maximality of q. This shows that $(0,0,0)$ is the only integer solution.

   Now let (x,y,z) be with rationals and L the least common multiple of the denominators of x,y,z. Then (Lx,Ly,L^2z) is an integer solution, i.e. $x=y=z=0$.

5. (HS-3) Let $x, yz \in \mathbb{R} - \{ 0 \}$ such that

$$x + \frac{y}{z} = 2 \ \ , \ \ y + \frac{z}{x} = 2 \ \ , \ \ z + \frac{x}{y} = 2$$

Show that $s := x + y + z$ can only have the values 3 or 7. You do not need to solve the equation system.

Reason: Puzzle..

Solution: The equations without denominators are

$$xz + y = 2z\,,\, yx + z = 2x\,,\, zy + x = 2y$$

hence $xz + yx + zy = 2(z + x + y) - (y + z + x) = x + y + z = s$. In the second step we multiply them

$$\begin{aligned} 1 &= \frac{y}{z} \cdot \frac{z}{x} \cdot \frac{x}{y} \\ &= (2 - x)(2 - y)(2 - z) \\ &= 8 - 4(x + y + z) + 2(xy + yz + zx) - xyz \\ &= 8 - 4s + 2s - xyz \\ &\Longrightarrow xyz = 7 - 2s \end{aligned}$$

From the first step we also get

$$xzy + y^2 = 2yz\ ,\ xyz + z^2 = 2xz\ ,\ xyz + x^2 = 2xy$$

and thus

$$\begin{aligned} 3xyz + x^2 + y^2 + z^2 &= 2(xy + xz + yz) \\ 3xyz + (x + y + z)^2 &= 4(xy + xz + yz) \\ 3xyz + s^2 &= 4s \end{aligned}$$

Now we have $3(7 - 2s) + s^2 = 4s$ or $(s - 3)(s - 7) = 0$ and s can only have the values $s \in \{ 3\,,\, 7 \}$.

There are actually only 4 solutions of the equation system:

$$(1, 1, 1), (\sqrt{7} \cot \frac{\pi}{7}, \sqrt{7} \cot \frac{2\pi}{7}, \sqrt{7} \cot \frac{4\pi}{7})$$

and the cyclic permutations. They all solve $(t-1)(t^3-7t^2+7t+7) = 0$.

31 July 2019

1. (a) Prove that every symmetric and positive definite matrix $A \in \mathbb{M}(n,\mathbb{R})$ can be uniquely written as $A = L \cdot L^\tau$, where L is a lower triangular matrix with positive diagonal elements.

   (b) Calculate L for $A = \begin{bmatrix} 4 & 2 & 4 & 4 \\ 2 & 10 & 17 & 11 \\ 4 & 17 & 33 & 29 \\ 4 & 11 & 29 & 39 \end{bmatrix}$.

Reason: Cholesky Decomposition

Solution:

(a) Induction over n. The statement is obviously true for $n = 1$. Let it be true for matrices in $\mathbb{M}(n-1,\mathbb{R})$. We will write

$$A = \begin{bmatrix} d & v^\tau \\ v & G \end{bmatrix}$$

Since A is positive definite, $x^\tau A x > 0$ for all $x \neq 0$. For $x = e_i := (0,\ldots,0,1,0,\ldots,0)^\tau$ we get $a_{ii} = e_i^\tau A e_i > 0$. Therefore we have $d > 0$.
With $H = G - \dfrac{vv^\tau}{d} \cdot I_{n-1}$ we get

$$A = \begin{bmatrix} d & v^\tau \\ v & G \end{bmatrix} = \begin{bmatrix} \sqrt{d} & 0 \\ \frac{v}{\sqrt{d}} & I_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & H \end{bmatrix} \cdot \begin{bmatrix} \sqrt{d} & \frac{v^\tau}{\sqrt{d}} \\ 0 & I_{n-1} \end{bmatrix}$$

H is symmetric by definition and also positive definite:

$$0 < \begin{bmatrix} -\frac{x^\tau v}{d} & x^\tau \end{bmatrix} \cdot \begin{bmatrix} d & v^\tau \\ v & G \end{bmatrix} \cdot \begin{bmatrix} -\frac{x^\tau v}{d} \\ x \end{bmatrix} = x^\tau \left(G - \frac{vv^\tau}{d} \right) x = x^\tau H x$$

Thus we can write $H = L_H L_H^\tau$ per induction assumption with a lower triangular matrix with positive diagonal elements. Finally we get

$$\begin{aligned} A &= \begin{bmatrix} \sqrt{d} & 0 \\ \frac{v}{\sqrt{d}} & I_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & L_H \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & L_H^\tau \end{bmatrix} \cdot \begin{bmatrix} \sqrt{d} & \frac{v^\tau}{\sqrt{d}} \\ 0 & I_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{d} & 0 \\ \frac{v}{\sqrt{d}} & L_H \end{bmatrix} \cdot \begin{bmatrix} \sqrt{d} & \frac{v^\tau}{\sqrt{d}} \\ 0 & L_H^\tau \end{bmatrix} \\ &= LL^\tau \end{aligned}$$

(b) $L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 2 & 3 & 5 & 1 \end{bmatrix}$

2. Let $L \subseteq H$ be a nonempty, closed, and convex set in a Hilbert space. Prove that there is an element of minimal norm in L.

   Reason: Completeness Properties.

   Solution: Let $d := \inf\{\,||f|| \,:\, f \in L\,\}$. Then there is a sequence $(f_n) \subseteq L$ such that $\lim_{n\to\infty} ||f_n|| = d$. By direct computation

$$\begin{aligned} \left|\left|\frac{f_n - f_m}{2}\right|\right|^2 &= 2\left|\left|\frac{f_n}{2}\right|\right|^2 + 2\left|\left|\frac{f_m}{2}\right|\right|^2 - \left|\left|\frac{f_n + f_m}{2}\right|\right|^2 \\ &\leq 2\left|\left|\frac{f_n}{2}\right|\right|^2 + 2\left|\left|\frac{f_m}{2}\right|\right|^2 - d^2 \end{aligned}$$

   where the inequality follows from convexity. Therefore

$$||f_n - f_m||^2 \leq 2||f_n||^2 + 2||f_m||^2 - 4d^2$$

   and so

$$\limsup_{n,m\to\infty} ||f_n - f_m||^2 \leq 2d^2 + 2d^2 - 4d^2 = 0$$

   which shows that (f_n) is a Cauchy sequence and as L is closed and therewith a complete subset in H, we conclude that there is $f \in L$ with $\lim_{n\to\infty} f_n$, which implies $||f|| = \lim_{n\to\infty} ||f_n|| = d$.

3. (HS-1) Is $N := 21^{39} + 39^{21}$ divisible by 45? Why, why not?

   Reason: Puzzle.

   Solution: $45 = 9 \cdot 5$ and $9|N$, so it remains to show that $5|N$. The last digit of 21^n is 1 and the last digit of 39^{2n+1} is 9 for any natural number n. Hence $10|N$ and especially $5|N$.

4. (HS-2) Let $0 < u, v, w < 1$. Show that among the numbers $u(1 - v)$, $v(1 - w)$, $w(1 - u)$ is at least one value not greater than $\frac{1}{4}$.

   Reason: Puzzle.

   Solution: Let us assume

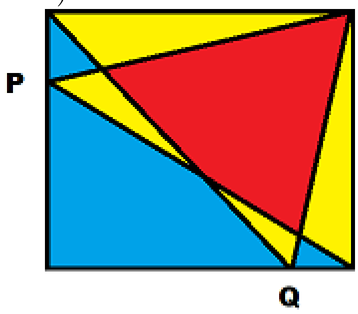
$$uvw(1-u)(1-v)(1-w) > \left(\frac{1}{4}\right)^3 = \frac{1}{64}$$

But $u(1-u) = \frac{1}{4} - (u - \frac{1}{2})^2 \leq \frac{1}{4}$ and likewise $v(1-v) \leq \frac{1}{4}$, and $w(1-w) \leq \frac{1}{4}$. Multiplying all three inequalities yields

$$u(1-u)v(1-v)w(1-w) \leq \frac{1}{64}$$

against our assumption.

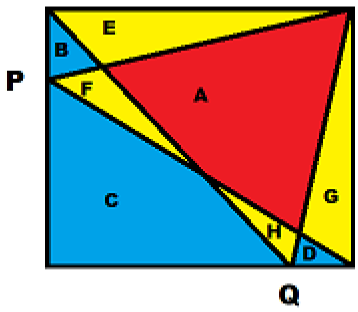
5. (HS-3) What is the ratio between the red and the blue area?



The points P and Q are anywhere on their edges.

Reason: Geometry.

Solution: Let's first label the areas and call the area of the square X.

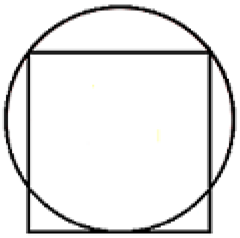


Since height and baseline of both triangles equal the side length of the square, their area is half of X:

$$A+E+H = \frac{1}{2}X = A+F+G = B+C+D+F+G = B+C+D+H+E$$

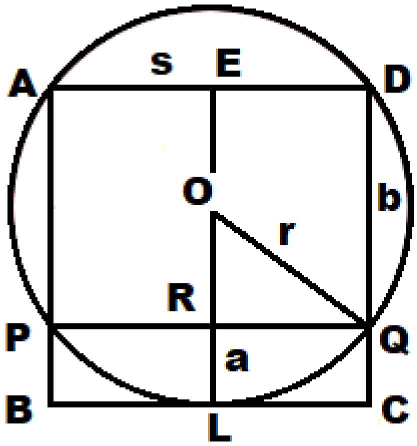
This means $\dfrac{A}{B+C+D} = \dfrac{A}{\frac{1}{2}X - F - G} = \dfrac{A}{A} = 1$

6. (HS-4) In what ratio does the circumference of the circle divide the left and right sides of the square?



Reason: Geometry.

Solution: Let's first label the graphic.



Pythagoras for ΔORQ gives us $r^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{s}{2}\right)^2$ and

$$s = \overline{AD} = \overline{AB} = \overline{EOL} = \mathrm{EO} + \mathrm{OL} = \frac{b}{2} + r$$

Thus $\left(s - \frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2 + \left(\frac{s}{2}\right)^2$ or $bs = \frac{3}{4}s^2$ or $\frac{s}{b} = \frac{4}{3}$.
From $a + b = s$ we get $\frac{a}{b} = \frac{1}{3}$.

32 June 2019

1. Let $\mathfrak{g}$ be a Lie algebra. Define
$$\mathfrak{A}(\mathfrak{g}) = \{\, \alpha \,:\, \mathfrak{g} \longrightarrow \mathfrak{g} \,|\, \forall\, X, Y \in \mathfrak{g} \,:\, 0 = [\alpha(X), Y] + [X, \alpha(Y)] \,\}$$
Show that $\mathfrak{A}(\mathfrak{g})$ is a Lie algebra and $X.\alpha(Y) = [X, \alpha(Y)] - \alpha([X, Y])$ defines a representation of $\mathfrak{g}$ on $\mathfrak{A}(g)$.

   Reason: Representation theory.

   Solution: Linearity is obvious for both cases, that $\mathfrak{A}(\mathfrak{g})$ is a vector space as well as that $X.\alpha$ is a linear transformation.

   (a) Let $\alpha, \beta \in \mathfrak{A}(\mathfrak{g})$.
$$\begin{aligned}[[\alpha,\beta]X, Y] &= [(\alpha\beta - \beta\alpha)X, Y] \\ &= -[\beta X, \alpha Y] + [\alpha X, \beta Y] \\ &= [X, \beta\alpha Y] - [X, \alpha\beta Y] \\ &= -[X, [\alpha,\beta]Y]\end{aligned}$$

   (b) A representation of $\mathfrak{g}$ on $\mathfrak{A}(\mathfrak{g})$ is a Lie algebra homomorphism $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{A}(\mathfrak{g}))$ and in our case $\varphi(X)(\alpha) := [\operatorname{ad} X, \alpha]$. Therefore we have to show that $\varphi(\alpha) \in \mathfrak{A}(\mathfrak{g})$ and $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$.
$$\begin{aligned}[(\varphi(X)(\alpha))(Y), Z] &= [[X, \alpha(Y)], Z] - [\alpha([X, Y]), Z] \\ &= -[[\alpha X, Y], Z] + [[X, Y], \alpha Z] \\ &= [[Y, Z], \alpha X] + [[Z, \alpha X], Y] - [[Y, \alpha Z], X] - [[\alpha Z, X], Y] \\ &= [[Y, Z], \alpha X] - [[\alpha Z, X], Y] - [[Y, \alpha Z], X] - [Y, [X, \alpha Z]] \\ &= -[[Z.\alpha X], Y] - [[\alpha X, Y], Z] - [[\alpha Z, X], Y] - [[Y, \alpha Z], X] - [Y, [X, \alpha Z]] \\ &= [[X, \alpha Y], Z] + [[\alpha Y, Z], X] - [Y, [X, \alpha Z]] \\ &= -[[Z, X], \alpha Y] - [Y, [X, \alpha Z]] \\ &= [Y, \alpha([X, Z])] - [Y, [X, \alpha Z]] \\ &= -[Y, (\varphi(X)(\alpha))(Z)]\end{aligned}$$

$$\begin{aligned}\varphi([X, Y])(\alpha) &= [\operatorname{ad}([X, Y]), \alpha] \\ &= [[\operatorname{ad}(X), \operatorname{ad}(Y)], \alpha] \\ &= -[[\operatorname{ad}(Y), \alpha], \operatorname{ad}(X)] - [[\alpha, \operatorname{ad}(X)], \operatorname{ad}(Y)] \\ &= [\operatorname{ad}(X), [\operatorname{ad}(Y), \alpha]] - [\operatorname{ad}(Y), [\operatorname{ad}(X), \alpha]] \\ &= [\operatorname{ad}(X), \varphi(Y)(\alpha)] - [\operatorname{ad}(Y), \varphi(X)(\alpha)] \\ &= \varphi(X)(\varphi(Y)(\alpha) - \varphi(Y)(\varphi(X)(\alpha) \\ &= ([\varphi(X), \varphi(Y)])(\alpha)\end{aligned}$$

2. Let R be a commutative ring with 1 and I an ideal. Show that R/I is an integral domain if and only if I is a prime ideal, and that R/I is a field if and only if I is a maximal ideal.

   Reason: Standard result in commutative algebra.

   Solution: R/I is an integral domain, if $\bar{a}\bar{b} = \bar{0}$ implies $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$, i.e. $ab \in I$ implies $a \in I$ or $b \in I$ which is the definition of a prime ideal I.

   Now let $I \triangleleft R$ be a maximal ideal and $r \notin I$. Then $I + rR = R$ so we have elements $a \in I$, $s \in R$ with $1 = a + rs$ and thus $\bar{1} = \bar{0} + \bar{r}\bar{s} = \bar{r}\bar{s}$, i.e. all elements of R/I different from zero are invertible.

   If on the other hand R/I is a field, and $I \triangleleft R$ is not maximal, then there is an ideal $I \subsetneq M \triangleleft R$ and an element $m \in M - I$ such that $\bar{m} \neq \bar{0}$ is invertible. This means there is a $r \in R$ such that $\bar{r}\bar{m} = \bar{1}$ or $rm - 1 \in I \subset M$. But this means $1 \in M$ and so $R = M$, which makes I a maximal ideal.

3. Solve $x^2y'' + xy' - y = x^3$ for positive x. **Reason:** ODE.

   Solution: Let $x = e^u$ so that $u = \log(x)$. Then

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\,\frac{1}{x} \\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{du}\,\frac{1}{x}\right) \\
&= \frac{dy}{dx}\frac{dy}{du}\,\frac{1}{x} - \frac{dy}{du}\,\frac{1}{x^2} \\
&= \frac{dy}{du}\,\frac{1}{x}\frac{dy}{du}\,\frac{1}{x} - \frac{dy}{du}\,\frac{1}{x^2} \\
&= \left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)\frac{1}{x^2}
\end{aligned}
$$

   Therefore,

$$
x^2y'' + xy' - y = \left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + \frac{dy}{du} - y = \frac{d^2y}{du^2} - y = e^{3u}
$$

   Considering the homogeneous case, we have $\lambda^2 - 1 = 0$ or $\lambda = \pm 1$. Therefore, $y_1 = e^u$ and $y_2 = e^{-u}$. Using undetermined coefficients, let

$y_p = Ae^{3u}$. Then $y' = 3Ae^{3u}$ and $y'' = 9Ae^{3u}$. Substitution gives

$$\underbrace{9Ae^{3u} - Ae^{3u}}_{8Ae^{3u}} = e^{3u}$$

so $A = \frac{1}{8}$ and the general solution is

$$\begin{aligned} y &= C_1e^u + C_2e^{-u} + \frac{1}{8}e^{3u} \\ &= C_1x + \frac{C_2}{x} + \frac{x^3}{8} \end{aligned}$$

4. Show that the Schwarzian Derivative

$$(Sf)(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

vanishes if and only if $f(z) = \dfrac{az+b}{cz+d}$ is a Möbius transformation.

Reason: Funny derivative.

Solution: Let $f(z) = \dfrac{az+b}{cz+d} = \dfrac{U(z)}{V(z)} = \dfrac{U}{V}$. Then

$$f'(z) = \frac{aV - cU}{V^2}$$

$$f''(z) = -2c\,\frac{aV - cU}{V^3}$$

and thus

$$(Sf)(z) = \left(\frac{-2c}{V}\right)' - \frac{2c^2}{V^2} = \frac{2c}{V^2}\,c - \frac{2c^2}{V^2} = 0$$

Let us now assume that $(Sf)(z) = 0$ and set $U(z) := \dfrac{f''(z)}{f'(z)}$, i.e. $2U' = U^2$ which means $U(z) = -\dfrac{2}{c_1+z}$, i.e. $f''(z) = -\dfrac{2}{c_1+z}\,f'(z)$. The solution to this differential equation is

$$\begin{aligned} f(z) &= \frac{c_2}{c_1+z} + c_3 \\ &= \frac{(c_2 + c_1c_3) + c_3z}{c_1+z} \\ &= \frac{(c_2c_4 + c_1c_3c_4) + c_3c_4z}{c_1c_4 + c_4z} \\ &= \frac{az+b}{cz+d} \end{aligned}$$

5. Let $x(t)$ be the height at time t, measured positively on the downward direction. If we consider only gravity, then $\ddot{x}(t) = \dfrac{d^2x}{dt^2} = a$ is a constant, denoted g, the acceleration due to gravity. Note that $F = ma = mg$. Air resistance encountered depends on the shape of the object and other things, but under most circumstances, the most significant effect is a force opposing the motion which is proportional to a power of the velocity $v(t) = \dot{x}(t)$. So

$$\ddot{x}(t) \cdot m = m \cdot g - k\dot{x}(t)^n$$

which is a second order differential equation, but there is no x term. So it is first order in $\dot{x}$. Therefore,

$$\frac{dv}{dt} = g - \frac{k}{m}v^n$$

This is not easy to solve, so we will make the simplifying approximation that $n = 1$ (if v is small, there is not much difference between v and v^n). Therefore, we have to solve

$$\frac{dv}{dt} + \frac{k}{m}v = g$$

Reason: ODE.

Solution: The integration factor is

$$I = e^{\int \frac{k}{m}\,dt} = e^{kt/m}$$

and thus

$$\begin{aligned}\left(\frac{dv}{dt} + \frac{k}{m}\,v\right) e^{kt/m} &= ge^{kt/m}\\ e^{kt/m}v &= \frac{gm}{k}\,e^{kt/m} + C\\ v &= \frac{mg}{k} + Ce^{-kt/m}\end{aligned}$$

with an arbitrary constant C. By $v(0) = v_0$ we get $C = v_0 - \dfrac{mg}{k}$

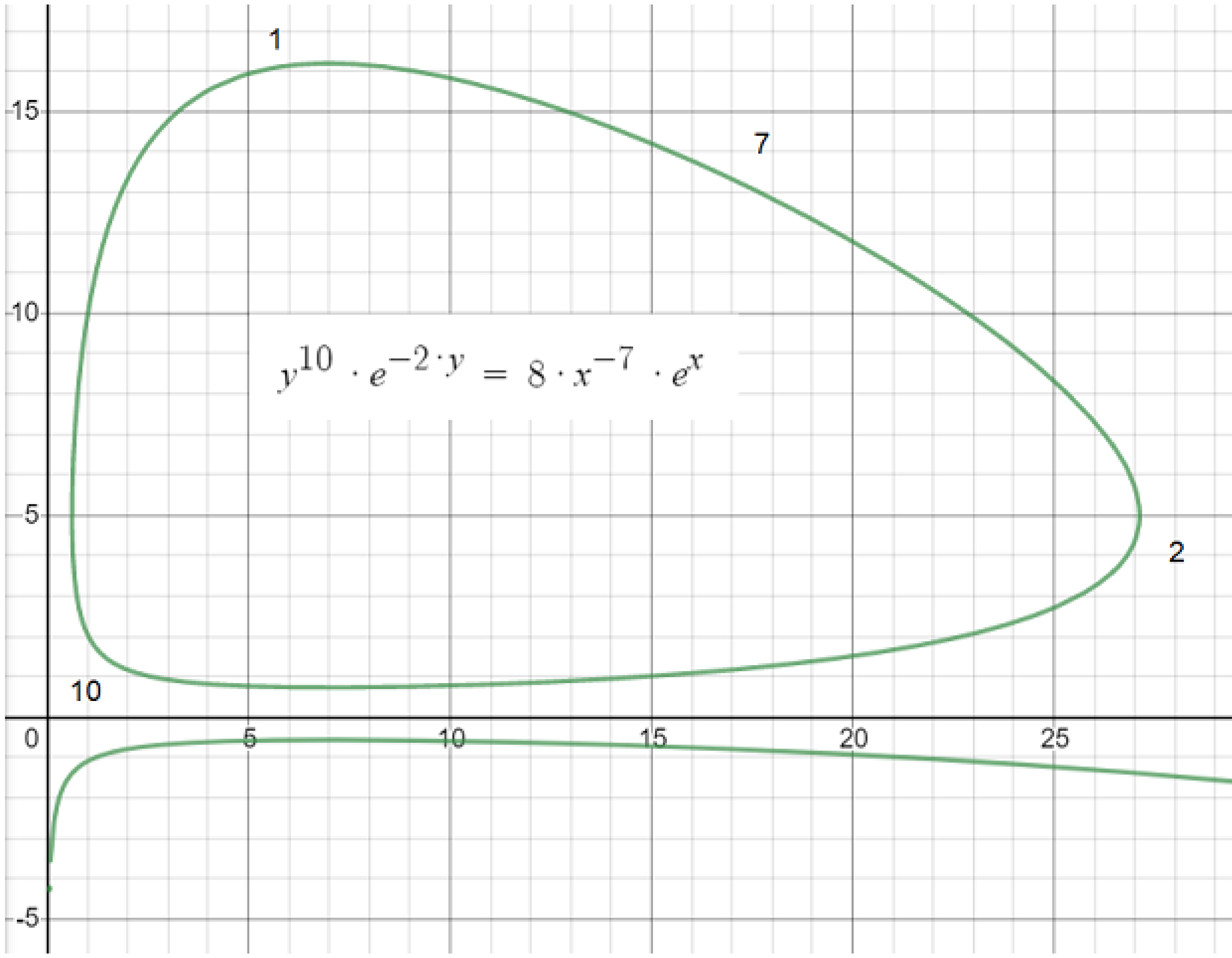
$$\begin{aligned}\int_{x_0}^{x} dx &= \int_0^t v\,dt\\ &= \int_0^t \left(\frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right) e^{-kt/m}\right)\\ &= \frac{mg}{k}\,t - \frac{m}{k}\,\left(v_0 - \frac{mg}{k}\right)\left(e^{-kt/m} - 1\right)\end{aligned}$$

that is
$$x(t) = x_0 + \frac{mg}{k}\,t + \frac{m}{k}\,\left(v_0 - \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right)$$

6. Consider a land populated by foxes and rabbits, where the foxes prey upon the rabbits. Let $x(t)$ and $y(t)$ be the number of rabbits and foxes, respectively, at time t. In the absence of predators, at any time, the number of rabbits would grow at a rate proportional to the number of rabbits at that time. However, the presence of predators also causes the number of rabbits to decline in proportion to the number of encounters between a fox and a rabbit, which is proportional to the product $x(t)y(t)$. Therefore, $dx/dt = ax - bxy$ for some positive constants a and b. For the foxes, the presence of other foxes represents competition for food, so the number declines proportionally to the number of foxes but grows proportionally to the number of encounters. Therefore $dy/dt = -cy + dxy$ for some positive constants c and d. The system
$$\dot{x}(t) = \frac{dx}{dt} = ax(t) - bx(t)y(t)\ ,\ \dot{y}(t) = \frac{dy}{dt} = -cy(t) + dx(t)y(t)$$
is our mathematical model. Eliminate the time parameter and find the relation between the population of foxes and the number of rabbits for parameters $a = 10\,,\ b = 2\,,\ c = 7\,,\ d = 1\,.$

Reason: Predator Prey Model.

Solution:
$$\begin{aligned} \frac{dy}{dx} &= \frac{-7y + xy}{10x - 2xy} \\ \Longrightarrow \frac{(10 - 2y)\,dy}{y} &= \frac{(-7 + x)\,dx}{x} \\ \Longrightarrow 10\log y - 2y &= -7\log x + x + C \\ \Longrightarrow y^{10}e^{-2y} &= k\,x^{-7}e^{x} \end{aligned}$$
with a constant positive parameter $k = e^C$.



7. Five vessels contain 100 balls each. Some vessels contain only balls of $10\,g$ mass, while the other vessels contain only balls of $11\,g$ mass. How can we determine with a single weighing which results in a mass, which vessels contain balls of $10\,g$ and which contain balls of $11\,g$? (It is allowed to remove balls from the vessels.) **Reason:** Riddle about the binary representation of numbers.

   Solution: We remove 2^k balls from vessel k and weigh those 31 balls. Let the result be $a\,g$. Thus we have an equation

   $$x_1 + 2x_2 + 4x_3 + 8x_4 + 16x_5 = a - 31 \cdot 10\,g \quad (x_k \in \{\,0, 1\,\})$$

   which is a unique binary representation and $x_k = 0$ are the vessels with $10\,g$ balls, $x_k = 1$ are the vessels with $11\,g$ balls.

8. Let $f \in L^1(\mathbb{R}^3)$ be rotation symmetric, i.e. $f(Rx) = f(x)$ for all $R \in \mathrm{SO}(3)$. Show that the Fourier transform $\mathcal{F}f$ is rotation symmetric, too, and calculate $\mathcal{F}f$ of $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by

$$f(x) = \frac{1}{|x|}\chi_{B_1(0)}(x)$$

with the Euclidean norm $|\,.\,|$, the unit ball $B_1(0)$ around the origin, and the characteristic function χ. **Reason:** Fourier transformation.

Solution: Since $\det R^\tau = 1$ we get with $y = R^\tau x$

$$\begin{aligned}\mathcal{F}f(R\xi) &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} f(x)\exp(-ix\cdot R\xi)\,d\lambda_3(x)\\ &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} f(R^\tau x)\exp(-iR^\tau x\cdot \xi)\,d\lambda_3(x)\\ &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} f(y)\exp(-iy\cdot \xi)\,d\lambda_3(x)\\ &= \mathcal{F}f(\xi)\end{aligned}$$

We use spherical coordinates for the second part, i.e. the function

$$\begin{aligned}\Phi \,:\, (0,1)\times(-\pi/2,\pi/2)\times(0,2\pi) &\longrightarrow B_1(0)\\ (r,\varphi,\theta) &\longmapsto (r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, r\sin\varphi)\end{aligned}$$

where $\det D\Phi = r^2\cos\varphi$. Then we get with $\xi = te_3$, $t\in\mathbb{R}$ and Fubini the Fourier transform

$$\begin{aligned}\mathcal{F}f(te_3) &= (2\pi)^{-3/2}\int_{B_1(0)} |x|^{-1}\exp(-itx_3)\,d\lambda_3(x)\\ &= (2\pi)^{-3/2}\int_0^1\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} r^2\cos\varphi\cdot r^{-1}\exp(-itr\sin\varphi)\,d\lambda_\theta\,d\lambda_\varphi\,d\lambda_r\\ &= (2\pi)^{-1/2}\int_0^1\int_{-\pi/2}^{\pi/2} r\cos\varphi\,\exp(-itr\sin\varphi)\,d\lambda_\varphi\,d\lambda_r\\ &\stackrel{u(\varphi)=r\sin\varphi}{=} (2\pi)^{-1/2}\int_0^1\int_{-r}^{r}\exp(-itu)\,d\lambda_u\,d\lambda_r\\ &= (2\pi)^{-1/2}\int_0^1\int_{-r}^{r}\cos(tu)\,d\lambda_u\,d\lambda_r\\ &= (2\pi)^{-1/2}\int_0^1 2\cdot\frac{\sin(tr)}{t}\,d\lambda_r\\ &= \sqrt{\frac{2}{\pi}}\,\frac{1-\cos t}{t^2}\end{aligned}$$

Since f is rotation symmetric we know

$$\mathcal{F}f(\xi) = \mathcal{F}f(|\xi|e_3) = \sqrt{\frac{2}{\pi}}\,\frac{1-\cos|\xi|}{|\xi|^2}$$

9. Solve $(3x^2y^2 + x^2)\,dx + (2x^3y + y^2)\,dy = 0$. **Reason:** Exact forms.

Solution:

$$\omega := \underbrace{(3x^2y^2 + x^2)}_{F}\,dx + \underbrace{(2x^3y + y^2)}_{G}\,dy$$

and observe that $\dfrac{\partial G}{\partial x} = 6x^2y = \dfrac{\partial F}{\partial y}$ so there exist a g such that $\omega = dg$ and

$$\frac{\partial g}{\partial x} = 3x^2y^2 + x^2\,,\;\frac{\partial g}{\partial y} = 2x^3y + y^2$$

Integrating the first yields $g = x^3y^2 + \dfrac{1}{3}x^3 + h(y)$ which differentiated with respect to y gives

$$\frac{\partial g}{\partial y} = 2x^3y + y^2 = 2x^3y + \frac{dh}{dy}$$

which means $h(y) = \dfrac{y^3}{3} + C$ for some arbitrary constant C. Hence $g = x^3y^2 + \dfrac{x^3}{3} + \dfrac{y^3}{3} + C$. With $\omega = dg = 0$ we have $x^3y^2 + \dfrac{x^3}{3} + \dfrac{y^3}{3} + C = C'$ for some arbitrary constant C' . With $D = C' - C$ which is still an arbitrary constant, the solution is

$$x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} = D$$

10. Calculate $\lim_{x\to 0} \dfrac{\cos^2 x - 1}{\sinh^2 x}$ and $\lim_{x\to 0} \dfrac{e^x + e^{-x} - 2 - x^2}{(\cos x - 1)^2}$ **Reason:** L'Hôpital.

Solution: For $f(x) := \cos^2 x - 1$ and $g(x) := \sinh^2 x$ we have $f'(x) = -2\cos x \sin x = -\sin(2x)$ and $g'(x) = 2\sinh x \cosh x = \sinh(2x)$. So $\lim_{x\to 0} f'(x) = f'(0) = 0$ and $\lim_{x\to 0} g'(x) = g'(0) = 0$ and we cannot apply the rule of L'Hôpital. However, the functions $F(x) := \sin(2x)$

and $G(x) := \sinh(2x)$ do fulfill the conditions in a neighborhood of $x = 0$ such that

$$\lim_{x\to 0} \frac{F'(x)}{G'(x)} = \lim_{x\to 0} \frac{-2\cos(2x)}{2\cosh(2x)} = -1$$

which means that $\dfrac{F(x)}{G(x)}$ has a limit for $x \to 0$ by L'Hôpital and we have again with L'Hôpital

$$-1 = \lim_{x\to 0} \frac{F'(x)}{G'(x)} = \lim_{x\to 0} \frac{F(x)}{G(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{\cos^2 x - 1}{\sinh^2 x}$$

By application of L'Hôpital four times we get

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x + e^{-x} - 2 - x^2}{(\cos x - 1)^2} &= \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{-2(\cos x - 1)\sin x} \\
&= \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{-\sin(2x) + 2\sin x} \\
&= \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{-2\cos(2x) + 2\cos x} \\
&= \lim_{x\to 0} \frac{e^x - e^{-x}}{4\sin(2x) - 2\sin x} \\
&= \lim_{x\to 0} \frac{e^x + e^{-x}}{8\cos(2x) - 2\cos x} \\
&= \frac{1}{3}
\end{aligned}$$

11. (HS-1) There are two bands in front of you. The two bands are of different lengths and made of different materials. But both take exactly an hour to burn from one end to the other. The burning speed is not constant, so the tape can burn fast at the beginning, then slower and faster, or randomly. You only have a box of matches and you should measure exactly 45 minutes with the help of the tapes. You must not cut the tapes, use a watch, etc.!

    Reason: Puzzle.

    Solution: You know that each band takes an hour to burn down. So you're lighting one band at both ends, and the other band at one end. When the first one has completely burned out, 30 minutes have passed, and you also ignite the second end of the other band. It takes exactly 15 minutes now to burn it down so that a total of 45 minutes has passed!

12. (HS-2) At the end of a one round chess tournament in which all players played once against each other we have the following result:

    1.Alan 2.Bernie 3.Chuck 4.David 5.Ernest

    The ranking is unambiguous, i.e. all have different scores, and as usual, a victory gets 1 point, a draw 1/2. Bernie is the only one who didn't lose, Ernest the only one who didn't win.

    Who played whom with which result?

    Reason: Puzzle.

    Solution:
    1. Alan has beaten Chuck, David and Ernest and lost against Bernie. (3)
    2. Bernie has beaten Alan and drew against the others. (2.5)
    3. Chuck has beaten David and drew against Bernie and Ernest. (2)
    4. David drew against Bernie and won against Ernest. (1.5)
    5. Ernest drew against Bernie and Chuck. (1)

13. (HS-3) A **unit** e is an element for which there is a multiplicative inverse, i.e. there is an e' such that $e \cdot e' = e' \cdot e = 1$. Units are divisors of 1 .
    An **irreducible** element $n \neq 0$ is an element, which cannot be written as $n = a \cdot b$ unless either a or b is a unit.
    A **prime** p is an element, which is not a unit and if $p \,|\, a \cdot b$ then either $p \,|\, a$ or $p \,|\, b$.
    Show that primes are irreducible, and irreducible elements are either units or primes. Bonus: If we think about integers, which essential property do we need?

    Reason: Primes and Proof Techniques.

    Solution: Assume p is irreducible and $p \,|\, a \cdot b$. Then $p = q \cdot a \cdot b$ and since p is irreducible, one of the factors has to be p and the others units. If $p = a$ or $p = b$ we are done, because then $p \,|\, a$ or $p \,|\, b$. If $p = q$, then $p \cdot (a \cdot b) - p = p \cdot (a \cdot b - 1) = 0$ and because we have an integral domain (no zero divisors), $a \cdot b = 1$ $(p \neq 0)$. Hence $p \,|\, 1$ and p is a unit.

    If p is prime then it is unequal 0 since we have an integral domain. Let $p = a \cdot b$ then $p \,|\, a$ or $p \,|\, b$, say $a = q \cdot p$. Thus $p = a \cdot b = q \cdot p \cdot b$ and again $p \cdot (1 - q \cdot b) = 0$ so that $q \cdot b = 1$ are units. Hence p cannot be written as $p = a \cdot b$ except one factor is a unit, in our case b .

14. (HS-4) The border collie Boy is at the end of a 1 km flock of sheep, which moves forward at a constant speed. As a control he now walks

- with a greater constant speed than the herd - from the end to the top of the herd and back to his place at the end of the flock. When he arrives back, the flock of sheep has walked exactly one kilometer further. Which distance did Boy run?

Reason: Puzzle.

Solution: Assume the flock is moving at a speed v_f and Boy at v_b. Boy's time to the top be t_1 and on the way back t_2, x the distance of the last sheep during t_1. Then we have for the first leg:

$$x = v_f \cdot t_1 \tag{3}$$
$$1 + x = v_b \cdot t_1 \tag{4}$$

and for the second leg

$$x = v_b \cdot t_2 \tag{5}$$
$$1 - x = v_f \cdot t_2 \tag{6}$$

Thus we have $(1-x)\cdot(1+x) = v_f v_b t_1 t_2 = x^2$ and $x = \dfrac{1}{\sqrt{2}}$ which means that Boy ran $1 + 2x = 1 + \sqrt{2} \approx 2.414$ km .

15. (HS-5) What is the smallest limit $L > \dfrac{\pi}{6}$ such that

$$\int_{\pi/6}^{L} \frac{dx}{\sin^2 x} = \int_{\pi/6}^{L} \frac{dx}{1-\cos x} + \int_{\pi/6}^{L} 6\,\frac{\cot x}{\sin x}\,dx$$

Reason: Trig Functions.

Solution:

$$\begin{aligned} 0 &= \int_{\pi/6}^{L} \left(\frac{1}{\sin^2 x} - \frac{1}{1-\cos x} - 6\,\frac{\cot x}{\sin x} \right) dx \\ &= \int_{\pi/6}^{L} \left(\frac{1}{1-\cos^2 x} \left(1 - (1+\cos x) - 6\,\cos x\right) \right) dx \\ &= -7 \int_{\pi/6}^{L} \frac{\cos x}{\sin^2 x}\,dx \\ &= 7 \left[\frac{1}{\sin x} \right]_{\pi/6}^{L} \\ &\Longleftrightarrow \end{aligned}$$

$$\sin L = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

and $L = \dfrac{5\pi}{6}$ is the smallest possible value L.

33 May 2019

1. (a) Let $(\mathfrak{su}(2,\mathbb{C}),\varphi,V)$ be a finite dimensional representation of the Lie algebra $\mathfrak{g}=\mathfrak{su}(2,\mathbb{C})$. Calculate $H^{\,0}(\mathfrak{g},\varphi)$ and $H^{\,1}(\mathfrak{g},\varphi)$ for the Chevalley-Eilenberg complex in the cases
   i. $(\varphi,V)=(\operatorname{ad},\mathfrak{g})$
   ii. $(\varphi,V)=(0,\mathfrak{g})$
   iii. $(\varphi,V)=(\pi,\mathbb{C}^2)$ is the natural representation on $\mathbb{C}^2$.

   (b) Consider the Heisenberg algebra $\mathfrak{g}=\mathfrak{h}=\left\{\left.\begin{pmatrix}0&a&c\\0&0&b\\0&0&0\end{pmatrix}\,\right|\,a,b,c\in\mathbb{R}\right\}$ and calculate $H^{\,0}(\mathfrak{h},\operatorname{ad})$ and $H^{\,1}(\mathfrak{h},\operatorname{ad})$.

Reason: Cohomology of Lie algebras.

Solution: The differentials of the cochains

$$C^{\,n}=C^{\,n}(\mathfrak{g},V)=\operatorname{Hom}(\wedge^n\mathfrak{g},V)\,,\;C^{\,-1}=\{\,0\,\}\,,\;C^{\,0}=V$$

are given by

$$d^{\,n}\,:\,C^{\,n}\longrightarrow C^{\,n+1}$$

$$\begin{aligned}&d^{\,n}(\omega).(X_1\wedge\ldots\wedge X_{n+1})\\&\quad=\sum_i(-1)^{i+1}\varphi(X_i)\left(\omega(X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge X_{n+1})\right)\\&\quad+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j]\wedge X_1\wedge\ldots\wedge\hat{X}_i\wedge\ldots\wedge\hat{X}_j\wedge\ldots\wedge X_{n+1})\end{aligned}$$

As $d^{\,n+1}d^{\,n}=0$ we have the cocycles $Z^{\,n}=Z^{\,n}(\mathfrak{g},\mathfrak{g})=\ker d^{\,n}$, the coboundaries $B^{\,n}=B^{\,n}(\mathfrak{g},\mathfrak{g})=\operatorname{im}d^{\,n-1}$ and the cohomology groups $H^{\,n}=H^{\,n}(\mathfrak{g},\mathfrak{g})=H^{\,n}(\mathfrak{g},\operatorname{ad})=Z^{\,n}/B^{\,n}$.

The relevant sequence is

$$\{\,0\,\}\stackrel{d^{\,-1}}{\longrightarrow}V\stackrel{d^{\,0}}{\longrightarrow}\operatorname{Hom}(\mathfrak{g},V)\stackrel{d^{\,1}}{\longrightarrow}\operatorname{Hom}(\mathfrak{g}\wedge\mathfrak{g},V)\stackrel{d^{\,2}}{\longrightarrow}\cdots$$

We want to know $H^{\,0}(\mathfrak{g},\varphi)=\ker d^{\,0}\,,\;H^{\,1}(\mathfrak{g},\varphi)=\ker d^{\,1}/\operatorname{im}d^{\,0}$.

(a) $$H^0(\mathfrak{g},\varphi)=\{v\in V\,|\,\forall\, X\in\mathfrak{g}\,:\,\varphi(X)(v)=0\}=\begin{cases}\mathfrak{Z}(\mathfrak{g})=\{\,\mathbf{0}\,\} & \text{if } \varphi=\operatorname{ad}\\ \mathfrak{g}=\mathfrak{su}(2,\mathbb{C}) & \text{if } \varphi=0\\ \{\,\mathbf{0}\,\} & \text{if } \varphi=\pi\end{cases}$$

$$\begin{aligned}B^1&=\operatorname{im} d^0\\ &=\{\omega\in C^1\,|\,\omega(X)=d^0(v)(X)=\varphi(X)(v) \text{ for a } v\in V\}\\ &=\begin{cases}\operatorname{ad}(\mathfrak{g})\cong\mathfrak{su}(2,\mathbb{C}) & \text{if } \varphi=\operatorname{ad}\\ \{\,\mathbf{0}\,\} & \text{if } \varphi=0\\ \mathbb{C}^2\quad(^*) & \text{if } \varphi=\pi\end{cases}\end{aligned}$$

$$\begin{aligned}Z^1&=\ker d^1\\ &=\{\omega\in C^1\,|\,d^1(\omega)(X,Y)=0\}\\ &=\{\omega\in C^1\,|\,\omega([X,Y])=\varphi(X)\omega(Y)-\varphi(Y)\omega(X)\}\\ &=\begin{cases}\operatorname{Der}(\mathfrak{g})=\operatorname{ad}(\mathfrak{g})\cong\mathfrak{su}(2,\mathbb{C}) & \text{if } \varphi=\operatorname{ad}\\ \{\,\mathbf{0}\,\} & \text{if } \varphi=0\\ \mathbb{C}^2\quad(^{**}) & \text{if } \varphi=\pi\end{cases}\end{aligned}$$

(*) According to the basis

$$i\sigma_1=\begin{bmatrix}0&i\\ i&0\end{bmatrix}\;,\;i\sigma_2=\begin{bmatrix}0&1\\ -1&0\end{bmatrix}\;,\;i\sigma_3=\begin{bmatrix}i&0\\ 0&-i\end{bmatrix}$$

of $\mathfrak{su}(2,\mathbb{C})$ we have $B^1(\mathfrak{g},\pi)=\left\{\begin{bmatrix}iz_2&z_2&iz_1\\ iz_1&-z_1&-iz_2\end{bmatrix}\,\middle|\,z_1,z_2\in\mathbb{C}\right\}\cong\mathbb{C}^2$.

(**) With the same basis as above and $\omega=\begin{bmatrix}A&B&C\\ U&V&W\end{bmatrix}$ we can solve the three equations

$$\omega([X,Y])=\pi(X)\omega(Y)-\pi(Y)\omega(X)=X\cdot\omega(Y)-Y\cdot\omega(X)$$

by using $[\mathfrak{su}(2,\mathbb{C}),\mathfrak{su}(2,\mathbb{C})]=\mathfrak{su}(2,\mathbb{C})$ and find

$$Z^1(\mathfrak{g},\pi)=\left\{\begin{bmatrix}A&-iA&U\\ U&iU&-A\end{bmatrix}\,\middle|\,A,U\in\mathbb{C}\right\}\cong\mathbb{C}^2$$

Therefore $H^1(\mathfrak{su}(2,\mathbb{C}),\varphi)=\{\,\mathbf{0}\,\}$ in all three cases of (φ,V).

(b) Let E_{ij} be a matrix with a 1 in $i-$th row and $j-$th column, and 0 elsewhere. Then we choose as basis $(A = E_{12}, B = E_{23}, C = E_{13})$ for $\mathfrak{h}$. As before we get

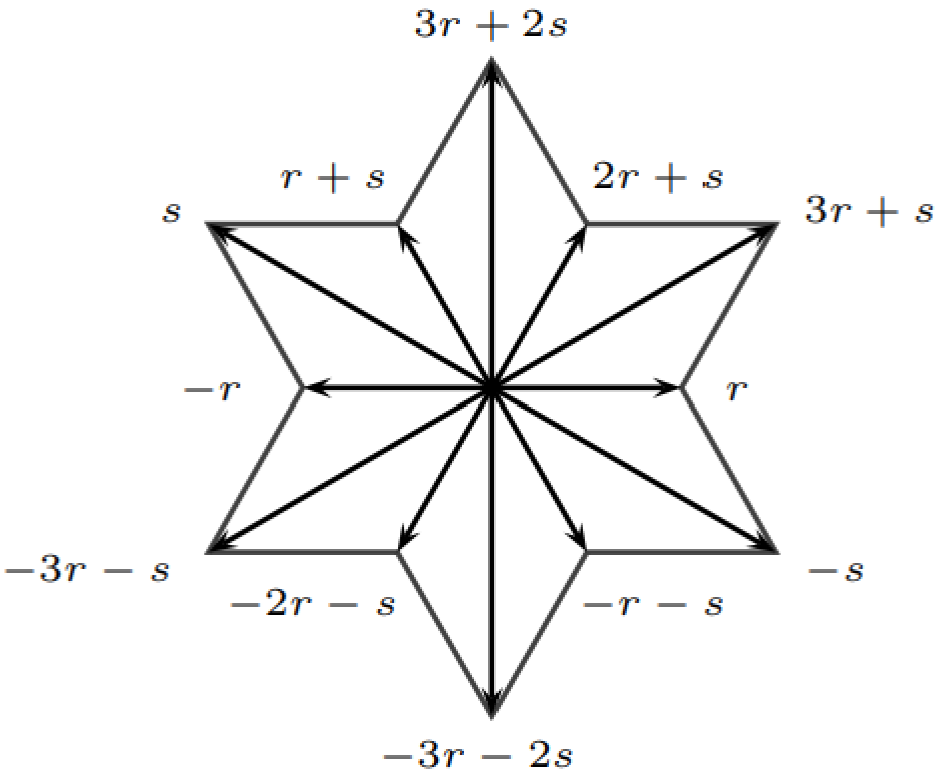
$$\begin{aligned}
H^0(\mathfrak{h}, \operatorname{ad}) &= \{ Y \in \mathfrak{h} \,|\, \forall X \in \mathfrak{h} \, : \, [X, Y] = 0 \} = \mathfrak{Z}(\mathfrak{h}) = \mathbb{R} \cdot C \\
B^1(\mathfrak{h}, \operatorname{ad}) &= \{ \omega \in C^1 \,|\, \omega(X) = \operatorname{ad}(X)(Y) \text{ for a } Y \in \mathfrak{h} \} \\
&= \{ \omega \in C^1 \,|\, \omega = -\operatorname{ad}(Y) \text{ for a } Y \in \mathfrak{h} \} \\
&= \operatorname{ad}(\mathfrak{h}) \\
&\cong \langle \operatorname{ad} A, \operatorname{ad} B \,|\, [\operatorname{ad} A, \operatorname{ad} B] = \operatorname{ad}[A, B] = \operatorname{ad} C = 0 \rangle \\
&= \langle E_{32} \, , \, -E_{31} \rangle \cong \mathbb{R}^2 \\
Z^1(\mathfrak{h}, \operatorname{ad}) &= \{ \omega \in C^1 \,|\, \omega([X, Y]) = [X, \omega(Y)] - [Y, \omega(X)] \} \\
&= \operatorname{Der}(\mathfrak{h}) \\
&= \left\{ \omega = \begin{pmatrix} \alpha & r_{12} & 0 \\ r_{21} & \beta & 0 \\ r_{31} & r_{32} & \alpha + \beta \end{pmatrix} \,\middle|\, \alpha, \beta, r_{ij} \in \mathbb{R} \right\} \\
H^1(\mathfrak{h}, \operatorname{ad}) &= Z^1(\mathfrak{h}, \operatorname{ad}) / B^1(\mathfrak{h}, \operatorname{ad}) \\
&= \left\{ \omega = \begin{pmatrix} \alpha & r_{12} & 0 \\ r_{21} & \beta & 0 \\ 0 & 0 & \alpha + \beta \end{pmatrix} \,\middle|\, \alpha, \beta, r_{ij} \in \mathbb{R} \right\} \\
&\cong \mathfrak{gl}(\mathbb{R}^2) \\
&\cong \mathfrak{gl} \left(\mathfrak{h} / [\mathfrak{h}, \mathfrak{h}] \right)
\end{aligned}$$

This demonstrates, that there are significant differences between semisimple and solvable Lie algebras (cp. Whitehead Lemmas).

2. Show that the dihedral group D_{12} of order twelve is the finite reflection group of the root system of type G_2.

Reason: Buildings.

Solution: The roots of G_2 are $\pm \{ r, s, r+s, 2r+s, 3r+s, 3r+2s \}$ (cp. https://www.physicsforums.com/insights/lie-algebras-a-walkthrough-the-structures/) which can be visualized by the following figure:



The covering transformations are generated by a rotation R of 30° and a reflection S at the axis $r \longleftrightarrow -r$ which is the group

$$\{\, R, S \,:\, S^2 = R^6 = 1\,,\, SRS = R^{-1} \,\} = D_{12}.$$

3. Consider the set

$$\mathcal{P}_n := \{\, \{2\}, \{4\}, \ldots, \{2n\} \,\} \subseteq \mathcal{P}(\mathbb{N})$$

and determine the $\sigma-$algebra $\mathcal{A}_\sigma(\mathcal{P}_n) \subseteq \mathcal{P}(\mathbb{N})$, and show that $\bigcup_{n\in\mathbb{N}} \mathcal{A}_\sigma(\mathcal{P}_n)$ isn't a $\sigma-$algebra.

Reason: Measure Theory.

Solution:

$$\begin{aligned}\mathcal{A}_\sigma(\mathcal{P}_n) = \{\, \emptyset, \mathbb{N} \,\} &\cup \{\, B \subseteq \mathbb{N} \,:\, B \subseteq \{2, 4, , \ldots, 2n\} \,\}\\ &\cup \{\, B \subseteq \mathbb{N} \,:\, 2k \in B \,\forall\, k > n \,\wedge\, 2k-1 \in B \,\forall\, k \in \mathbb{N} \,\}\end{aligned}$$

Assume $\bigcup_{n\in\mathbb{N}} \mathcal{A}_\sigma(\mathcal{P}_n)$ is a $\sigma-$algebra. Since for all $n \in \mathbb{N}$ we have $B_n := \{\, 2, 4, \ldots, 2n \,\} \in \mathcal{A}_\sigma(\mathcal{P}_n)$, the union $\bigcup_{k\in\mathbb{N}} B_k \in \bigcup_{n\in\mathbb{N}} \mathcal{A}_\sigma(\mathcal{P}_n)$

which contradicts $\bigcup_{k\in\mathbb{N}} B_k \notin \mathcal{A}_\sigma(\mathcal{P}_n)$ for all $n \in \mathbb{N}$.

https://www2.mathematik.hu-berlin.de/~geomanal/teaching/bruening/analysis3_WS0809/I

4. Linear Operators. (Only solutions to both count!)

   (a) Show that eigenvectors to different eigenvalues of a self-adjoint linear operator are orthogonal and the eigenvalues real.

   (b) Given a real valued, bounded, continuous function $g \in C([0,1])$ with
$$m = \inf_{t\in[0,1]} g(t)\ ,\ M = \sup_{t\in[0,1]} g(t)$$
   and an operator $T_g(f)(t) := g(t)f(t)$ on the Hilbert space $\mathcal{H} = L^2([0,1])$. Calculate the spectrum of T_g.

   Reason: Spectrum of Operators.

   Solution:
$$\lambda\langle x,y\rangle = \langle T(x),y\rangle = \langle x,Ty\rangle = \bar{\mu}\langle x,y\rangle \Longrightarrow \langle x,y\rangle = 0$$
$$\lambda\langle x,x\rangle = \langle T(x),x\rangle = \langle x,Tx\rangle = \bar{\lambda}\langle x,x\rangle \Longrightarrow \lambda = \bar{\lambda}$$

   From the boundaries of g we get that m, M are a lower, resp. upper bound of T_g. Hence $\sigma(T_g) \subseteq [m,M]$. According to the mean value theorem for continuous functions we know, that g takes every value in $[m,M]$ at least once, i.e for every $\mu \in [m,M]$ there is a real number $t_\mu \in [0,1]$ such that $g(t_\mu) = \mu$. Thus
$$T_g(f)(t_\mu) = g(t_\mu)f(t_\mu) = \mu \cdot f(t_\mu)$$
   and $T - \mu$ isn't bounded invertible, hence $\mu \in \sigma(T_g)$ and $\sigma(T_g) = [m,M]$.

5. Let $\mathbb{F}$ be a field. Then for a polynomial $f \in \mathbb{F}[X_1,\ldots,X_n]$ we define $D(f) = \{\, q \in \mathbb{A}^n(\mathbb{F}) \,|\, f(q) \neq 0 \,\}$. Show that these sets build a basis of the Zariski topology on $\mathbb{A}^n(\mathbb{F})$ and decide whether finitely many of them are sufficient.

   Reason: Affine Variety.

   Solution: Recall that $\mathbb{F}[V] = \mathbb{F}[X_1,\ldots,X_n]/I(V)$ with $I(V) = \{\, f \in \mathbb{F}[X_1,\ldots,X_n] \,|\, f(p) = 0\, \forall\, p \in V \,\}$ is the coordinate ring of the affine variety $V \subseteq \mathbb{A}^n(\mathbb{F})$ and $V = V(I(V))$, i.e. V is the vanishing set of polynomials in the ideal $I(V)$.

(a) For a point $p \in \mathbb{A}^n(\mathbb{F}) - V$ outside an affine variety there is a polynomial $f \in \mathbb{F}[X_1, \ldots, X_n]$ such that $f(p) = 1$ and $f(q) = 0$ for all $q \in V$:
Since $p \notin V = V(I(V))$ there is a polynomial $g \in I(V)$ with $g(p) \neq 0$. Then $f := g(p)^{-1} \cdot f$ has the required properties.

(b) For each open set $U \subseteq \mathbb{A}^n(\mathbb{F})$ and point $p \in U$ there is a polynomial $f \in \mathbb{F}[X_1, \ldots, X_n]$ such that $p \in D(f_p) \subseteq U$:
Let $V = \mathbb{A}^n(\mathbb{F}) - U$. Then $V \neq \mathbb{A}^n(\mathbb{F})$ since $p \in U$ and so $U \neq \emptyset$. By the previous statement (a) we have a polynomial $f_p \in I(V)$ with $f_p(p) = 1$. Hence $p \in D(f_p) = \mathbb{A}^n(\mathbb{F}) - V(f_p) \subseteq \mathbb{A}^n(\mathbb{F}) - V = U$.

(c) Now let $\emptyset \neq U \subseteq \mathbb{A}^n(\mathbb{F})$ be an open set. By the previous statement (b) there are polynomials $f_p \in \mathbb{F}[X_1, \ldots, X_n]$ for every point $p \in U$ such that $p \in D(f_p) \subseteq U$. Hence $U = \bigcup_{p\in U} D(f_p)$ and

$$\bigcup_{p\in U} D(f_p) = \mathbb{A}^n(\mathbb{F}) - \bigcap_{p\in U} V(f_p) = \mathbb{A}^n(\mathbb{F}) - V(\{\, f_p \,|\, p \in U \,\})$$

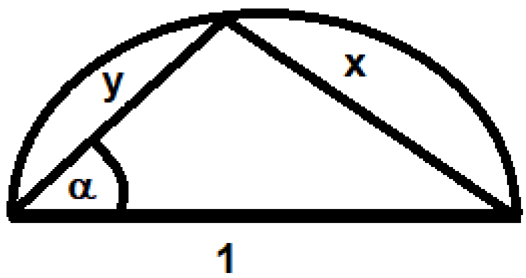
Since $\mathbb{F}[X_1, \ldots, X_n]$ is Noetherian, there are finitely many $f_1, \ldots, f_m$ with $V(\{\, f_p \,|\, p \in U \,\}) = V(f_1, \ldots, f_m)$ and $U = \bigcup_{i=1}^m D(f_i)$ and finitely many are sufficient.

6. Let $R := \mathbb{Q}[x, y]/\langle x^2 + y^2 - 1\rangle$ and $\varphi \in \operatorname{Der}(R)$ a $\mathbb{Q}-$linear derivation such that $\varphi(x) = y$, $\varphi(y) = -x$. A derivation $\varphi : R \longrightarrow R$ of an algebra R is a linear function with $\varphi(p \cdot q) = \varphi(p) \cdot q + p \cdot \varphi(q)$.

(a) Determine the kernel of φ.

(b) Solve $\varphi^2 + \mathrm{id} = 0$.

(c) Since $x^2 + y^2 = 1$ we can apply Thales' theorem and identify $(x, \alpha), (y, \alpha)$ with the sides of a right triangle with hypotenuse (diameter) 1 according to an angle α. Show that



$$(x, \alpha + \beta) = (x, \alpha)(y, \beta) + (x, \beta)(y, \alpha)$$

Reason: Sine and Cosine.

Solution: $\varphi(1) = \varphi(1\cdot 1) = \varphi(1)\cdot 1 + 1\cdot\varphi(1) = 2\varphi(1)$ and thus $\varphi(1) = 0$. Since φ is $\mathbb{Q}-$linear, we get $\varphi(\lambda) = \lambda\cdot\varphi(1) = 0$ for all $\lambda\in\mathbb{Q}$, i.e. $\mathbb{Q}\subseteq\ker\varphi$. It can be shown by induction that

$$\varphi(x^n y^m) = nx^{n-1}y^{m+1} - mx^{n+1}y^{m-1}$$

and especially

$$\varphi(x^n) = nx^{n-1}y$$
$$\varphi(y^m) = -mxy^{m-1}$$

Every polynomial $p(x,y)\in R$ can be written as $p(x,y) = f(x)+y\cdot g(x)$ with $f(x),g(x)\in\mathbb{Q}[x]$. Now let

$$\begin{aligned}
0 &= \varphi(p)\\
&= \varphi(f)+\varphi(y)\cdot g + y\cdot\varphi(g)\\
&= y\sum_{i=1}^{n} f_i(ix^{i-1}) - x\cdot g(x) + y^2\cdot\sum_{j=1}^{m} g_j\cdot(jx^{j-1})\\
&= y\sum_{i=1}^{n} f_i(ix^{i-1}) - xg_0 - \sum_{j=1}^{m}\left(g_jx^{j+1}+(x^2-1)jg_jx^{j-1}\right)\\
&= y\sum_{i=1}^{n} f_i(ix^{i-1}) - xg_0 + \sum_{j=1}^{m} jg_jx^{j-1} - \sum_{j=1}^{m}(j+1)\,g_jx^{j+1}\\
&= y\sum_{i=1}^{n} f_i(ix^{i-1}) - xg_0 + g_1 + 2g_2x - mg_{m-1}x^m - (m+1)g_mx^{m+1}\\
&\quad+\sum_{j=2}^{m-1}\left((j+1)g_{j+1}-jg_{j-1}\right)x^j
\end{aligned}$$

This means $f_i = 0$ for all $i>0$, $g_1 = g_{m-1} = g_m = 0$ and $(j+1)g_{j+1} = jg_{j-1}$ for $j = 2,\ldots,m-1$. Backwards substitution yields $g_j = 0$ for all $j\geq 0$ and $p(x,y) = f(x)+y\cdot g(x) = f_0$, i.e. $\ker\varphi\subseteq\mathbb{Q}$.

Suppose $\lambda x^n + \tau yx^m$ is the term of highest degree in a solution of $\varphi^2(p(x,y))+p(x,y) = \varphi^2(\lambda f(x)+\tau yg(x))+\lambda f(x)+\tau yg(x) = 0$. Then

$$\begin{aligned}
\varphi^2(\lambda\cdot x^n+\tau\cdot yx^m) &= x^n\cdot\left(-\lambda n^2\right)+x^{n-2}\cdot(\ldots)\\
&\quad+yx^m\cdot\left(-\tau\,(m+1)^2\right)+yx^{m-2}(\ldots)
\end{aligned}$$

and φ^2 cannot raise the degree. Thus we have modulo terms of lower degree from $\varphi^2(p) = -p$

$$\lambda n^2 = \lambda \text{ and } \tau(m+1)^2 = \tau$$

and $p(x,y)=\lambda x+\tau y$ are the only solutions:

$$\varphi(\lambda x+\tau y)=-\tau x+\lambda y\,,\;\varphi(-\tau x+\lambda y)=-\lambda x-\tau y$$

Since $(x,\alpha)=\sin\alpha$ and $(y,\beta)=\cos\beta$ the formula

$$(x,\alpha+\beta)=(x,\alpha)(y,\beta)+(x,\beta)(y,\alpha)$$

is simply the addition theorem of the sine function.

7. For all $a,b,c\in\mathbb{R}$ holds

$$a>0,b>0,c>0\Longleftrightarrow a+b+c>0,ab+ac+bc>0,abc>0\,.$$

Reason: Vieta.

Solution: Set $p(x)=x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$. Then $p(x)<0$ if $x\leq 0$ so the roots a,b,c of $p(x)$ are all positive.

8. Let $a,b\in L^2\left(\left[-\frac{\pi}{2},+\frac{\pi}{2}\right]\right)$ given as

$$a(x)=11\sin(x)+8\cos(x)\;,\;b(x)=4\sin(x)+13\cos(x)$$

Calculate the angle $\varphi=\sphericalangle(a,b)$ between the two vectors.

Reason: Hilbert Space.

Solution: We define $f(x)=\sin(x)-6\cos(x)\;,\;g(x)=6\sin(x)+\cos(x)$ and observe, that $\{\,f,g\,\}$ is a orthogonal basis for a two dimensional subspace of $L^2\left(\left[-\frac{\pi}{2},+\frac{\pi}{2}\right]\right)$ with $\gamma:=|f|=|g|=\sqrt{\dfrac{37\pi}{2}}$. As we are interested in an angle, we won't have to bother the length of our coordinate vectors, i.e. we do not need to normalize them. Now we have $a=-f+2g\,,\,b=-2f+g$ and

$$\begin{aligned}\cos\varphi&=\cos(\sphericalangle(a,b))\\&=\cos(\sphericalangle(-f+2g,-2f+g))\\&=\frac{\langle -f+2g,-2f+g\rangle}{|-f+2g|\cdot|-2f+g|}\\&=2\,\frac{\langle f,f\rangle+\langle g,g\rangle}{\sqrt{(|f|^2+4|g|^2)}\cdot\sqrt{(4|f|^2+|g|^2)}}\\&=2\,\frac{\gamma^2+\gamma^2}{\sqrt{5\gamma^2\cdot 5\gamma^2}}\\&=\frac{4}{5}\end{aligned}$$

and $\varphi\approx 80.8^\circ\approx 0.45\pi$

9. Let $\varepsilon_k := \begin{cases} 1 & , \text{ if the decimal representation of } k \text{ has no digit 9} \\ 0 & , \text{ otherwise} \end{cases}$

Show that $\sum_{k=1}^{\infty} \dfrac{\varepsilon_k}{k}$ converges.

Reason: Series.

Solution: The numbers $10^n \leq k < 10^{n+1}$ have $n+1$ digits. On the first place are 8 distinct digits unequal 9 possible, 9 such digits for the others. Thus we have $8 \cdot 9n$ numbers within the interval without the digit 9 and

$$\sum_{k=10^n}^{10^{n+1}-1} \frac{\varepsilon_k}{k} \leq \frac{8 \cdot 9n}{10^n}$$

and the partial sums $\sum_{k=1}^{10^n-1} \dfrac{\varepsilon_k}{k} \leq \sum_{j=0}^{n-1} 8 \cdot \left(\dfrac{9}{10}\right)^n = 80$ are bounded, hence the series converges.

10. Let $x_0 \in [a,b] \subseteq \mathbb{R}$ and $f \,:\, [a,b] \longrightarrow \mathbb{R}$ continuous and differentiable on $[a,b] - \{x_0\}$. Furthermore exists the limit $c := \lim_{x \to x_0} f\,'(x)$. Then $f(x)$ is differentiable in x_0 with $f\,'(x_0) = c$.

Proof: Let $x \in [a,b] - \{x_0\}$. According to the mean value theorem for differentiable functions there is a

$$\xi(x) \in (\min\{x, x_0\}, \max\{x, x_0\})$$

with $f\,'(\xi(x)) = \dfrac{f(x) - f(x_0)}{x - x_0}$. Because $\lim_{x \to x_0} \min\{x, x_0\} = \lim_{x \to x_0} \max\{x, x_0\} = x_0$ we must have $\lim_{x \to x_0} \xi(x) = x_0$ and by assumption $\lim_{x \to x_0} f\,'(\xi(x)) = c$, hence $\lim_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = c$.

What has to be regarded in this proof, and is there a way to avoid this hidden assumption?

Reason: Axiom of choice.

Solution: Let

$$\Lambda(x) := \left\{ \xi \in (\min\{x, x_0\}, \max\{x, x_0\}) \,:\, \frac{f(x) - f(x_0)}{x - x_0} = f\,'(\xi) \right\}$$

The mean value theorem guarantees us that all $\Lambda(x) \neq \emptyset$, but we need more: namely a function

$$\xi \,:\, [a,b] - \{x_0\} \longrightarrow \bigcup_{x \in [a,b] - \{x_0\}} \Lambda(x)$$

i.e. we made use of the axiom of choice.

To avoid AC, let $\varepsilon > 0$. Then there is a $\delta > 0$ such that $|f'(x) - c| < \varepsilon$ whenever $x \in [a,b] - \{x_0\}$ with $|x - x_0| < \delta$. By the mean value theorem, $\Lambda(x) \neq \emptyset$ and we can choose $(*)$ an arbitrary element $\xi \in \Lambda(x)$ and get $|\xi - x_0| < |x - x_0| < \delta$ and thus

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - c \right| = |\, f'(\xi) - c\, | < \varepsilon$$

$(*)$ In this version we only used $\Lambda(x) \neq \emptyset$ for a single value x given by the mean value theorem. To select a single element from a nonempty set does not require AC. This point is given via Rolle's theorem, which again uses the existence of an extremal point in the interior of a closed interval, which again uses the theorem of Bolzano-Weierstraß, which is proven constructively via induction and the completeness of $\mathbb{R}$.

11. (HS-1) A house H and a rosary R are near a circular lake L. The Gardener walks with two watering cans from the house to the lake, fills the cans and goes to the rosary. We assume $\overline{HR} \cap L = \emptyset$. At which point S of the shore does he have to get water, so that his path length is minimal, and why?

    Reason: Reflection.

    Solution: You choose the point S on the circle, such that the tangent t to the circle in S is a mirror which bisects the angle $\sphericalangle HSR$ of his path. This is the shortest way from H to the circle and on to R.

    (a) Huygens - Fresnel principle to prove the law of reflection.

    (b) Let $H = (0,h), S = (s,0), R = (p,q)$. Then the path length is

    $$L = \sqrt{h^2 + s^2} + \sqrt{(p-s)^2 + q^2} \text{ and}$$
    $$\frac{dL}{ds} = \frac{s}{\sqrt{h^2 + s^2}} - \frac{p - s}{\sqrt{(p-s)^2 + q^2}} = \cos\alpha - \cos\beta$$

    with the incident angle α and the reflected angle β. If they are equal we get the minimum which corresponds to the bisection of the normal to the tangent at S.

12. (HS-2) How long is the distance on a direct flight from London to Los Angeles and where is its most northern point? How long will it last by an assumed average speed of 494 knots over ground? We neglect the influence of weather, esp. wind.

We take the values $51^\circ\,28'\,39''\,N$, $0^\circ\,27'\,41''\,W$ for LHR in London, $33^\circ\,56'\,33''\,N$, $118^\circ\,24'\,29''\,W$ for LAX in Los Angeles, and a radius of $3,958$ miles for earth.

Reason: Spherical Trigonometry.

Solution: $9,070.546\,\text{km}$, $5,636.165\,\text{mi}$, $9\,\text{h}\;55\,\text{min}$, $61^\circ\,22'\,53''\,N$, $47^\circ\,11'\,36''\,W$

LHR: $51.4775^\circ\,\text{N}, 0.4614^\circ\,\text{W} = 0.898452\,\text{N}, 0.008053\,\text{W}$
LAX: $33.9425^\circ\,\text{N}, 118.408^\circ\,\text{W} = 0.533168\,\text{N}\;2.066609\,\text{W}$
$494\,\text{kn} = 494 \cdot 1.15078\,\text{mph} \approx 568.49\,\text{mph}$

The formula for the spherical distance is given by the spherical law of cosine as

$$D = R{\cdot}\zeta = R{\cdot}\arccos\left(\sin(\phi_A)\cdot\sin(\phi_B) + \cos(\phi_A)\cdot\cos(\phi_B)\cdot\cos(\lambda_B - \lambda_A)\right)$$

which in our case is

$$\begin{aligned} D &= 3,958 \cdot \arccos\left(\sin(0.898452)\cdot\sin(0.533168)\right. \\ &\qquad \left. + \cos(0.898452)\cdot\cos(0.533168)\cdot\cos(2.066609 - 0.008053)\right)\text{mi} \\ &\approx 5,636.165\,\text{mi} \approx 9.9143\,\text{h} \approx 9\,\text{h}\;55\,\text{min} \end{aligned}$$

The most northern point is given with

$$\begin{aligned} \alpha_A &= \arccos\left(\frac{\cos(\phi_A)\cdot\sin(\phi_B) - \cos(\lambda_A - \lambda_B)\cdot\cos(\phi_B)\cdot\sin(\phi_A)}{\sqrt{1 - \left(\cos(\lambda_A - \lambda_B)\cdot\cos(\phi_A)\cdot\cos(\phi_B) + \sin(\phi_A)\cdot\sin(\phi_B)\right)^2}}\right) \\ &\approx 0,8773446 \end{aligned}$$

by

$$\begin{aligned} P_N &= (\phi_N, \lambda_N) \\ &= \left(\arccos\left(\sin(|\alpha_A|)\cdot\cos(\phi_A)\right), \lambda_A + \operatorname{sgn}(\alpha_A)\cdot\left|\arccos\left(\frac{\tan(\phi_A)}{\tan(\phi_N)}\right)\right|\right) \\ &\approx (1.071307\,,\,0.823681) \approx (61^\circ\,22'\,53''\,N\,,\,47^\circ\,11'\,36''\,W) \end{aligned}$$

which is in SW-Greenland near Qassimiut, Ivigtut, and Kangilinnguit.

13. (HS-3) Trial before an American district court. The witness claims he saw a blue cab drive off after a night accident. The judge decides to test the reliability of the witness. Result: The witness recognizes the color correctly in the dark in 80% of all cases. A survey also found that

85% of taxis in the city are green and 15% are blue.
With what probability has the taxi actually been blue?

Reason: Bayes' Theorem.

Solution: 15 out of 100 taxis are blue. The witness identifies 80% as blue, which are 12 taxis (and 3 taxis falsely as green). 85 taxis are green, and the witness actually identifies 80% as green, that's 68 (and 17 as blue). In total, the witness identifies 29 taxis as blue. The probability that a taxi identified as blue by the witness is actually blue is thus $12/29 = 41.38\%$.

The probability that a taxi identified as blue by the witness is actually blue is, according to Bayes: $(0.8 \cdot 0.15)/(0.8 \cdot 0.15 + 0.2 \cdot 0.85) = 41.38\%$.

14. (HS-4) A monk climbs a mountain. He starts at 8 a.m. on 1000 m above sea level and reaches the peak at 8 p.m. at 3000 m. After a bivouac on top of the mountain, he returns to the valley the next morning and again starts at 8 a.m. and returns at 8 a.m.

    (a) If he wants to avoid being at the same time of day at the same place as the day before when he climbed upwards, which strategy must he use downwards, and why?

    (b) Assume he climbed at a rate of height $u(t)$ proportional to the square root of time, determine his path in dependence of hourly noted time.

    (c) Assume he follows the same path downwards and his height is given by $d_1(t) = \dfrac{125}{9}(t-20)^2 + 1000$ in the first three hours and $d_2(t) = -125\,t + 3500$ for the rest of his way, when will he be at the same point as the day before and at which height.

    Reason: Homework.

    Solution:

    (a) He has to use an alternative route downwards, because if he climbs down the way he climbed up, then he will cross a certain height at the same time as the day before; just imagine he would simultaneously climb up and down. He will have to meet himself then.

    (b) We know $u(8) = 1,000$, $u(20) = 3,000$, and $u(t) \sim \sqrt{t}$. So we can write $u(t) = \alpha\sqrt{t-\beta} + 1,000$ and get

$$\beta = 8\ ,\ \alpha = \frac{2,000}{\sqrt{12}} = \frac{1,000}{\sqrt{3}} \text{ and } u(t) = \frac{1,000}{\sqrt{3}}\sqrt{t-8} + 1,000$$

(c) After three hours he has reached the height $u(11) = 2000\,m$ upwards, and the height $d_1(11) = 9 \cdot 125\,m + 1,000\,m = 2,125\,m$ downwards. He therefore reaches the same height and location on his second leg downwards, i.e. we have to solve $u(t) = d_2(t)$ or

$$0 = t^2 + \left(-\frac{64}{3} - 40\right)t + \left(\frac{512}{3} + 400\right)$$
$$t = \frac{92}{3} - \frac{1}{3}\sqrt{92^2 - 5136} = \frac{1}{3}\left(92 - 16\sqrt{13}\right)$$
$$t \approx 11.437 \approx 11^h\ 26^m\ 13^s$$

and

$$d_2(t) \approx 2,070.37\,m$$

15. (HS-5) I'm annoyed by my two new alarm clocks. They both are powered by the grid. One leaps two minutes an hour and the other one runs a minute an hour too fast. Yesterday I took the effort and set them to the correct time. This morning, I assume there was a power loss, one clock showed exactly $6\,a.m$ while the other one showed $7\,a.m.$ When did I set the clocks and how long did they run?

Reason: Equation of uniform movement.

Solution: One clock runs by $v_1(t) = \dfrac{29}{30}t + t_0$ and the other one by $v_2(t) = \dfrac{61}{60}t + t_0$. We know that $v_1(t_1) \equiv 6 \bmod 24$ and $v_2(t_1) \equiv 7 \bmod 24$. From this we get, that the clocks ran $t_1 = 20$ hours, and I set them at $t_0 \equiv 7 - \dfrac{61}{60} \cdot 20 \equiv 31 - \dfrac{61}{3} \equiv 10^h\,40^m\,(a.m.)$ the previous day.

34 April 2019

1. Find the area A enclosed by the asteroid $(x,y) = (\cos^3 t, \sin^3 t)$ for $0 \le t \le 2\pi$.

   Reason: Integral. **Solution:** By symmetry we have with $dx = 3 \cdot \cos^2 t \cdot \sin t\, dt$

$$
\begin{aligned}
A &= 4\int_0^1 y\,dx \\
&= 4\int_0^{\pi/2} (\sin^3 t)(3\cdot\cos^2 t\cdot \sin t)\,dt \\
&= 12\int_0^{\pi/2} \left(\frac{1-\cos 2t}{2}\right)^2\left(\frac{1+\cos 2t}{2}\right)dt \\
&= \frac{3}{2}\int_0^{\pi/2} (1-2\cos 2t+\cos^2 2t)(1+\cos 2t)\,dt \\
&= \frac{3}{2}\int_0^{\pi/2} (1-\cos 2t-\cos^2 2t+\cos^3 2t)\,dt \\
&= \frac{3}{2}\left[\left(t-\frac{1}{2}\sin 2t\right)-\frac{1}{2}\left(t+\frac{1}{4}\sin 4t\right)+\frac{1}{2}\left(\sin 2t-\frac{1}{3}\sin^3 2t\right)\right]_0^{\pi/2} \\
&= \frac{3\pi}{8} \\
&\approx 1.1781
\end{aligned}
$$

2. Two surface ships on maneuvers are trying to determine a submarine's course and speed to prepare for an aircraft intercept. Ship A is located at $(4,0,0)$, whereas ship B is located at $(0,5,0)$. All coordinates are given in thousands of feet. Ship A locates the submarine in the direction of the vector $2\mathbf{i}+3\mathbf{j}-(1/3)\mathbf{k}$, and ship B locates it in the direction of the vector $18\mathbf{i}-6\mathbf{j}-\mathbf{k}$. Four minutes ago, the submarine was located at $A=(2,-1,-1/3)$. The aircraft is due in 20 minutes. Assuming that the submarine moves in a straight line at a constant speed, to what position should the surface ships direct the aircraft?

   Reason: Navigation. **Solution:** Information from ship A indicates the submarine is now on the line $L_1 : (x,y,z) = (4+2t, 3t, -\frac{1}{3}t)$; information from ship B indicates the submarine is now on the line $L_2 : (x,y,z) = (18s, 5-6s, -s)$. The current position of the sub is at the intersection of both lines at $C = (6,3,-1/3)$ with $t=1, s=1/3$. The straight line path of the submarine contains both points A and C; the line representing this path is $L : (x,y,z) = (2+4t, -1+4t, -1/3)$.

The submarine traveled the distance between A and C in 4 minutes, i.e. at a speed of $\frac{1}{4}|AC| = \frac{1}{4}\sqrt{32} = \sqrt{2}$ thousand feet per minute. In 20 minutes the submarine will move $20\sqrt{2}$ thousand feet from C along the line L.

For the rendezvous point $R \in L$ we thus have for $t > 0$

$$20\sqrt{2} = |RC| = \sqrt{(-4+4t)^2 + (-4+4t)^2} \Longrightarrow 25 = (t-1)^2 \Longrightarrow t = 6$$

and the submarine will be located at $R = (26, 23, -1/3)$ in 20 minutes.

3. Calculate the following:

   (a) $\displaystyle\int \frac{\sqrt{(x^2-1)^3}}{x}\,dx$

   (b) The arc length L of $y = -\dfrac{x^2}{8} + \log x$ for $1 \leq x \leq 2$

   Reason: Integral. **Solution:**

   (a) We substitute $x = \sec\varphi$, $dx = \sec\varphi\tan\varphi d\varphi$ to get

$$\begin{aligned}\int \frac{(x^2-1)^{3/2}}{x}\,dx &= \int \frac{(\sec^2\varphi - 1)^{3/2}}{\sec\varphi}\sec\varphi\tan\varphi\,d\varphi\\ &= \int \tan^4\varphi\,d\varphi\\ &= \int \tan^2\varphi\,(\sec^2\varphi - 1)\,d\varphi\\ &= \int (\tan^2\varphi\,\sec^2\varphi - (\sec^2\varphi - 1))\,d\varphi\\ &= \frac{1}{3}\tan^3\varphi - \tan\varphi + \varphi + C\\ &= \frac{1}{3}\sqrt{(x^2-1)^3} - \sqrt{x^2-1} + \operatorname{arcsec} x + C\end{aligned}$$

(b)

$$\begin{aligned} L &= \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_1^2 \sqrt{1+\left(\frac{1}{x}-\frac{1}{4}x\right)^2}\,dx \\ &= \int_1^2 \left(\frac{1}{x}+\frac{x}{4}\right)dx \\ &= \frac{3}{8}+\log 2 \\ &\approx 1.068 \end{aligned}$$

4. Find the similarity transformations to diagonalize the following matrices:

(a) $A = \begin{pmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$

(b) $B = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

Reason: Linear Algebra. **Solution:**

(a) The characteristic polynomial of A is

$$\chi(A;\lambda) = -\lambda^3+2\lambda^2-4\lambda+8 = -(\lambda-2)(\lambda+2i)(\lambda-2i)$$

and $\frac{1}{2}A \in \mathrm{SO}(3,\mathbb{R})$ with $\det(A)=8$ and $\operatorname{tr}(A)=2$. The eigenvector for $\lambda=2$ is $(1,0,1)^\tau$ and since A is orthogonal, the eigenvectors for $\pm 2i$ are of the form $(1,a,-1)^\tau$ which yields $a=\mp i\sqrt{2}$. After normalization we get

$$S^{-1}AS = \begin{pmatrix} 2i & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2i \end{pmatrix},\ S=\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ -i\sqrt{2} & 0 & i\sqrt{2} \\ -1 & \sqrt{2} & -1 \end{pmatrix}$$

(b) The characteristic polynomial of B is

$$\chi(B;\lambda) = \lambda^2-2\lambda\cos\varphi+1 = (\lambda-\cos\varphi-i\sin\varphi)(\lambda-\cos\varphi+i\sin\varphi)$$

that is eigenvalues $\lambda \in \{\cos\varphi \pm i\sin\varphi\} = \{e^{\pm i\varphi}\}$ with eigenvectors $(1,\mp i)^\tau$. Normalization yields

$$S^{-1}BS = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix},\ S=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$$

5. Suppose that $\mathbb{F}$ is a finite field with say $|\mathbb{F}| = p^m = q$ and that V is a vector space of finite dimension n over $\mathbb{F}$. Find the order of $\operatorname{GL}(V)$.
Reason: Combinatorics. **Solution:** There are $|V| = q^n$ elements in V and for any fixed basis $\{ v_1, \ldots , v_n \}$ there is a unique element $\varphi \in \operatorname{GL}(V)$ that transforms it into another basis $\{ v_1, \ldots , v_n \}$ and vice versa. So how many possibilities do we have to choose such a basis? For w_1 we have $q^n - 1$ possibilities, as the zero vector cannot be chosen. For w_2 we can choose any vector, which isn't one of the q multiples of w_1. For w_3 we may choose all vectors, which are not in one of the q^2 many linear combinations of the former, etc. So all in all we have

$$|\operatorname{GL}(V)| = (q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{n-1})$$

6. (HS-4) Can the numbers $1, 2, 3, \ldots , 16$ be arranged in a row so that each two adjacent numbers add up to a square number?

   Example: $2, 7, 9, 16, \ldots$ would be a possibility for the first four numbers $(2 + 7 = 9, 7 + 9 = 16, 9 + 16 = 25)$; but then we get stuck.

   Reason: Puzzle (66).

   Solution: $16, 9, 7, 2, 14, 11, 5, 4, 12, 13, 3, 6, 10, 15, 1, 8$

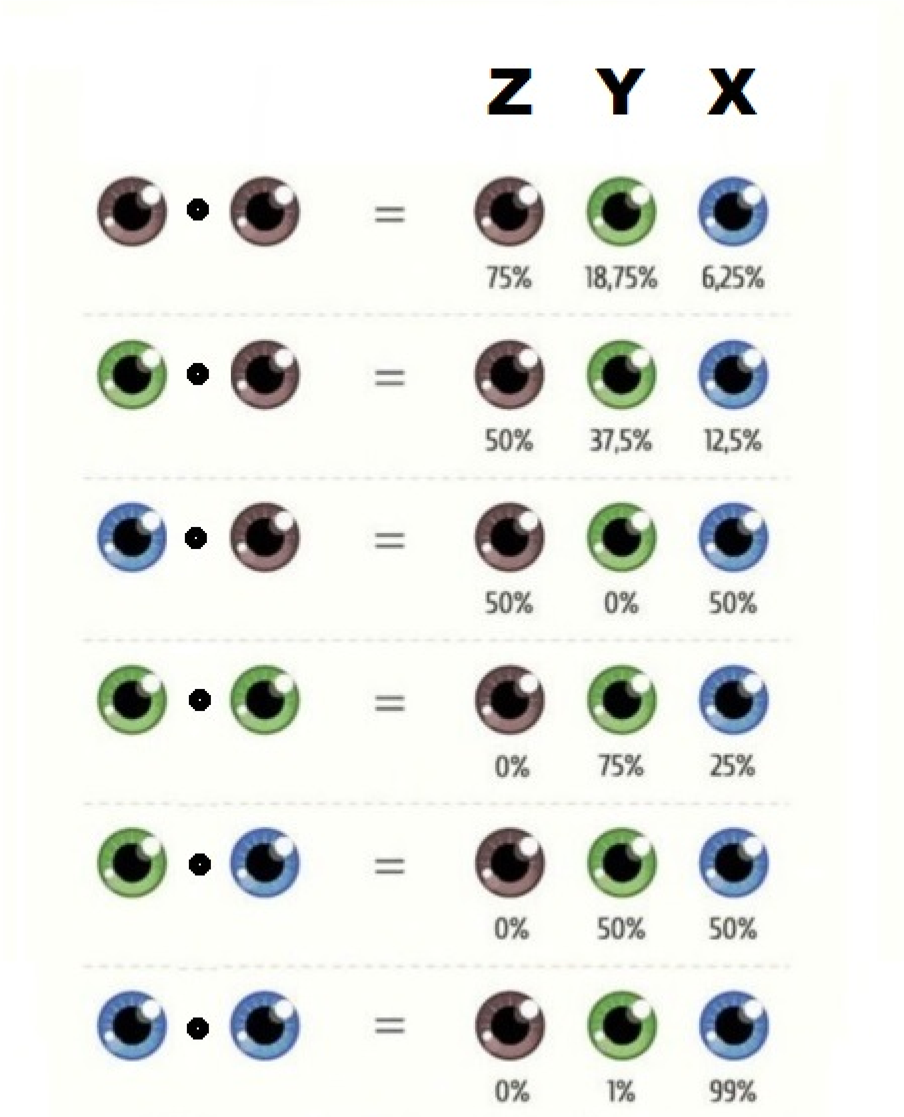
7. (HS-5) We are looking for a ten-digit number N, where the first digit indicates how many zeros occur in N, the second digit, how many ones appear in N, the third digit, how many doubles occur in N, ... and the tenth digit, how many nines appear in N.

   Reason: Puzzle (74).

   Solution: $N = 6, 210, 001, 000$

35 March 2019

1. The graphic of eye colors shows us the probability of baby's eye color in dependency of the parents'. This yields the following multiplication table:



$$
\begin{array}{lcrcrcr}
x \cdot x & = & 3/4\,x & + & 3/16\,y & + & 1/16\,z \\
x \cdot y & = & 1/2\,x & + & 3/8\,y & + & 1/8\,z \\
x \cdot z & = & 1/2\,x & & & + & 1/2\,z \\
y \cdot y & = & & & 3/4\,y & + & 1/4\,z \\
y \cdot z & = & & & 1/2\,y & + & 1/2\,z \\
z \cdot z & = & & & & & z
\end{array}
$$

which we extend to a real, commutative, distributive, three dimensional algebra A.

(a) Is A an associative algebra?

(b) Prove, that A is a baric algebra, i.e. show that there is a nontrivial algebra homomorphismus $\omega : A \longrightarrow \mathbb{R}$, the weight function.

(c) Determine a basis for $\ker \omega$ and rewrite the multiplication table according to this new basis.

(d) Prove that there is an ideal N of codimension one in A, such that $A^2 \nsubseteq N$.

(e) A algebra is called genetic, if there is a basis $\{ u_i \}$ such that the structure constants λ_{ijk} defined by

$$u_i \cdot u_j = \sum_{k=1}^{n} \lambda_{ijk} u_k$$

fulfill the following conditions:

- $\lambda_{111} = 1$
- $\lambda_{1jk} = 0$ for all $j > k$
- $\lambda_{ijk} = 0$ for all $i, j > 1$ and $k \leq \max\{i, j\}$

Prove that all genetic algebras are baric algebras.

(f) Show that A is no genetic algebra.

(g) Determine all idempotent elements of A. Is there a basis of A with idempotent elements?

Reason: Algebras, from a biological point of view.

Solution:

(a)

$$
\begin{aligned}
(x \cdot x) \cdot y &= \left(\frac{3}{4}x + \frac{3}{16}y + \frac{1}{16}z\right) y \\
&= \frac{3}{8}x + \frac{9}{32}y + \frac{3}{32}z + \frac{9}{64}y + \frac{3}{64}z + \frac{1}{32}y + \frac{1}{32}z \\
&= \frac{3}{8}x + \frac{29}{64}y + \frac{11}{64}z \\
x \cdot (x \cdot y) &= x \cdot \left(\frac{1}{2}x + \frac{3}{8}y + \frac{1}{8}z\right) \\
&= \frac{3}{8}x + \frac{3}{32}y + \frac{1}{32}z + \frac{3}{16}x + \frac{9}{64}y + \frac{3}{64}z + \frac{1}{16}x + \frac{1}{16}z \\
&= \frac{5}{8}x + \frac{15}{64}y + \frac{9}{64}z
\end{aligned}
$$

hence A is not associative.

(b) We define $\omega(x) = \omega(y) = \omega(z) = 1$ and observe, that the sums of coefficients on the right hand sides of our multiplication table are all equal to one, i.e. ω is an algebra homomorphism. Per definition it is not the zero homomorphism.

(c) We set $a := x - z$, $b := y - z$, $c := z$, such that $\ker \omega = \mathbb{R}a + \mathbb{R}b$ and $\omega(c) = 1$. The new multiplication table then is

$$
\begin{aligned}
a^2 &= -\frac{1}{4}\,a + \frac{3}{16}\,b & a \cdot b &= -\frac{1}{8}\,b \\
b^2 &= -\frac{1}{4}\,b & a \cdot c &= \frac{1}{2}\,a \\
c^2 &= c & b \cdot c &= \frac{1}{2}\,b
\end{aligned}
$$

(d) $N := \ker \omega$ is a proper ideal of A with codimension 1. Since $A^2 = A$ we have $A^2 \nsubseteq N$.

(e) Let B be a genetic algebra with basis $\{ u_k \}$ and structure constants $\{ \lambda_{ijk} \}$ and

$$
\omega\left(\sum_{k=1}^{n} \mu_k u_k\right) = \mu_1
$$

Hence

$$\begin{aligned}\omega(\mu_i)\cdot\omega(\nu_j) &= \mu_1\cdot\nu_1\\ \omega(\mu_i\cdot\nu_j) &= \omega\left(\sum_i \mu_i \sum_j \nu_j \sum_k \lambda_{ijk}\right)\\ &= \sum_i \mu_i \sum_j \nu_j\cdot\lambda_{ij1}\\ &= \sum_i \mu_i\cdot\nu_1\cdot\lambda_{i11}\\ &= \mu_1\cdot\nu_1\cdot\lambda_{111}\\ &= \mu_1\cdot\nu_1\end{aligned}$$

(f) From

$$(\alpha a+\beta b+\gamma c)^2 = \left(-\frac{1}{4}\alpha^2+\alpha\gamma\right)a+\left(\frac{3}{16}\alpha^2-\frac{1}{4}\beta^2-\frac{1}{4}\alpha\beta+\beta\gamma\right)b+\gamma^2 c$$

we get for $(\alpha a+\beta b+\gamma c)^2=0$ successively $\gamma=0,\alpha=0,\beta=0$, i.e. $0\in A$ is the only element whose square vanishes. On the other hand, we have for the element $u_n\in B-\{0\}$ that $u_n^2=0$ in any genetic algebra B, hence A can't be one.

(g) With the same calculation as before, we get from $(\alpha a+\beta b+\gamma c)^2 = \alpha a+\beta b+\gamma c$ the following cases:

i. $\gamma^2=\gamma\Longrightarrow\gamma\in\{\,0,1\,\}$

ii. $\alpha=0\,,\,\beta\neq 0\Longrightarrow -\frac{1}{4}\beta+\gamma=1\Longrightarrow\gamma=0\,,\,\beta=-4$

iii. $\alpha\neq 0\Longrightarrow -\frac{1}{4}\alpha+\gamma=1\Longrightarrow\gamma=0\,,\,\alpha=-4\Longrightarrow\beta^2=12$

The set of all idempotent elements of A is therefore

$$\{\,0\,,\,c\,,\,-4b\,,\,-4a\pm 2\sqrt{3}b\,\}$$

which spans the entire algebra. Note that this doesn't mean, that A is a Boolean algebra, since not every element is idempotent.

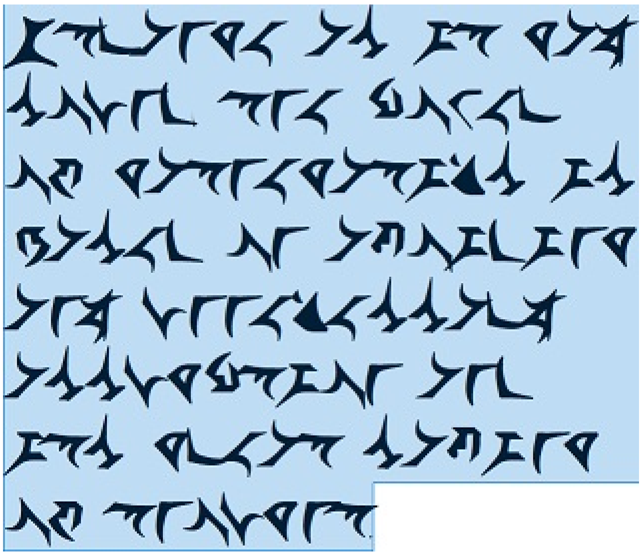
2. Prove that starting with $\frac{1}{1}$ the following binary tree

$$\frac{a}{b}\qquad \swarrow\quad\searrow\qquad \frac{a}{a+b}\qquad\qquad \frac{a+b}{b}$$

defines a counting of all positive rational numbers without repetition and all quotients canceled out.

Reason: Calkin - Wilf counting.

Solution: We define a norm of these elements by $N(p/q) = p+q$. The parent quotient of $\frac{p}{q} \neq \frac{1}{1}$ is either $\frac{p}{q-p}$ or $\frac{p-q}{q}$. The norm of the child has strictly increased in both cases.
Assume we have uncanceled quotients of value $\frac{p}{q}$. Then there is one among them of minimal norm. If $d > 1$ is a common divisor of p, q, then d divides the two possible parent knots, too, which contradicts minimality.
Assume we had more than one knot of value $\frac{p}{q}$. Then there is one among them of minimal norm. However, its possible parent knots would occur more than once as well; again with a smaller norm. (This also follows from the previous step, as only canceled quotients can occur.)
Let $\frac{p}{q}$ be a quotient that doesn't occur. Again by minimality this quotient wouldn't have a parent knot of smaller norm.

3. Who said what here: 
   Reason: Decryption.

   Solution: "Strange as it may sound, the power of mathematics is based on avoiding any unnecessary assumption and its great saving of thought." (Ernst Mach, physicist)

4. Calculate $I := \int_0^\infty \dfrac{\sqrt{x^{e-2}}}{x^e + 1}\,dx$

   Reason: Interesting Integration Trick.

   Solution: First we get rid of the inconvenient denominator, so for

$x \geq 0$ we have

$$\int_0^\infty e^{(-x^e-1)y} dy = \left[-\frac{e^{(-x^e-1)y}}{x^e+1} \right]_{y=0}^{y=\infty} = \frac{1}{x^e+1}$$

and for our integral $I = \int_0^\infty \int_0^\infty x^{\frac{e}{2}-1} e^{-x^e y} e^{-y}\, dy\, dx$. In the next step, we clean up the powers of the exponential function, that is we substitute $z = x^e y$ and get

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty e^{-z} e^{-y} x^{\frac{e}{2}-1} \frac{1}{e} y^{-1} x^{1-e}\, dz\, dx \\ &= \frac{1}{e} \int_0^\infty \int_0^\infty e^{-z} e^{-y} y^{-1} x^{-\frac{e}{2}}\, dz\, dx \\ &= \frac{1}{e} \int_0^\infty \int_0^\infty e^{-z} e^{-y} \sqrt{\frac{y}{z}} y^{-1}\, dz\, dx \\ &= \frac{1}{e} \int_0^\infty z^{-\frac{1}{2}} e^{-z}\, dz \int_0^\infty y^{-\frac{1}{2}} e^{-y}\, dy \\ &= \frac{1}{e} \Gamma\left(\frac{1}{2}\right)^2 \\ &= \frac{\pi}{e} \end{aligned}$$

5. An algebra A is a vector space with a binary distributive multiplication. An example are group algebras, i.e. the distributive extension of the formal basis vectors $g \in G$ such as

   $A := \mathbb{R}[S_3] = \mathbb{R} \cdot (1) + \mathbb{R} \cdot (12) + \mathbb{R} \cdot (13) + \mathbb{R} \cdot (23) + \mathbb{R} \cdot (123) + \mathbb{R} \cdot (132)$

   (a) Find the center $Z(A) = \{\, z \in A \,|\, zv = vz \text{ for all } v \in A \,\}$ of A, and (b) determine the structure of A, i.e. its decomposition into direct factors and the corresponding isomorphisms.

   Reason: Group Algebras.

   Solution: We can identify an element $v = \sum_{\sigma \in S_3} v_\sigma \cdot \sigma \in A$ with the function $v : S_3 \longrightarrow \mathbb{R}$ given by $v(\sigma) = v_\sigma$. Multiplication can then be written as

$$(vw)(\sigma) = \sum_{\alpha \cdot \beta = \sigma} v_\alpha w_\beta = \sum_{\alpha \in S_3} v_\alpha w_{\alpha^{-1}\sigma} = \sum_{\alpha \in S_3} v_{\sigma\alpha^{-1}} w_\alpha$$

and for the function $A \supset S_3 \ni \alpha \longleftrightarrow \chi_\alpha \,:\, \sigma \longmapsto \delta_{\alpha\sigma}$

$$(\chi_\alpha \cdot v)(\sigma) = \sum_{\beta \in S_3} \chi_\alpha(\beta) v(\beta^{-1}\sigma) = v(\alpha^{-1}\sigma)$$
$$(v \cdot \chi_\alpha)(\sigma) = \sum_{\beta \in S_3} v(\sigma\beta^{-1}) \chi_\alpha(\beta) = v(\sigma\alpha^{-1})$$

If $v \in Z(A)$, then $v(\alpha^{-1}\beta\alpha) = (\chi_\alpha v)(\beta\alpha) = (v\chi_\alpha)(\beta\alpha) = v(\beta\alpha\alpha^{-1}) = v(\beta)$ and vice versa if $v(\alpha^{-1}\beta\alpha) = v(\beta)$ then $[\chi_\alpha, v] = 1$ and $v \in Z(A)$. Hence v is a central element if and only if it is constant on the conjugacy classes $\beta^{S_3} = \{\, \alpha^{-1}\beta\alpha \,|\, \alpha \in S_3 \,\}$ of S_3. Since conjugation doesn't change the cycle length, we have three conjugacy classes $\{\,(1)\,\}$, $\{\,(12),(13),(23)\,\}$, $\{\,(123),(132)\,\}$ and $|Z(A)| = 3$. Thus

$$Z(A) = Z(\mathbb{R}[S_3]) = \mathbb{R}\cdot(1) + \mathbb{R}\cdot((12)+(13)+(23)) + \mathbb{R}\cdot((123)+(132))$$

Group algebras and all their modules are semisimple by Maschke's theorem, and the theorem of Wedderburn and Artin states, that semisimple algebras are direct sums of full, simple matrix algebras $\mathbb{M}(n, D)$ over a division ring D which in our case are the real numbers $D = \mathbb{R}$. Since we always have the trivial module $A.m = m$, we always have the trivial component $\mathbb{M}(1, \mathbb{R})$ as direct factor of A. A comparison of dimensions yields $6 = 1 + n_1^2 + \ldots + n_s^2$ and so $6 = 1+1+1+1+1+1$ or $6 = 1+1+2^2$ as only possibilities. In the first case, we would have $A \cong \mathbb{R}^6$ which isn't possible, as A is non Abelian, so that

$$A \cong \mathbb{R} \times \mathbb{R} \times \mathbb{M}(2, \mathbb{R})$$

is the only decomposition into simple factors possible. This also fits to our previous result, that $Z(A) \cong \mathbb{R}^3$.

The corresponding representations of S_3 on $\mathbb{R}^3$ are given by

$$\pi_1(\sigma) := \mathrm{id}_{\mathbb{R}^3}$$
$$\pi_2(\sigma) := (-1)^{\mathrm{sgn}(\sigma)} \cdot \mathrm{id}_{\mathbb{R}^3}$$
$$\pi_3((1)) := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \pi_3((12)) := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$\pi_3((13)) := \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} \qquad \pi_3((23)) := \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$
$$\pi_3((123)) := \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \qquad \pi_3((132)) := \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

where π_3 is restricted to $\mathbb{R}^2 = \{\,(x_1, x_2, x_3) \,|\, x_1 + x_2 + x_3 = 0\,\} \subseteq \mathbb{R}^3$.

6. Let $B \subseteq \mathbb{R}^n$ be measurable and $P = (a_1, \ldots, a_n, b) \in \mathbb{R}^{n+1}$ a point with $b > 0$ and $C_B = \{\, P + t(Q - P) \mid Q \in B \times \{0\}_{n+1}\,,\, t \in [0,1]\,\}$ the cone above the basis B with the peak P. Prove the measure formula

$$\lambda^{n+1}(C_B) = \frac{b}{n+1} \cdot \lambda^n(B)$$

Reason: Integration Transformation Theorem.

Solution: Define

$$\begin{aligned} \varphi \,:\, \mathbb{R}^n \times [0,b] &\longrightarrow \mathbb{R}^n \times [0,b] \\ (x_1, \ldots, x_n, t) &\longmapsto (x_1, \ldots, x_n, 0) + \frac{t}{b}\,(a_1 - x_1, \ldots, a_n - x_n, b) \end{aligned}$$

and observe that φ is a bijection on $\mathbb{R}^n \times [0,b)$ with

$$D\varphi = \begin{bmatrix} 1-\frac{t}{b} & 0 & \cdots & 0 & \frac{a_1}{b} \\ 0 & 1-\frac{t}{b} & \cdots & 0 & \frac{a_2}{b} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-\frac{t}{b} & \frac{a_n}{b} \\ 0 & 0 & \vdots & 0 & 1 \end{bmatrix}$$

with determinant $|D\varphi| = \left(1 - \dfrac{t}{b}\right)^n = \dfrac{1}{b^n}\,(b-t)^n$ and φ is a diffeomorphism on $\mathbb{R}^n \times (0,b)$ with

$$\varphi\,(B \times (0,b)) = C_B - \{\, P, B \,\}$$

which are both P, B of Lebesgue measure zero. Thus we can apply the transformation theorem for integrals

$$\lambda^n(\varphi(S)) = \int_S |\,D\varphi\,| \; d\lambda^n$$

and get

$$\begin{aligned} \lambda^{n+1}(C_B) &= \int_{B\times[0,b]} \frac{1}{b^n}|b-t|^n \, d\lambda^{n+1} \\ &= \frac{1}{b^n} \cdot \int_B 1 \, d\lambda^n \cdot \int_0^b |b-t|^n \, dt \\ &= \frac{1}{b^n} \cdot \lambda^n(B) \cdot \int_0^b u^n \, du \\ &= \frac{1}{b^n} \cdot \lambda^n(B) \cdot \frac{1}{n+1} \cdot b^{n+1} \\ &= \frac{b}{n+1} \cdot \lambda^n(B) \end{aligned}$$

7. Show that
$$x \cdot y = \frac{2xy - x - y}{xy - 1}$$
defines a one dimensional, real, local Lie group G around $0 \in \mathbb{R}$ and compute the vector field of left multiplication by an element $g \in \mathbb{R}$.

Reason: Lie Groups.

Solution: The neutral element of G is 0 and the inverse $x^{-1} = \frac{x}{2x} - 1$ which can be verified along with associativity by simple calculations. The operations are also well-defined on the open sets $U = \{ x \in \mathbb{R} : |x| < 1 \}$ and $U_0 = \{ x \in \mathbb{R} : |x| < \frac{1}{2} \}$ for the inversion. The group operations are also analytic on suited neighborhoods of 0, so G is actually a Lie group. For the left multiplication $L_g : x \longmapsto g \cdot x$ we get
$$DL_g(x_0) = \left.\frac{d}{dx}\right|_{x=x_0} L_g(x) = \frac{(g-1)^2}{(gx_0 - 1)^2}$$

8. (HS-1) Two numbers a, b are called amicable, if the sum of all proper divisors of one is the other number (1 is included). The smallest example is
$$(a, b) = (220, 284) = (1{+}2{+}4{+}71{+}142, 1{+}2{+}4{+}5{+}10{+}11{+}20{+}22{+}44{+}55{+}110)$$
Let $n \in \mathbb{N}$ and $(x, y, z) = (3 \cdot 2^n - 1\,,\, 3 \cdot 2^{n-1} - 1\,,\, 9 \cdot 2^{2n-1} - 1)$. Prove that if x, y, z are all odd primes, then $(a, b) = (2^n \cdot x \cdot y\,,\, 2^n \cdot z)$ are amicable numbers.

Hint: First find a formula for the sum of all divisors $\sigma(n)$ given the prime decomposition of n.

Reason: Theorem of Thabit Ibn Qurra. (9th century, Mesopotamia)

Solution: For $n = p_1^{k_1} \cdots p_r^{k_r}$ then the sum of all divisors is
$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

$$\begin{aligned}\sigma(a)-a&=\sigma(2^n\cdot x\cdot y)-2^n\cdot x\cdot y\\&=\left(2^{n+1}-1\right)(x+1)(y+1)-2^nxy\\&=\left(2^{n+1}-1\right)(3\cdot 2^n)(3\cdot 2^{n-1})-2^n(3\cdot 2^n-1)(3\cdot 2^{n-1}-1)\\&=\left(2^{n+1}-1\right)\cdot 9\cdot 2^{2n-1}-2^n\left(9\cdot 2^{2n-1}-9\cdot 2^{n-1}+1\right)\\&=2^n\cdot\left(9\cdot 2^{2n}-9\cdot 2^{n-1}-9\cdot 2^{2n-1}+9\cdot 2^{n-1}-1\right)\\&=2^n\cdot\left(9\cdot 2^{2n-1}-1\right)\\&=2^n\cdot z\\&=b\end{aligned}$$

and by an analogue calculation $\sigma(b)-b=a$.

9. (HS-2) A number is called perfect, if it equals the sum of all its divisors except itself, e.g. $6=1+2+3$ and $28=1+2+4+7+14$ are perfect. If 2^k-1 is a prime number, then $2^{k-1}(2^k-1)$ is a perfect number and every even perfect number has this form.

Reason: Mersenne Numbers.

Solution: Let $n=2^{k-1}(2^k-1)$ and $p=2^k-1$ prime. Then

$$\begin{aligned}\sigma(n)&=(2^k-1)\cdot\frac{p^2-1}{p-1}\\&=(2^k-1)\cdot\frac{2^{2k}-2^{k+1}}{2^k-2}\\&=(2^k-1)\cdot 2^k\cdot 1\\&=2\cdot 2^{k-1}\cdot(2^k-1)\\&=2n\end{aligned}$$

and n is perfect.
If on the other hand $n=2^{k-1}m$ is an even perfect number, $k>1$ and m is odd, then

$$\begin{aligned}\sigma(n)=&(2^k-1)\cdot\sigma(m)\\&=2n\\&=2^km\end{aligned}$$

and $(2^k-1)\,|\,m$, say $(2^k-1)M=m$. Hence

$$\begin{aligned}\sigma(m)&=\frac{2^km}{2^k-1}\\&=\frac{2^k(2^k-1)M}{2^k-1}\\&=2^kM\\&\geq m+M\\&=(2^k-1)M+M\\&=2^kM\end{aligned}$$

since both m,M divide m. Thus equality holds everywhere and m,M are the only divisors of m, i.e. $m=(2^k-1)M$ is prime. As $k>1$ this is only possible, if $M=1$ and $m=2^k-1$ is of the desired form.

Numbers 2^k-1 are called **Mersenne numbers** and primes 2^k-1 **Mersenne primes**, in which case k has to be prime, too. It is unclear (but suspected), whether there are infinitely many Mersenne primes. The highest known number is currently $2^{82,589,933}-1$ with $24,862,048$ digits. It is also unclear whether there are infinitely many perfect numbers.

10. (HS-3)

   (a) What is the smallest five-digit number n such that n and $2n$ together contain all 10 digits from 0 to 9?

   (b) On how many zeros does the number $1000\,!$ end?

   (c) For which six-digit number $ABCDEF$ do we have:
   $ABCDEF\cdot 1=ABCDEF$
   $ABCDEF\cdot 3=BCDEFA$
   $ABCDEF\cdot 2=CDEFAB$
   $ABCDEF\cdot 6=DEFABC$
   $ABCDEF\cdot 4=EFABCD$
   $ABCDEF\cdot 5=FABCDE$

Reason: Number Puzzle.

Solution:

   (a) $n=13485$ with $2n=26970$. Other solutions are e.g. $n=13548,13845$ which are bigger.

(b) We have as many zeros at the end as there are factors 5, so $1000/5 + 1000/5^2 + 1000/5^3 + \lfloor 1000/5^4 \rfloor = 200 + 40 + 8 + 1 = 249$.

(c) If σ notes the cyclic shift by one digit ($\sigma(ABCDEF) = BCDEFA$) we get with $x = ABCDEF$

$$x \cdot 10^k \equiv \sigma^k(x) \bmod 7$$

i.e. σ acts like the multiplication by 10 in $\mathbb{Z}_7$, $10 : 7 = 1.42857$, and 142857 is the solution.

36 February 2019

1. A little number theory.

   - Compute the last three digits of 3^{2405} .
   - Show that there is an integer $a \in \mathbb{Z}$ such that $64959 \mid (a^2-7)$.

   Reason: Practice for computer science.

   Solution:

   (a) We need the result of $3^{2405} \equiv x \operatorname{mod} 1000$ to compute the last three digits. Since 3 and 1000 are coprime, we have $3^{\varphi(1000)} \equiv 1 \operatorname{mod} 1000$ by Euler's theorem. Now

   $$\varphi(10^3)=\varphi(8)\varphi(125)=\varphi(2^3)\varphi(5^3)=2^3\left(1-\frac{1}{2}\right)5^3\left(1-\frac{1}{5}\right)=400$$

   Thus we have $3^{2405}=\left(3^{400}\right)^6\cdot 3^5 \equiv 1^6\cdot 243 \equiv 243 \operatorname{mod} 1000$ as the last three digits.

   (b) It is $64959=59\cdot 1101=59\cdot 3\cdot 367$.

   - $\left(\dfrac{7}{3}\right)=\left(\dfrac{1}{3}\right)=1$
   - $\left(\dfrac{7}{59}\right)=-\left(\dfrac{59}{7}\right)=-\left(\dfrac{3}{7}\right)=\left(\dfrac{7}{3}\right)=\left(\dfrac{1}{3}\right)=1$
   - $\left(\dfrac{7}{367}\right)=-\left(\dfrac{367}{7}\right)=-\left(\dfrac{3}{7}\right)=\left(\dfrac{7}{3}\right)=\left(\dfrac{1}{3}\right)=1$

   Hence there are integers a_1,a_2,a_3 with $a_1^2 \equiv 7 \operatorname{mod} 3$, $a_2^2 \equiv 7 \operatorname{mod} 59$, $a_3^2 \equiv 7 \operatorname{mod} 367$ and by the Chinese remainder theorem an integer a such that $a \equiv a_1 \operatorname{mod} 3$, $a \equiv a_2 \operatorname{mod} 59$, $a \equiv a_3 \operatorname{mod} 367$. This is still true for the squared equations $a^2 \equiv a_1^2 \operatorname{mod} 3$, $a^2 \equiv a_2^2 \operatorname{mod} 59$, $a^2 \equiv a_3^2 \operatorname{mod} 367$ so again by the Chinese remainder theorem $a^2 \equiv 7 \operatorname{mod} 64959$.

2. Let $f(x)=\dfrac{(\cos\varphi-\sqrt{3}\sin\varphi+1)x+2\sqrt{3}\sin\varphi}{x^2}$
   and $g(x)=\dfrac{(\cos\varphi-\sqrt{3}\sin\varphi-1)x+2\sqrt{3}\sin\varphi}{x^2}$.

   For which values of φ are $f \perp g$ in $L^2([1,\infty))$?

   Reason: Thales.

   Solution: The norm in $L^2([1,\infty))$ is defined by the inner product

$||h(x)||^2 = \langle h(x), h(x)\rangle = \int_1^\infty h(x)^2 dx$. We define $p(x) = x^{-1}$ and $q(x) = \sqrt{3}(2-x)x^{-2}$. Then $\{ p, q \}$ define a orthonormal basis of the subspace V they span in $L^2([1,\infty))$. As f, g can be written as

$$f(x) = p(x)+p(x)\cos\varphi+q(x)\sin\varphi\,,\; g(x) = -p(x)+p(x)\cos\varphi+q(x)\sin\varphi$$

which means they point to the same point on the unit circle of V from the left and from the right intersection with the diameter, the statement follows by the theorem of Thales, i.e. all values of φ fulfill the condition.

3. (HS-1) A man wants to figure out the length of an escalator, i.e. the number of steps [N] if it was out of order. Since it wasn't out of order, he counted 60 steps if he walks with the stairs and 90 steps if he walks in the opposite direction. What is [N]?

   Reason: School-1.

   Solution: Let's measure velocity in steps per second and distance in steps. Let v_M be the man's velocity and v_T the escalator's. We have two equations for the distance:

$$x(t) = (v_M + v_T) \cdot \frac{60N}{v_M} = (v_M - v_T) \cdot \frac{90N}{v_M}$$

   which is $v_M = 5v_T$ and $x(t) = 360N \cdot \frac{v_T}{v_M} = 72N$.

4. (HS-2) We are looking for a number with eight digits: two of each 1,2,3,4. The ones are separated by one other number, the twos by two, the threes by three, and the fours by four other numbers.

   Reason: School-2.

   Solution: 23421314 or backwards 41312432.

5. (HS-3) Which of you four threw the ball in my window? A says: It was E. E says: It was G. F says: It was not me. G says: E lied a.) If only one of the four lied, who threw the ball? b.) If only one person has told the truth, who was the culprit?

   Reason: School-3.

   Solution: If only one lied, then E was the culprit If only one told the truth, then F was the culprit.

6. (HS-4) Choose any two but different natural numbers and form their sum, their difference and product. Prove that among these three numbers at least one is divisible by 3.

   Reason: School-4.

   Solution: If one of the two numbers is divisible by 3, so is the product. If the two numbers divided by 3 have the same remainder, then their difference is divisible by 3. If a number divided by 3 leaves the remainder 1, the other the remainder 2, then their sum is a multiple of 3.

7. (HS-5) Prove that the remainder in dividing any prime by 30 is either 1 or prime again. Is this also true when dividing a prime number by 60?

   Reason: School-5.

   Solution: Every prime number p can be written as $p = 30q + r$, q and r are natural numbers with $1 \leq r \leq 29$. For all numbers r divisible by $2, 3$, or 5 then $p = 30q + r$ is not prime. Therefore only $1, 7, 11, 13, 17, 19, 23, 29$ are possible remainders.
   Now let $p = 60q + r$ with $1 \leq r \leq 59$. Since the prime 109 is in the form $109 = 60 \cdot 1 + 49$ and 49 is not prime, the statement does not hold for 60.

37 January 2019

1. Given the surface

$$f(t,\varphi) = ((1+t^2)\cos\varphi\ ,\ (1+t^2)\sin\varphi\ ,\ t) \quad (t \in \mathbb{R}\ ,\ 0 \leq \varphi \leq 2\pi)$$

   (a) Compute the first fundamental form of this surface.

   (b) Compute the second fundamental form and the Gauss curvature of this surface.

   (c) Compute the geodesic curvature κ_g and the normal curvature κ_n of the circular latitude at $t = 1$.

   Only solutions to all three parts will be accepted.

   Reason: Curvatures.

   Solution:

   (a) $f_t = (2t\cos\varphi, 2t\sin\varphi, 1)\ ,\ f_\varphi = (-(1+t^2)\sin\varphi, (1+t^2)\cos\varphi, 0)$

$$\begin{aligned} I(af_t + bf_\varphi, cf_t + df_\varphi) &= \langle (a,b)^\tau, (c,d)^\tau \rangle \\ &= (a,b)\begin{bmatrix} A & B \\ B & C \end{bmatrix}(c,d)^\tau \\ &= ac\langle f_t, f_t\rangle + (ad+bc)\langle f_t, f_\varphi\rangle + bd\langle f_\varphi, f_\varphi\rangle \\ &= ac\cdot A + (ad+bc)\cdot B + bd\cdot C \\ &= ac(1+4t^2) + (ad+bc)(0) + bd((1+t^2)^2) \end{aligned}$$

   and $g(t,\varphi) = I = \begin{bmatrix} 1+4t^2 & 0 \\ 0 & (1+t^2)^2 \end{bmatrix}$

   (b) $\vec{n} = \dfrac{f_t \times f_\varphi}{||f_t \times f_\varphi||} = \dfrac{1}{\sqrt{1+4t^2}} \cdot (-\cos\varphi, -\sin\varphi, 2t)^\tau$

$$\begin{aligned} f_{tt} &= (2\cos\varphi, 2\sin\varphi, 0)^\tau \\ f_{t\varphi} &= (-2t\sin\varphi, 2t\cos\varphi, 0)^\tau \\ f_{\varphi\varphi} &= (-(1+t^2)\cos\varphi, -(1+t^2)\sin\varphi, 0)^\tau \end{aligned}$$

$$h(t,\varphi) = II = \begin{bmatrix} f_{tt}\cdot\vec{n} & f_{t\varphi}\cdot\vec{n} \\ f_{\varphi t}\cdot\vec{n} & f_{\varphi\varphi}\cdot\vec{n} \end{bmatrix} = \frac{1}{\sqrt{1+4t^2}}\begin{bmatrix} -2 & 0 \\ 0 & 1+t^2 \end{bmatrix}$$

   and $\kappa_G(t,\varphi) = \dfrac{\det h}{\det g} = \dfrac{-2}{(1+4t^2)^2(1+t^2)}$

(c) The circular latitude at $t = 1$ is $c(\varphi) = f(1, \varphi) = (2\cos\varphi, 2\sin\varphi, 1)^\tau$ which is a circle with radius $R = 2$ and so its curvature κ_R is

$$\kappa(\varphi) = \frac{||c'(\varphi) \times c''(\varphi)||}{||c'(\varphi)||^3} = \frac{4}{8} = \frac{1}{2} = \frac{1}{R}$$

For the normal curvature we get

$$\kappa_n = \frac{h(c'(\varphi), c'(\varphi))}{g(c'(\varphi), c'(\varphi))} = \frac{(0,1)\, II_{t=1}(0,1)^\tau}{(0,1)\, I_{t=1}(0,1)^\tau} = \frac{2}{\sqrt{5}} \cdot \frac{1}{4} = \frac{1}{2\sqrt{5}}$$

and the geodesic curvature is

$$\kappa_g = \sqrt{\kappa_R^2 - \kappa_n^2} = \sqrt{\frac{1}{4} - \frac{1}{20}} = \frac{1}{\sqrt{5}}$$

2. Three pirates are stranded on an island and find that there are only a few monkeys besides drinking water and coconuts. After collecting coconuts for a whole day, they want share them the next morning. At night, one of the pirates awakes and hides his third of the coconuts. But since an odd number of nuts is left, he gives one to a monkey. The second pirate awakens shortly afterwards and hides his third of the remaining coconuts. Again an odd number of coconuts remains, so he gives one to a monkey. The third does the same thing a short time later and gives a leftover nut to a monkey. The next morning they divided the few remaining coconuts among each other. Now the question: How many coconuts did the three pirates at least collect the day before and how are they distributed on each?

   Reason: Riddle.

   Solution: We have to solve $N_{k+1} = \frac{2}{3}N_k - 1$ for $k = 1, 2, 3$ and $N_3 = 3R$ where N_0 is the number of coconuts collected and R the remaining share for each in the morning. Solving this recursion results in

$$3R = \frac{2^3}{3^3}\left(N_0 - \sum_{k=0}^{2}(\frac{2}{3})^k\right) \Longleftrightarrow N_0 = \frac{1}{8}(81 \cdot R + 57) \in \mathbb{Z}$$

   The smallest solution to $81R + 57 \equiv R + 1 \equiv 0 \bmod 8$ is $R = 7$ which yields $N_0 = 78$. The first pirate receives 33, the second 24, the third 18, and the monkeys 3 coconuts.

3. A cyclist drives along a railway track. Every 30 minutes, he is overtaken by a train and every 20 minutes he is met by a train. At which frequency

do the trains travel on this connection?

Reason: Riddle.

Solution: Imagine she rides one hour in one direction and one hour in the other. Then she meets three trains in the first hour and is overtaken by two in the second hour. So the frequency is thus 5 trains per direction in 120 minutes, i.e. every 24 minutes a train.

4. The Heisenberg group $H = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \; : \; a,b,c \in \mathbb{Z}^3 \right\}$ operates discontinuously on $\mathbb{R}^3$ by

$$h(p) = h(x,y,z) = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ z+ay+c \end{bmatrix}$$

Show that the Heisenberg manifold $\mathbb{R}^3/H$ is orientable.

Reason: Manifolds.

Solution: A manifold is orientable, if and only if there is an atlas, such that for all charts $(U,\varphi),(V,\psi)$ with a nonempty intersection and all points p in the domain of $\varphi \circ \psi^{-1}$

$$\det(D_p(\varphi \circ \psi^{-1})) > 0$$

Therefore we get with the trivial atlas $\psi = id_{\mathbb{R}^3}$ on $\mathbb{R}^3$:

$$\det(D_p(\varphi \circ \psi^{-1})) = \det(D_p(\psi \circ h \circ \psi^{-1}(\psi(p))) = \det D_p(h) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix} = 1$$

5. Solve $x'(t) = \dfrac{2t+2x(t)}{3t+x(t)}$, $x(2)=0\,.$

Reason: Initial Value Problem.

Solution: With $y(t) = \dfrac{x(t)}{t}$ we get $x' = \dfrac{2+2y}{3+y}$ and $ty' = x' - y\,.$ Hence

$$\begin{aligned} ty' &= \frac{2+2y}{3+y} - y = \frac{-y^2-y+2}{3+y} = t \cdot \frac{dy}{dt} \\ \frac{dt}{t} &= \frac{-3-y}{y^2+y-2} \cdot dy = -\frac{4}{3} \cdot \frac{1}{y-1}\,dy + \frac{1}{3} \cdot \frac{1}{y+2}\,dy \end{aligned}$$

and $\log|t| = \frac{1}{3}\log|y+2| - \frac{4}{3}\log|y-1| + C$ or

$$t^3 = C \cdot \frac{y+2}{(y-1)^4} \Longleftrightarrow (x-t)^4 = C \cdot (x+2t) \text{ and } C = 4$$

6. Show that $T : C([1,2]) \longrightarrow C([1,2])$ defined by

$$T(y)(t) := 1 + \int_1^t \frac{y(s)}{2s}\,ds$$

has at least one fixed point and determine them.

Reason: Fixed points.

Solution:

$$|T(y)(t) - T(z)(t)| \leq \int_1^t \frac{2s}{|y(s)-z(s)|}\,ds \leq \int_1^t \frac{||y-z||_\infty}{2}$$
$$\leq \frac{t-1}{2}||y-z||_\infty \leq \frac{1}{2}||y-z||_\infty$$

Differentiation of $Ty = y$ yields $y'(t) = \dfrac{y(t)}{2t}$ or $y(t) = C \cdot \sqrt{t}$. Hence $C \cdot \sqrt{t} = y = Ty = 1 + \dfrac{C}{2}\displaystyle\int_1^t \frac{1}{\sqrt{s}}\,ds = 1 + C \cdot (\sqrt{t} - 1)$ and thus $C = 1$. The only fixed point of T is $y(t) = \sqrt{t}$.

7. Compute $\exp(tA)$ where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and determine the behavior of $\det(\exp(tA))$ for $t \to \pm\infty$.

Reason: Matrix Exponentiation.

Solution: With $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} = E_{13} - E_{31}$ we have $A = 1 + B$. Since $[1, B] = 0$ we get

$$\exp(tA) = \exp(t \cdot 1)\exp(tB) = e^t \cdot \exp(tE_{13} - tE_{31})$$

$$\begin{aligned}
(tE_{13}-tE_{31})^0 &= t^0(E_{11}+E_{22}+E_{33})\\
(tE_{13}-tE_{31})^1 &= t^1(E_{13}-tE_{31})\\
(tE_{13}-tE_{31})^2 &= t^2(-E_{11}-E_{33})\\
(tE_{13}-tE_{31})^3 &= t^3(-E_{13}+E_{31})\\
(tE_{13}-tE_{31})^4 &= t^4(E_{11}+E_{33})\\
(tE_{13}-tE_{31})^5 &= t^5(E_{13}-E_{31})\\
&\ldots
\end{aligned}$$

which is cyclic of order four in the matrix component and $n \longmapsto t^n$ for the factor. If we now add the separate positions divided by $n!$

$$\begin{aligned}
&(1,1)\,:\,1-\frac{t^2}{2!}+\frac{t^4}{4!}\mp\ldots=\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{(2k)!}=\cos t\\
&(1,3)\,:\,t-\frac{t^3}{3!}+\frac{t^5}{5!}\mp\ldots=\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{(2k+1)!}=\sin t\\
&(2,2)\,:\,1\\
&(3,1)\,:\,-t+\frac{t^3}{3!}-\frac{t^5}{5!}\pm\ldots=-\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{(2k+1)!}=-\sin t\\
&(3,3)\,:\,1-\frac{t^2}{2!}+\frac{t^4}{4!}\mp\ldots=\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k}}{(2k)!}=\cos t
\end{aligned}$$

then $\exp(tB)=\begin{bmatrix}\cos t & 0 & \sin t\\ 0 & 1 & 0\\ -\sin t & 0 & \cos t\end{bmatrix}$ and $\exp(tA)=\begin{bmatrix}e^t\cos t & 0 & e^t\sin t\\ 0 & e^t & 0\\ -e^t\sin t & 0 & e^t\cos t\end{bmatrix}$

Thus $\det(\exp(tA))=e^{\operatorname{tr}(tA)}=e^{3t}$ from which the behavior towards $\pm\infty$ is obvious.

8. Let G be a group generated by $\sigma,\varepsilon,\delta$ with $\sigma^7=\varepsilon^{11}=\delta^{13}=1$.

   (a) Show that there is no transitive operation of G on a set with 8 elements.

   (b) Is there are group G with the above properties, that operates transitively on a set with 12 elements?

Reason: Groups.

Solution:

(a) Assume G operates transitively on $M = \{1, 2, \ldots, 8\}$ via $\varphi : G \longrightarrow S_8$. As the order of $\varphi(\varepsilon)$ is a common divisor of 11 and $|S_8| = 8!$, both numbers are coprime and thus $\varphi = 1$. The same argument applies to $\varphi(\delta)$ hence $\varphi(\sigma)$ generates $\varphi(G)$, which is a cyclic group of order 1 or 7. By the orbit-stabilizer theorem and a transitive operation we would have $8 \,|\, |\varphi(\sigma)| = |\varphi(G)| \in \{1, 7\}$ which is impossible.

(b) Let $\sigma = (1\,2\,3\,4\,5\,6\,7)$ and $\varepsilon = (2\,3\,4\,5\,6\,7\,8\,9\,10\,11\,12)$. Both cycles generate a subgroup $H \leq S_{12}$ which operates transitively on $M = \{1, 2, \ldots 12\}$. Now $(h, z).m := h.m$ is a transitive operation of

$$G := H \times \mathbb{Z}/13\,\mathbb{Z}$$

on M, too, and G is generated by $(\sigma, 0)$, $(\varepsilon, 0)$, $(1, 1 + 13\,\mathbb{Z})$.

9. Let R, S be rings and $\varphi : R \longrightarrow S$ a ring epimorphism. Further let $J \subseteq S$ be an ideal.

(a) Define an ideal $I \subseteq R$ such that $R/I \cong S/J$.

(b) Is the preimage of the center of S equal to the center of R?

Reason: Rings.

Solution:

(a) Let $\pi : S \twoheadrightarrow S/J$ be the canonical projection. Then $\pi \circ \varphi : R \longrightarrow S/J$ is also surjective and $I := \varphi^{-1}(J) = \ker \pi \circ \varphi$. The statement follows by the homomorphism theorem.

(b) No. Let $S = \{0\}$. Then $\ker \varphi = R$ which is the center of R if and only if R is commutative. So every non commutative ring provides a counterexample, e.g. a matrix ring.

10. A lie algebra $\mathfrak{g}$ is called reductive, if $\mathfrak{g} = \mathfrak{Z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ is the direct sum of its center and its derived algebra. (This is an important class of Lie algebras, as they are exactly those whose representations split into a direct sum of irreducible representations. Semisimple and in particular the simple, classical matrix Lie algebras are reductive.)

Show that the Lie algebra $\mathfrak{gl}(V)$ of all endomorphisms of a finite dimensional complex vector space is reductive.

Reason: Lie algebras.

Solution: $\mathfrak{Z}(\mathfrak{g}) = \mathbb{C} \cdot 1$ by Schur's Lemma and we can write every matrix $X \in \mathfrak{gl}(V)$ as $X = c \cdot 1 + S_X$ where $S_X \in \mathfrak{sl}(V)$, the simple Lie algebra of all endomorphisms of V with zero trace. For dimensional reasons, we get

$$\mathfrak{gl}(V) = \mathbb{C} \cdot 1 \oplus \mathfrak{sl}(V) = \mathfrak{Z}(\mathfrak{g}) \oplus \mathfrak{sl}(V)$$

Since $\mathfrak{sl}(V)$ is a simple Lie algebra, we have

$$[\mathfrak{gl}(V), \mathfrak{gl}(V)] = [\mathfrak{sl}(V), \mathfrak{sl}(V)] = \mathfrak{sl}(V)$$

and thus

$$\mathfrak{gl}(V) = \mathfrak{Z}(\mathfrak{g}) \oplus [\mathfrak{gl}(V), \mathfrak{gl}(V)]$$

Part I

December, 2018

38 December 2018

1. Find an integer with ten digits, such that the first n digits $(1 \leq n \leq 10)$ are divisible by n.

   Reason: Easy test of divisibility properties.

   Solution:

   - An integer is divisible by 1 without remainder.
   - An integer is divisible by 2 without remainder if the last digit is even.
   - An integer is divisible by 3 without remainder if its checksum is divisible by 3.
   - An integer is divisible by 4 without remainder if the last two digits are divisible by 4.
   - An integer is divisible by 5 without remainder if the last digit is divisible by 5.
   - An integer is divisible by 6 without remainder if its checksum is divisible by 3 and the last digit by 2.
   - An integer is divisible by 8 without remainder if the last three digits are divisible by 8.

- An integer is divisible by 9 without remainder if its checksum is divisible by 9.
- An integer is divisible by 10 without remainder if the last digit is a 0.
- For divisibility by 7, there is unfortunately no such simple condition.

1	2	3	4	5	6	7	8	9	10	digits at
1	2	1	2	5	2	1	2	1	0	cond. 5,10,2,4,6,8
3	4	3	4		4	3	4	3		
7	6	7	6		6	7	6	7		
9	8	9	8		8	9	8	9		
1	4	1	2	5	4	1	2	1	0	cond. digits 4 and 8
3	8	3	6		8	3	6	3		
7		7				7		7		
9		9				9		9		
1	4	1	2	5	8	1	2	1	0	digits 1-3 and 4-6
3	8	3	6	5	4	3	6	3		
7		7				7		7		
9		9				9		9		
1	4	1	2	5	8	1	6	1	0	digits 8-10
3	8	3	2	5	8	9	6	3		
7		7	6	5	4	3	2	7		
9		9	6	5	4	7	2	9		
1	4	1	2	5	8	9	6	3	0	digits 4-6 and 7-9
3	8	3	6	5	4	3	2	1		
7		7	6	5	4	3	2	7		
9		9	6	5	4	7	2	3		
			6	5	4	7	2	9		
1	4	7	2	5	8	9	6	3	0	digits 1-3
1	8	3	6	5	4	3	2	1		
1	8	9	6	5	4	3	2	7		
3	8	1	6	5	4	7	2	3		
3	8	7	6	5	4	7	2	9		
7	4	1								
7	8	3								
7	8	9								
9	8	1								
9	8	7								

By consecutive elimination of possibilities, and in the last step the doubles, we get ten possible numbers

1472589630 , 1836547290 , 1896547230 , 1896547290 , 3816547290 , 7412589630 , 7896543210 , 9816543270 , 9816547230 , 9876543210

where only 3816547290 is divisible by 7.

2. Is there always a position on a continuous floor for a rectangular table with four equal legs, such that the table does not wiggle?

Reason: Fun with the mean value theorem.

Solution:

- Quadratic case.

We consider the heights h of its legs at a certain point x on the floor measured by its angle to a fixed point (radial coordinates). The table doesn't wiggle, if the sum of two opposite heights are equal, i.e. if $h(x)+h(x+\pi)=h(x+\frac{\pi}{2})+h(x+\frac{3\pi}{2})$. The function $f\,:\,\mathbb{R}\longrightarrow\mathbb{R}$ defined by

$$f(x)=h(x)-h(x+\frac{\pi}{2})+h(x+\pi)-h(x+\frac{3\pi}{2})$$

is continuous and we assume that our table wiggles, so w.l.o.g. $f(x_0)>0$. Thus

$$\begin{aligned}0>-f(x_0)&=-h(x_0)+h(x_0+\frac{\pi}{2})-h(x_0+\pi)+h(x_0+\frac{3\pi}{2})\\&=-h(x_0+2\pi)+h(x_0+\frac{\pi}{2})-h(x_0+\pi)+h(x_0+\frac{3\pi}{2})\\&=h(x_0+\frac{\pi}{2})-h(x_0+\pi)+h(x_0+\frac{3\pi}{2})-h(x_0+2\pi)\\&=f(x_0+\frac{\pi}{2})\end{aligned}$$

With $f(x_0+\frac{\pi}{2})<0<f(x_0)$ we get a point $\xi\in[x_0,x_0+\frac{\pi}{2}]$ such that $f(\xi)=0$ by the mean value theorem, and the table does not wiggle there.

- Rectangular case.

Now we have to consider $f(x):=h(x)+h(x+\pi)-h(x+d)-h(x+d+\pi)$ for an angle $d\in(0,\pi)$. By the periodicity of $h(x)$ we get for $H:=\int_0^{2\pi}h(x)h(x)\,dx$ that $H=\int_0^{2\pi}h(x+c)\,dx$ for all

$c \in \mathbb{R}$. Hence

$$\begin{aligned}\int_0^{2\pi} f(x)\,dx &= \int_0^{2\pi} h(x)\,dx + \int_0^{2\pi} h(x+\pi)\,dx \\ &\quad - \int_0^{2\pi} h(x+d)\,dx - \int_0^{2\pi} h(x+d+\pi)\,dx \\ &= H + H - H - H \\ &= 0\end{aligned}$$

and by the mean value theorem for integration there is a point $\xi \in [0, 2\pi]$ with

$$0 = \int_0^{2\pi} f(x)\,dx = f(\xi)(2\pi - 0) \Longrightarrow f(\xi) = 0$$

3. Two mathematicians meet by chance on the plane: "Did not you have three sons?" asks one, "how old are they?" "The product of years is 36," is the answer, "and the sum of years is exactly today's date." "Hmm, that's not enough for me," says the colleague. "Oh, right," says the second mathematician, "I forgot to mention that my eldest son has a dog." How old are the three sons?

   Reason: Logic Puzzle.

   Solution: There are eight combinations for the product, as each son is at least 1. One of them adds to 38=1+1+36 which is no date. Since product and sum weren't sufficient, there have to be two combinations with the same sum, which leaves two possibilities: 1+6+6=2+2+9, from which only one has an oldest son. He is 9.

4. Find functions $y(t)$ and $z(t)$ which locally solve the equations

$$\begin{cases} e^t + \tan y(t) &= 1 \\ t^2 + z(t)^3 + z(t) &= 0 \end{cases}$$

   in a neighborhood of $t = 0$ and investigate their behavior with respect to monotony (where defined). It is sufficient to determine the functions up to a differential equation. It's a mathematical problem, so existence will do.

   Reason: Implicit Function Theorem.

   Solution: The equation system is solvable at $t = 0$ with $y(0) = z(0) =$

0. Furthermore is $\vec{g}(t,\vec{x}) = \begin{pmatrix} e^t + \tan y(t) \\ t^2 + z(t)^3 + z(t) \end{pmatrix}$ with $\vec{x} = (y,z)^\tau$ continuously differentiable with

$$\frac{\partial \vec{g}}{\partial \vec{x}} = \begin{bmatrix} \dfrac{\partial g_1}{\partial y} & \dfrac{\partial g_1}{\partial z} \\ \dfrac{\partial g_2}{\partial y} & \dfrac{\partial g_2}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{1+y^2} & 0 \\ 0 & 3z^2+1 \end{bmatrix}$$

with determinant one at $(t,\vec{x}) = (0,0,0)$. By the implicit function theorem, we thus have a neighborhood $U = U_\varepsilon(0) = (-\varepsilon, \varepsilon)$ and functions $y(t), z(t) \in C^1(U)$ which solves our equations for all $t \in (-\varepsilon, \varepsilon)$. Additionally we get for $\vec{f} = (y(t), z(t))^\tau$

$$\begin{aligned} \frac{\partial \vec{f}}{\partial t} &= (y(t)', z(t)')^\tau \\ &= -\left(\frac{\partial \vec{g}}{\partial \vec{x}}(t, \vec{f}(t)) \right)^{-1} \cdot \frac{\partial \vec{g}}{\partial t}(t, \vec{f}(t)) \\ &= -\begin{bmatrix} 1+y(t)^2 & 0 \\ 0 & 3z(t)^2+1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} e^t \\ 2t \end{bmatrix} \\ &= -\begin{bmatrix} \dfrac{1}{1+y(t)^2} & 0 \\ 0 & \dfrac{1}{3z(t)^2+1} \end{bmatrix}^{-1} \cdot \begin{bmatrix} e^t \\ 2t \end{bmatrix} \\ &= -\left[\frac{e^t}{1+y(t)^2}, \frac{2t}{3z(t)^2+1} \right]^\tau \end{aligned}$$

Since $y(t)' < 0$ on $(-\varepsilon, \varepsilon)$ the function $y(t)$ is strictly monotone decreasing, whereas the function $z(t)$ due to the nominator of $z(t)'$ is strictly monotone increasing on $(-\varepsilon, 0)$ and strictly monotone decreasing on $(0, \varepsilon)$.

5. Areas and Volumes.

   (a) Show that the paraboloid

$$P = \{ (x,y,z)^\tau \in \mathbb{R}^3 \mid x^2 + y^2 = z, \ x, y \in [-1,1] \} \subseteq \mathbb{R}^3$$

   and the hyperboloid

$$H = \{ (x,y,z)^\tau \in \mathbb{R}^3 \mid x^2 - y^2 = z, \ x, y \in [-1,1] \} \subseteq \mathbb{R}^3$$

   have equal areas.

(b) Bring $M = \{ (x,y,z)^\tau \in \mathbb{R}^3 \,|\, x^2 \leq y^4 \leq z^8 \leq 1 \}$ into a normal form and calculate its volume.

Reason: Integration.

Solution:

(a) We choose the parameterization $\Phi(u,v) = (u,v,u^2+v^2)^\tau\ ,\ u,v \in [-1,1]$ for P and get the normal vector

$$\begin{aligned} \vec{N}_P(u,v) &= \frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v} \\ &= (1,0,2u)^\tau \times (0,1,2v)^\tau \\ &= (-2u,-2v,1)^\tau \end{aligned}$$

and we get for the area

$$|P| = \int_\Phi 1\,d\sigma = \int_{[-1,1]^2} ||\vec{N}_P(u,v)||\,d(u,v) = \int_{[-1,1]^2} \sqrt{4u^2+4v^2+1}\,d(u,v)$$

The parameterization of H is given by $\Psi(u,v) = (u,v,u^2-v^2)^\tau$ whose normal vector $\vec{N}_H(u,v) = (-2u,2v,1)^\tau$ has the same norm as $\vec{N}_P$ and thus yields the same area integral.

(b) $x \mapsto \sqrt[n]{x}$ is strictly monotone increasing for all even n on $[0,\infty)$ so

$$\begin{aligned} z^8 \leq 1 &\Longleftrightarrow z \in [-1,1] \\ y^4 \leq z^8 &\Longleftrightarrow y \in [-z^2,z^2] \\ x^2 \leq y^4 &\Longleftrightarrow x \in [-y^2,y^2] \end{aligned}$$

A normal form of M is thus given by

$$M = \{ (x,y,z)^\tau \in \mathbb{R}^3 \,|\, -1 \leq z \leq 1\ ,\ -z^2 \leq y \leq z^2\ ,\ -y^2 \leq x \leq y^2 \}$$

and

$$
\begin{aligned}
|M| &= \int_M d(x,y,z)\\
&= \int_{-1}^{1}\int_{-z^2}^{z^2}\int_{-y^2}^{y^2} dx\,dy\,dz\\
&= \int_{-1}^{1}\int_{-z^2}^{z^2} [x]_{-y^2}^{y^2}\, dy\,dz\\
&= \int_{-1}^{1}\int_{-z^2}^{z^2} 2y^2\,dy\,dz\\
&= \int_{-1}^{1}\left[\frac{2}{3}y^3\right]_{-z^2}^{z^2} dz\\
&= \int_{-1}^{1}\frac{4}{3}z^6\,dz\\
&= \left[\frac{4}{21}z^7\right]_{-1}^{1}\\
&= \frac{8}{21}
\end{aligned}
$$

6. The table cards at a rotatable round table with 12 seats are set up for expected 12 people. However, the persons ignore the cards and randomly distribute themselves to the seats.

   Is it always possible with a single turn of the table to make sure that at least two people sit in front of their table cards?

   Reason: Modular arithmetics.

   Solution: If we denote the distance of a guest to his expected seat by $\pi(i) - i$ for the given permutation $\pi \in S_{12}$, then the question is: Are there always two numbers i, j such that for any $\pi \in S_{12}$ we have $\pi(i) - i = \pi(j) - j$?

   Assume this is not the case. Then $\{\,\pi(i) - i\,|\,0 < i < 13\,\} \cong \mathbb{Z}_{12}$ and $\sum_{i=1}^{12}(\pi(i) - i) = 0$. On the other hand is $\sum_{\mathbb{Z}_{12}} i = 78 \equiv 6 \bmod 12$ which cannot both be true.

7. Kummer and Bertrand.

   Let $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ be a sequence of positive real numbers and $A = \sum_{n=1}^{\infty} a_n$. Prove the following statements:

(a) If there is a sequence $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ of positive real numbers, such that there is an index N for which $b_{n-1} \cdot \dfrac{a_{n-1}}{a_n} - b_n \geq C$ for a constant $C > 0$ and all $n > N$, then A converges.

(b) If there is a sequence $(b_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ of positive real numbers, such that the series $\sum_{n=1}^{\infty} \dfrac{1}{b_n}$ diverges, and there is an index $N \in \mathbb{N}$ such that $b_{n-1} \cdot \dfrac{a_{n-1}}{a_n} - b_n \leq 0$ for all $n > N$, then A diverges.

(c) We define the sequence of real numbers by

$$b_n := \left(n \cdot \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right) \log(n)$$

and $B := \lim_{n\to\infty} b_n \in \overline{\mathbb{R}} = \mathbb{R} \cup \{ \pm\infty \}$.

Then A converges if $B > 1$ and diverges if $B < 1$.

Solution:

(a) (Kummer's convergence criterion)

Given $0 < C \cdot a_n \leq b_{n-1}a_{n-1} - b_n a_n$ for all $n > N$ we get

$$0 < C \cdot \sum_{n=N+1}^{M} a_n \leq \sum_{n=N+1}^{M} (b_{n-1}a_{n-1} - b_n a_n) = b_N a_N - b_M a_M < b_N a_N$$

and thus $\sum_{n=N+1}^{M} a_n < \dfrac{b_N a_N}{C}$ and we have a sequence of partial sums $A_M = \sum_{n=1}^{M} a_n$ which is strictly monotone increasing for $M > N$ and bounded from above.

(b) (Kummer's divergence criterion)

Our condition now reads $0 < b_N a_N \leq \ldots \leq b_{n-1}a_{n-1} \leq b_n a_n$ for all $n > N$, resp. $a_n \geq \dfrac{b_N}{b_n} a_N$. Hence $\sum_{n=N+1}^{M} a_n \geq b_N a_N \sum_{n=N+1}^{M} \dfrac{1}{b_n}$ which diverges for $M \to \infty$ and so does A by the minority criterion.

(c) (Bertrand's criterion)

Let $c_n := n\log(n)$ for $n > 1$. The series $\sum_{n=2}^{\infty} \dfrac{1}{c_n}$ diverges by the integral criterion ($\int \dfrac{1}{x\log(x)}\,dx \sim \log\log x$). For $f(x) := \dfrac{1}{x\log(x)}$ we have $f(n) = c_n$ and $f(x)$ is monotone decreasing for $x \geq 2$ with $\lim_{x\to\infty} f(x) = 0$. Furthermore we have

$$\begin{aligned}\lim_{R\to\infty}\int_2^R \frac{1}{x\log(x)}\,dx &= \lim_{R\to\infty}\int_2^R \frac{\frac{d}{dx}\log(x)}{\log(x)}\,dx \\ &= \lim_{R\to\infty}\left[\log(\log(R)) - \log(\log(2))\right] = \infty\end{aligned}$$

Now we define

$$\begin{aligned}K_n &:= c_n \cdot \frac{a_n}{a_{n+1}} - c_{n+1} \\ &= n\log(n)\cdot\frac{a_n}{a_{n+1}} - (n+1)\log(n+1) \\ &= n\log(n)\cdot\frac{a_n}{a_{n+1}} - n\log(n+1) - \log(n+1) \\ &= n\log(n)\cdot\frac{a_n}{a_{n+1}} - n\left(\log\left(1+\frac{1}{n}\right) + \log(n)\right) - \left(\log\left(1+\frac{1}{n}\right)+\log(n)\right) \\ &= n\log(n)\cdot\frac{a_n}{a_{n+1}} - (n+1)\log\left(1+\frac{1}{n}\right) - n\log(n) - \log(n) \\ &= \log(n)\cdot\left(n\frac{a_n}{a_{n+1}} - n - 1\right) - \log\left(1+\frac{1}{n}\right)^{n+1} \\ &= B_n - \log\left(1+\frac{1}{n}\right)^{n+1}\end{aligned}$$

Since the logarithm is continuous, we get

$$K := \lim_{n\to\infty} K_n = B - \log(e) = B - 1 \in \overline{\mathbb{R}} = \mathbb{R}\cup\{\pm\infty\}$$

and we can apply Kummer's criteria (see previous parts) and A converges if $K > 0$, that is $B > 1$, and diverges, if $K < 0$, that is $B < 1$.

8. Let $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \in \mathbb{R}[x]$ be a polynomial where all roots are negative. Prove that

$$\int_1^{\infty} \frac{1}{p(x)}\,dx$$

converges absolutely if and only if $n > 1$.

Reason: Integrals.

Solution: We start with $n \geq 2$.

$$\lim_{x\to\infty} \frac{p(x)}{x^n} = \lim_{x\to\infty} \left(1 + a_{n-1}\frac{1}{x} + a_{n-2}\frac{1}{x^2} + \ldots + a_0\frac{1}{x^n}\right) = 1$$

Multiplication by $\sqrt{x}$ yields

$$\lim_{x\to\infty} \frac{p(x)}{x^{n-\frac{1}{2}}} = \lim_{x\to\infty} \sqrt{x}\left(1 + a_{n-1}\frac{1}{x} + a_{n-2}\frac{1}{x^2} + \ldots + a_0\frac{1}{x^n}\right) = \infty$$

so there is a point $x_0 \in [0,\infty)$ such that $p(x) > x^{n-\frac{1}{2}}$ $(*)$ for all $x > x_0$, especially $p(x) > 0$ in this range. For the integral of absolute values we get

$$\int_1^\infty \left|\frac{1}{p(x)}\right| dx = \int_1^{x_0} \frac{dx}{|p(x)|} + \int_{x_0}^\infty \frac{dx}{|p(x)|} = C + \int_{x_0}^\infty \frac{dx}{|p(x)|} \qquad (**)$$

The integral $\int_{x_0}^\infty x^{\frac{1}{2}-n}\,dx$ converges because $\frac{1}{2} - n < -1$. From $(*)$ we get that this integral is a convergent majorant for the second term in $(**)$ which therefore converges absolutely.

For $\deg(p) = n = 0$ we have $p(x) = 1$ and

$$\int_1^\infty \frac{dx}{p(x)} = \int_1^\infty dx = \lim_{\xi\to\infty} [x]_1^\xi = \infty$$

For $\deg(p) = n = 1$ we have $p(x) = x + c$ and

$$\int_1^\infty \frac{dx}{p(x)} = \int_1^\infty \frac{dx}{x+c} = \lim_{\xi\to\infty} [\log|x+c|]_1^\xi = \infty$$

We have $c > 0$ and the logarithm is defined, for otherwise $p(-c) = 0$ and $p(x)$ would have a positive root which is against our assumption.

9. Let $\emptyset \neq U \subseteq \mathbb{R}^+$ be an open set, and $x_0 \in U$. We define the **quotient logarithm** of a function $f : U \longrightarrow \mathbb{R}^+$ at $x = x_0$ by

$$f^{\,-}(x_0) := \lim_{x\to x_0} \frac{\log f(x) - \log f(x_0)}{\log x - \log x_0}$$

Solve the *differential equation* $f^- = f$.

Reason: Funny differential.

Solution: By the mean value theorem for the logarithm, we find $\xi \in (x, x_0)$, $\eta \in (f(x), f(x_0)$ such that

$$\begin{aligned} f^-(x_0) &= \lim_{x\to x_0} \frac{\log f(x) - \log f(x_0)}{\log x - \log x_0} \\ &= \lim_{x\to x_0} \frac{\frac{1}{\eta}\,(f(x) - f(x_0))}{\frac{1}{\xi}\,(x - x_0)} \\ &= \frac{x_0}{f(x_0)} \cdot f\,'(x_0) \end{aligned}$$

So we get

$$\begin{aligned} f &= f^- \\ f &= \frac{x}{f}\,f\,' \\ f^2 &= x\,f\,' \\ \frac{df}{dx} &= \frac{f^2}{x} \\ \int \frac{df}{f^2} &= \int \frac{dx}{x} \\ -\frac{1}{f} &= \log x + \log c\;,\; c \in \mathbb{R}^+ \\ f &= -\frac{1}{\log cx} \end{aligned}$$

10. Let $p > q$ be prime numbers such that $p \not\equiv 1 \bmod q$.
Prove that each group with pq elements is cyclic.

Reason: Sylow's theorems.

Solution: Let $n \geq 1$ be the number of Sylow p-subgroups of G, and $m \geq 1$ be the number of Sylow q-subgroups of G.

(i) By Sylow's third theorem $n\,|\,q$ and $n \equiv 1 \bmod p$. Since 1 and q are the only divisors of q, and $1 < q < p$ we can rule out $n = q$ and conclude $n = 1$.

(ii) Similarly we have $m\,|\,p$ and $m \equiv 1 \bmod q$. The condition $p \not\equiv 1 \bmod q$ rules out $m = p$ so we can conclude $m = 1$.

(iii) Say H is the Sylow p-subgroup and K the Sylow q-subgroup.
Since gHg^{-1} is a Sylow p-subgroup, and gKg^{-1} is a Sylow q-subgroup,

too, both subgroups H, K have to be normal, and $HK \leq G$. Because $H, K \subsetneq HK$, i.e. $p, q \mid |HK| \mid |G| = pq$ with $|HK| > p+1 > q+1$ we get $|HK| = pq$, resp. $G = HK$. We also have $|H \cap K| = 1$ as $|H \cap K| \,|\, |H| = p$ and $|H \cap K| \,|\, |K| = q$, so $G = H \times K$ is a direct product of normal subgroups H, K of prime order p, resp. q. However, groups of prime order are cyclic and we get

$$G = H \times K = \mathbb{Z}_p \times \mathbb{Z}_q \cong \mathbb{Z}_{pq}$$

11. If we multiply our five digits number by four, we will get the same number in reverse order. What's the number?

    Reason: Arithmetic Riddle.

    Solution: Let x be the number we are looking for. Since $x, 4x$ both have five digits, x has to be between $9,999$ and $25,000$, i.e. x will start with a one or a two. As $4x$ has to be even, we get $x = 2----$ and $4x \geq 80,000$. So the last digit of x has to be eight or nine. As $4x$ ends with a two, we have $x = 2---8$. The second digit of x cannot create an overflow or we would have $4x \geq 90,000$. Thus the second digit of x is in $\{0,1,2\}$ so we are looking for an $x \in \{20--8, 21--8, 22--8\}$ with $4x \in \{8--02, 8--12, 8--22\}$. Say $x = ---c8$, then $4x = y + (40c + 30 + 2)$ and testing for $4c + 3 \in \{0,1,2\}$ yields $c \in \{2,7\}$ and with $4 \cdot 28 = 112$, $4 \cdot 78 = 312$ a second but last digit one, i.e. $x = 21--8$. Furthermore the second but last digit of x has to be two or seven. If it was a two, then $4x = 82-12$ but $x > 21,000$ and $4x > 84,000$. Thus we have $x = 21-78$ and $4x = 87-12$. From $4x = 87c12 = 4 \cdot 21c78 = 84,000 + 400c + 312$ we get $30 + c = 4c + 3$ or $c = 9$ and $x = 21,978$.

12. Let $\mathcal{B}$ be a Boolean ring with 1, i.e. each element of $\mathcal{B}$ is idempotent. Show that each prime ideal is maximal.

    Reason: Abstract Algebra.

    Solution: Let $P \subseteq \mathcal{B}$ be a prime ideal. Then $\mathcal{B}/P$ is an integral domain. We show that $\mathcal{B}/P \cong \mathbb{Z}_2$ which is a field, and therefore P is maximal.

    Let $x, y \in \mathcal{B}/P - \{0\}$ and $z := x \cdot y$. Then $xz = x(xy) = (xx)y = xy$ and $0 = x(z-y)$. As $\mathcal{B}/P$ is an integral domain and $x \neq 0$, we have $y = z$ and similarly $x = z$. So all elements different from 0 are identical. Because $P \neq \mathcal{B}$ we get $\mathcal{B}/P \cong \mathbb{Z}_2$.

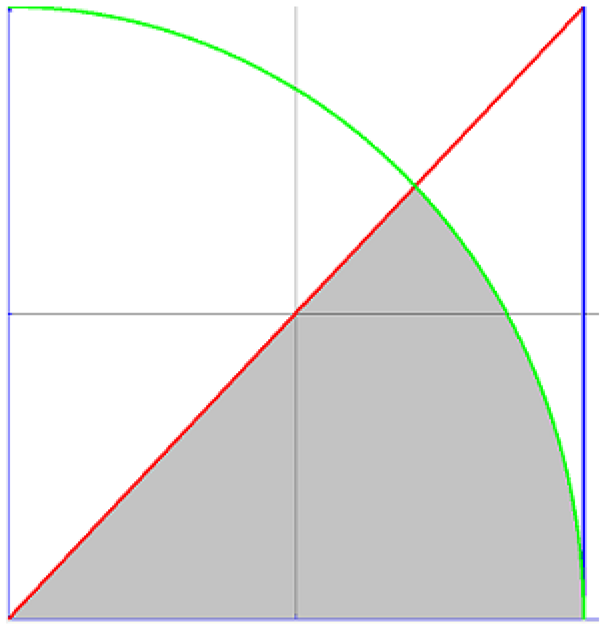
13. On a state fair is a booth where you can shoot at a square. You get 1 point for each hit and 2 points if you hit closer to the center than to the boundary. How big is your chance to get the extra point?

Reason: Geometry.

Solution: We want to derive the curve within the square which marks the limiting condition. Think of a Cartesian coordinate with the square's center as its origin, and a side length of 2 . We get for a point $P = (x, y)$ which lies in the first quadrant and fulfills the boundary condition

$$1 - x = \sqrt{x^2 + y^2} \Longrightarrow y = \pm\sqrt{1 - 2x}$$

On the diagonal $x = y$ it is the point $y = x = \sqrt{2} - 1 =: x_0$ and $(0.5\,,\,1)$ on the $x-$axis.



The area A is therefore

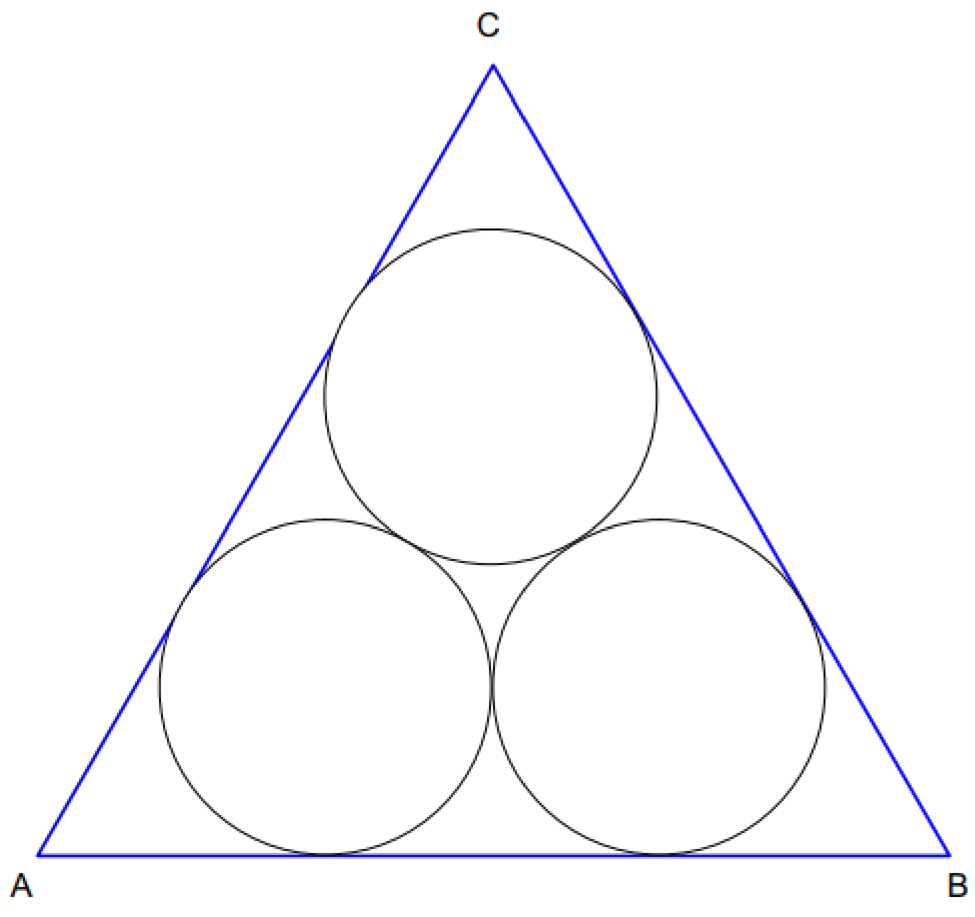
$$\begin{aligned} A &= \int_0^{x_0} x\,dx + \int_{x_0}^{\frac{1}{2}} \sqrt{1-2x}\,dx \\ &= \frac{1}{2}x_0^2 + \frac{1}{3}\,(1-2x_0)^{\frac{3}{2}} \\ &= \frac{3}{2} - \sqrt{2} + \frac{5}{3}\sqrt{2} - \frac{7}{3} \\ &= \frac{2}{3}\sqrt{2} - \frac{5}{6} \end{aligned}$$

As we have 8 such areas and the square has an area of 4, we get a total chance of

$$p = \frac{8}{4}\left(\frac{2}{3}\sqrt{2} - \frac{5}{6}\right) = \frac{4}{3}\sqrt{2} - \frac{5}{3} \approx 0.218951416... \approx 21.9\,\%$$

14. The Italian mathematician G.F.Malfatti presented the following in 1803, because of its degree of difficulty well-known task: Construct three circles into a given triangle so that the total area of the circles is maximal.

For the equilateral triangle, Malfatti found the solution



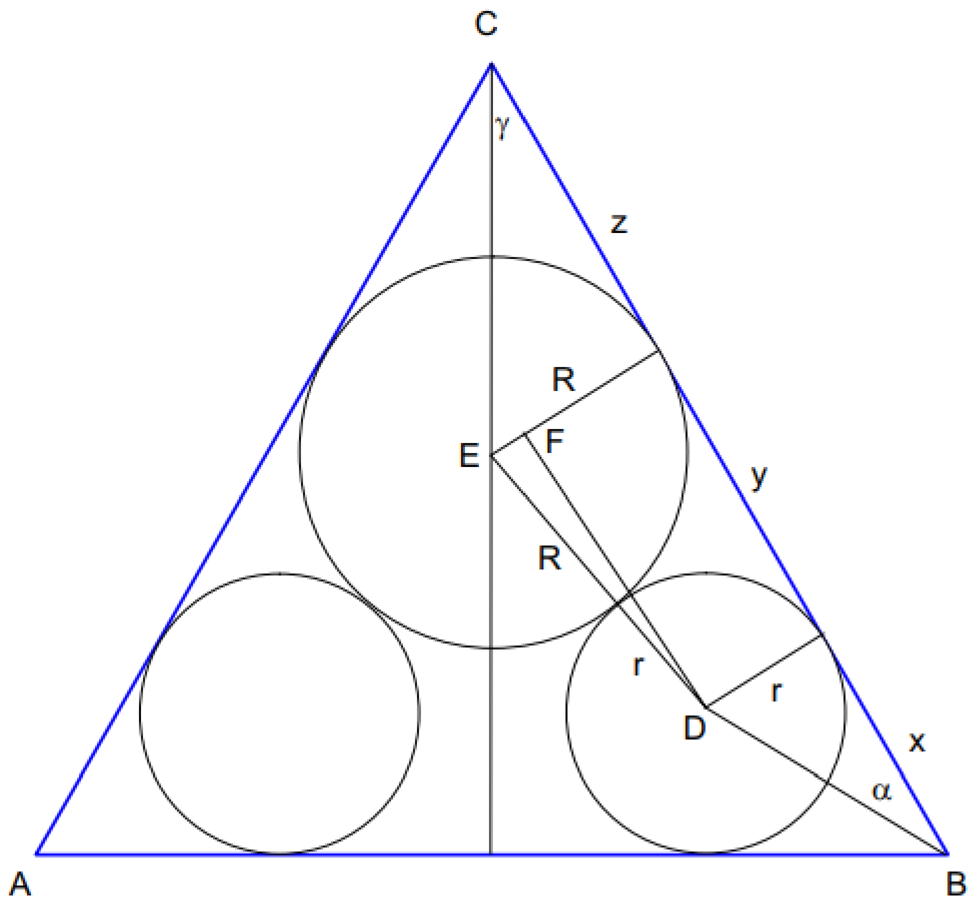
It wasn't until 1929 when the mathematicians Lob and Richmond showed that Malfatti had made a mistake here.

Show that there is a better solution for the equilateral triangle.

Reason: Geometry.

Solution: Problem and graphics from
http://www.matheraetsel.de/archiv/Extremwerte/Malfatti1/malfatti1_2.pdf.

We examine all symmetric solutions in which
a) each circle center lies on an angle bisector and
b) two circles touch each other.



We calculate the distances x and z with the tangent theorem in a right triangle

$$\tan\alpha = \frac{r}{x}\ ,\ \alpha = \frac{1}{6}\pi\ ,\ x = r\sqrt{3} \qquad (1) \tag{7}$$
$$\tan\gamma = \frac{R}{z}\ ,\ \gamma = \frac{1}{6}\pi\ ,\ z = R\sqrt{3} \qquad (2) \tag{8}$$

From the touching condition of the circles we get a right triangle DEF with $y = DF$ such that

$$(R+r)^2 = (R-r)^2 + y^2 \longrightarrow y = 2\sqrt{Rr} \tag{9}$$

The sum of all three sections x, y, z are the side length of the triangle

$$\overline{BC} =: a = x + y + z = r\sqrt{3} + 2\sqrt{Rr} + R\sqrt{3} \tag{10}$$

In case of equal circles, i.e. $r = R$ we then have $R = \dfrac{a}{2(1+\sqrt{3})}$ and for the sum of their areas

$$A_1 = 3\cdot\pi R^2 = \frac{3\pi}{4(1+\sqrt{3})^2}a^2 \approx 31.567\,\%\,a^2$$

Next we chose R to be the radius of the inscribed circle, which for an equilateral triangle is

$$R_0 = \frac{a}{2\sqrt{3}} \tag{11}$$

and with equation (4)

$$a = r\sqrt{3} + 2\sqrt{\frac{ar}{2\sqrt{3}}} + \frac{a}{2} \Longrightarrow r = \frac{a}{6\sqrt{3}} \tag{12}$$

For the sum of all three discs we thus get

$$A_2 = \pi R_0^2 + 2\pi r^2 = (\frac{\pi}{12} + \frac{\pi}{54})a^2 = \frac{11\pi}{108}a^2 \approx 31.9977\,\%\,a^2 \tag{13}$$

15. Find all three digits numbers $x = [abc] = 100a + 10b + c$ such that all powers $x^n\ ,\ n \in \mathbb{N}$ end on $[...abc]$, too.

    Reason: Puzzle.

    Solution: The problem is trivial for $n = 1$ so let us assume we had solved it for $n = 2$. Then $x^3 = [abc]^3 = [abc]\cdot[abc]^2 = [abc]\cdot[...abc] =$

$[abc] \cdot (1000d + 100a + 10b + c)$ and we are back to the power of two, as the multiplication by $1000d$ doesn't contribute anything to the last three digits. Thus it is sufficient to solve the problem for $n = 2$.

$x^2 = (100a + 10b + c)^2 = 10 \cdot d + c^2$ and the only digits which end up squared by the same digit are $c \in \{ 0, 1, 5, 6 \}$.

(a) $c = 0$.
$x^2 = (100a + 10b)^2 = 10000a^2 + 2000ab + 100b^2$
The last but one digit is zero, so $b = 0$ and $x^2 = 10000a^2$ which means for the second before last digit $a = 0$, but $x = 0$ hasn't three digits.

(b) $c = 1$.
$x^2 = (100a+10b+1)^2 = 10000a^2+2000ab+200a+100b^2+20b+1$
Thus $b = 2b \bmod 10$, i.e. $b = 0$ and $x^2 = 1000d + 200a + 1$.
Therefore $2a = a \bmod 10$ or $a = 0$ which isn't possible.

(c) $c = 5$.
$(100a + 10b + 5)^2 = 10000a^2 + 2000ab + 1000a + 100b^2 + 100b + 25$
The result shows, that $b = 2$ so $x^2 = 1000d + 625$ which is a solution, since $625^2 = 390625$.

(d) $c = 6$.
$(100a + 10b + 6)^2 = 10000a^2 + 2000ab + 1200a + 100b^2 + 120b + 36$
The result shows, that $b \in \{ 1, 3, 5, 7, 9 \}$ is an odd number. Squaring $[b6]$ yields, that only $[76]$ is a possible solution, i.e. $b = 7$ and $x^2 = 1000d + 1200a + 776$. Again we thus have only odd values for a and testing them leaves $[abc] = 376$.

Therefore there are two three digits numbers which fulfill the condition: 625 and 376 .

16. The teacher writes a number less than 50,000 on the board.

The first student finds that n is divisible by 2.
The second student finds that n is divisible by 3.
The third student finds that n is divisible by 4.
...
The twelfth student finds that n is divisible by 13.

Ten of the students told the truth, two lied. The two liars have made their statements immediately after each other.

What number did the teacher write on the board?

Reason: Logic Puzzle.

Solution: The least common multiple of $1, 2, 3, \ldots, 13$ is $360, 360$. The following students couldn't have lied, because it resulted in another lie of a student who is not their neighbor:

$$\begin{array}{rlrlrl} n=2 & (4) & n=3 & (6) & n=4 & (8) \\ n=5 & (10) & n=6 & (12) & n=10 & (2 \text{ or } 5) \\ n=12 & (3 \text{ or } 4) & & & & \end{array}$$

Thus the only combination of liars left are $(7, 8)$ and $(8, 9)$.
$(360, 360 : 7) : 2 = 25, 740$ and $(360, 360 : 3) : 2 = 60, 060 > 50, 000$, i.e. the teacher wrote $25, 740$ on the board.

17. Calculate curvature and torsion of the curve

$$x : [0, a] \longrightarrow \mathbb{R}^3 , \, x(t) = \left(t, t^2, \frac{2}{3}t^3\right)^\tau$$

Reason: Physicist's Practice.

Solution: Since $\dfrac{d\sigma}{dt} = ||\dot{x}(t)||_2 = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2$ we get for the tangent unit vector

$$T(t) = \frac{dx}{d\sigma} = \frac{dx/dt}{d\sigma/dt} = \frac{1}{1+2t^2}\begin{pmatrix} 1 \\ 2t \\ 2t^2 \end{pmatrix}$$

$$\frac{dT}{d\sigma} = \frac{dT/dt}{d\sigma/dt} = \frac{1}{(1+2t^2)^3} = \begin{pmatrix} -4t \\ 2-4t^2 \\ 4t \end{pmatrix}$$

The curvature is therefore

$$\kappa(t) = \left|\left|\frac{dT}{d\sigma}\right|\right|_2 = \frac{(16t^2 + 4(1-2t^2)^2 + 16t^2)^{\frac{1}{2}}}{(1+2t^2)^3} = \frac{2}{(1+2t^2)^2}$$

the normal vector

$$N(t) = \frac{1}{\kappa(t)} \cdot \frac{dT}{d\sigma} = \frac{1}{1+2t^2}\begin{pmatrix} -2t \\ 1-2t^2 \\ 2t \end{pmatrix}$$

and the binormal vector

$$B(t) = T(t) \times N(t) = \frac{1}{1+2t^2}\begin{pmatrix} 2t^2 \\ -2t \\ 1 \end{pmatrix}$$

From that we get

$$\frac{dB}{d\sigma} = \frac{dB/dt}{d\sigma/dt} = \frac{1}{(1+2t^2)^3}\begin{pmatrix} 4t \\ 4t^2-2 \\ -4t \end{pmatrix}$$

and the torsion is

$$\tau(t) = -\frac{dB}{d\sigma}\cdot N(t) = \frac{2}{(1+2t^2)^2}$$

18. In which country in Europe originated this custom? Write down the numbers behind your answers from left to right and post your result.

    Reason: Quiz.

    Solution: $27, 81, 81, 9, 24, 34, 12, 14, 22, 23$: for $n = 0, 1, ..., 5$
    $2^n + 7^n + 8^n + 18^n + 19^n + 24^n = 3^n + 4^n + 12^n + 14^n + 22^n + 23^n$

    (a) From Nikulden to Budni Vecher is lent in this country. At Christmas you can then taste Kravai, the traditional Christmas bread. The presents on Christmas Eve brings the Djado Koleda ("Grandfather Christmas").

    Russia 56 - Bulgaria 27 : Bulgaria 27

    (b) Christmas is Jul and a house elf named Nisse (Julenisse) is even more important in this country than Santa Claus. It is said that he lives in stables and in barns and takes care of the animals there. He likes to play a little prank on the children.

    Norway 87 - Denmark 81 : Denmark 81

    (c) The Christmas holidays are also called "beer festivals" in this country. Traditionally, they were celebrated rather quietly in the circle of the family. Even visitors were rather undesirable on the Christmas holidays, female visit on the 2nd Christmas holiday was once even considered a particularly bad omen. Christmas dinner in this country includes dishes such as roast goose, sauerkraut, potatoes, or ginger cookies.

    Estonia 81 - Germany 42 : Estonia 81

    (d) Ilex and mistletoe are important symbols of Christmas in this country, as is the robin that is most often seen on Christmas cards.

    England 9 - Netherlands 3 : England 9

(e) At Christmas, a log is burned in the fireplace and a cake shaped like a log is made, according to old customs. Otherwise, you will dine in this country rather nobly with selected delicacies. Even the smell of roasted sweet chestnuts must not be missing in the run-up to Christmas.

France 24 - Spain 67 : France 24

(f) An absolute must at Christmas in this country is the Joulukinkku, the Christmas ham. Christmas peace is proclaimed in this country on 12/24 and deceased family members are remembered on Christmas Eve. For Christmas dinner you will be served traditional rice pudding with cinnamon, sugar and an almond, which should bring good luck.

Latvia 16 - Finland 34 : Finland 34

(g) This country put the Christmas tree into the focus of Christmas for the first time. Also edible tree decoration of former times was replaced for the first time by glass balls. There are not just one, but several official gift bringers.

Germany 12 - Czech Republic 2 : Germany 12

(h) 14 days before Christmas, they turn up, the thirteen charming, but also somewhat sneaky Christmas goblins - the last one, called the "Thirteenth", will not disappear until January 6th. However, the children of this country have to be especially careful of the troll woman Grýla, the mother of the thirteen kobolds - who was not good, is caught by her and consumed without further ado. An absolute must at Christmas is the traditional Christmas drink Jólaöl.

Iceland 14 - Greenland 9 : Iceland 14

(i) Christmas is referred to in the language of this country as "Winter Festival" and Christmas Eve is the "Winter Festive Evening". On this day, according to Christian custom, the birth of Christ is celebrated, and according to ancient pagan custom, the return of the virgin of the sun. A popular Christmas ornament is Puzuri, a kind of mobile made of straw.

Latvia 22 - Poland 33 : Latvia 22

(j) An important pre-Christmas custom in this country is the celebration of Lucia Day: on the day of Saint Lucia, December 13, the eldest daughter of the house plays the "Lucia Bride" and wakes the whole family for breakfast. On Christmas Eve, which is often called the "Dopparedan" (one-day's day), sausages, potato casserole with anchovies or Lutfisk, marinated cod are often served. The gifts are not brought by Santa Claus or the Christkind, but the Julbock.

Sweden 23 - Lithuania 31 : Sweden 23

19. Ten DYK Christmas gifts.

Reason: Facts.

Solution:

10: The image of Santa Claus flying his sleigh began in 1819 and was created by Washington Irving,
(https://en.wikipedia.org/wiki/Washington_Irving),
the same author who dreamt up the Headless Horseman.

09: Clement Moore's poem introduced eight more reindeer for Santa's sleigh and their names were Dasher, Dancer, Prancer, Vixen, Comet, Cupid, Duner and Blixem (for the German words for thunder (Donner) and lightning (Blitz)). These later evolved into Donner and Blitzen.

08: Some leave food out for Santa Claus' reindeer as Norse children did, leaving hay and treats for Odin's eight-legged horse Sleipnir hoping they would stop by during their hunting adventures. Dutch children adopted this same tradition, leaving food in their wooden shoes for St. Nicholas' horse.

07: America's first batch of eggnog was made in the Jamestown settlement in 1607. Its name comes from the word "grog", meaning any drink made with rum. Non-alcoholic eggnog is popular as well.

06: Between the 16th and 19th centuries global temperatures were significantly lower than normal in what was known as a "little ice age". Charles Dickens grew up during this period and experienced snow for his first eight Christmases. This "White Christmas" experience influenced his writing and began a tradition of expectation for the holidays.

05: The Christmas wreath
(https://en.wikipedia.org/wiki/Wreath#Advent_and_Christmas_wreaths)
was originally hung as a symbol of Jesus. The holly represents his crown of thorns and the red berries the blood he shed.

04: Tinsel was invented in 1610 in Germany and was once made of real silver.

03: A Christmas tree
(https://en.wikipedia.org/wiki/Christmas_tree)
is a decorated tree, usually an evergreen conifer such as spruce, pine, or fir or an artificial tree of similar appearance, associated with the celebration of Christmas. The modern Christmas tree was developed in medieval Livonia (present-day Estonia and Latvia) and early modern Germany, where Protestant Germans brought decorated trees into their homes. It acquired popularity beyond the Lutheran areas of Germany and the Baltic countries during the second half of the 19th century, at first among the upper classes.

02: The tradition of hanging stockings comes from a Dutch legend. A poor man had three daughters for whom he could not afford to provide a dowry. St. Nicholas dropped a bag of gold down his chimney and gold coins fell out and into the stockings drying by the fireplace. The daughters now had dowries and could be married, avoiding a life on the streets.

01: In 1914 during World War I there was a now famous Christmas truce
(https://en.wikipedia.org/wiki/Christmas_truce)
in the trenches between the British and the Germans. They exchanged gifts across a neutral no man's land, played football together, and decorated their shelters.

Part II
October, 2018

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1. Calculate

(a)
$$\sum_{n=0}^{\infty}\left(\frac{2}{2+3i}\right)^n$$

(b)
$$\sum_{n=0}^{\infty}\left(2\sqrt{n}-4\sqrt{n+1}+2\sqrt{n+2}\right)$$

(c)
$$\sum_{n=3}^{\infty}\frac{8n}{(n^2-1)^2}$$

(d)
$$\lim_{x\to 0}\frac{\cos(x^2)-\sqrt{1+x^3}}{x^3}$$

Reason: Easy.

Solution:

(a)
$$\begin{aligned}S_n&=\sum_{k=0}^{n}\left(\frac{2}{2+3i}\right)^k\\&=\left[1-\left(\frac{2}{2+3i}\right)^{n+1}\right]\cdot\left[1-\frac{2}{2+3i}\right]^{-1}\\&=\left[1-\left(\frac{2}{2+3i}\right)^{n+1}\right]\cdot\frac{2+3i}{3i}\end{aligned}$$

Because we have $\left|\frac{2}{2+3i}\right|=\frac{2}{\sqrt{13}}<1$ we get

$$\sum_{n=0}^{\infty}\left(\frac{2}{2+3i}\right)^n=\frac{1}{1-\frac{2}{2+3i}}=\frac{2+3i}{3i}=1-\frac{2}{3}i$$

(b) It follows by induction that the $n-$th partial sum is

$$S_n = -2(1+\sqrt{n+1}-\sqrt{n+2})$$

so we get

$$\begin{aligned}\sum_{n=0}^{\infty}\left(2\sqrt{n}-4\sqrt{n+1}+2\sqrt{n+2}\right) &= \lim_{n\to\infty} S_n \\ &= -2-2\cdot\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n+2}) \\ &= -2+2\cdot\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n+2})} \\ &= -2\end{aligned}$$

(c) With the partial fraction $\dfrac{8n}{(n^2-1)^2}=\dfrac{2}{(n-1)^2}-\dfrac{2}{(n+1)^2}$ we have

$$\begin{aligned}S_n &= \sum_{k=3}^{n}\frac{4k}{(k^2-1)^2} \\ &= 2\sum_{k=3}^{n}\left(\frac{1}{(k-1)^2}-\frac{1}{(k+1)^2}\right) \\ &= 2\left(\sum_{k=2}^{n-1}\frac{1}{k^2}-\sum_{k=4}^{n+1}\frac{1}{k^2}\right) \\ &= \frac{13}{18}-\frac{2}{n^2}-\frac{2}{(n+1)^2}\end{aligned}$$

and $\sum_{n=3}^{\infty}\dfrac{8n}{(n^2-1)^2}=\lim_{n\to\infty}S_n=\dfrac{13}{18}$

(d) We have the Taylor expansions $\cos(x^2)=1-\dfrac{x^4}{2}+O(x^6)$ and

$\sqrt{1+x^3} = 1 + \frac{x^3}{2} - \frac{x^6}{8} + O(x^9)$ at $x = 0$ and thus

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos(x^2) - \sqrt{1+x^3}}{x^3} &= \lim_{x\to 0} \frac{1 - \frac{x^4}{2} + O(x^6) - 1 - \frac{x^3}{2} + \frac{x^6}{8} + O(x^9)}{x^3} \\
&= \lim_{x\to 0} \frac{-\frac{x^3}{2} - \frac{x^4}{2} + \frac{x^6}{8} + O(x^6) + O(x^9)}{x^3} \\
&= \lim_{x\to 0} \left(-\frac{1}{2} - \frac{1}{2}x + \frac{1}{8}x^3 + O(x^3) + O(x^6) \right) \\
&= -\frac{1}{2} + \lim_{x\to 0} O(x^3) \\
&= -\frac{1}{2}
\end{aligned}$$

2. (a) Determine $\int_1^\infty \frac{\log(x)}{x^3}\,dx$.

   (b) Determine for which α the integral $\int_0^\infty x^2 \exp(-\alpha x)\,dx$ converges.

   (c) Find a sequence of functions $f_n : \mathbb{R} \longrightarrow \mathbb{R}$, $n \in \mathbb{N}$ such that

$$\sum_{\mathbb{N}} \int_{\mathbb{R}} f_n(x)\,dx \neq \int_{\mathbb{R}} \left(\sum_{\mathbb{N}} f_n(x) \right) dx$$

   (d) Find a family of functions $f_r : \mathbb{R}^+ \longrightarrow \mathbb{R}$, $r \in \mathbb{R}$ such that

$$\lim_{r\to 0} \int_{\mathbb{R}} f_r(x)\,dx \neq \int_{\mathbb{R}} \lim_{r\to 0} f_r(x)\,dx$$

   (e) Find an example for which

$$\frac{d}{dx} \int_{\mathbb{R}} f(x,y)\,dy \neq \int_{\mathbb{R}} \frac{\partial}{\partial x} f(x,y)\,dy$$

Reason: Integrals and commutativity with other operations.

Solution:

(a) By partial integration ($u = \log(x)$, $v' = x^{-3}$) we get

$$\begin{aligned}\int_1^\infty \frac{\log(x)}{x}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\log(x)}{x^3}\,dx\\ &= -\lim_{t\to\infty}\left.\frac{\log(x)}{2x^2}\right|_1^t + \lim_{t\to\infty}\int_1^t \frac{1}{2x^3}\,dx\\ &= -\lim_{t\to\infty}\frac{\log(t)}{2t^2} - \lim_{t\to\infty}\left.\frac{1}{4x^2}\right|_1^t\\ &= \frac{1}{4}\end{aligned}$$

(b) For $\alpha = 0$ we get an infinite integral. Let us now assume $\alpha \neq 0$. By partial integration twice, we get

$$\begin{aligned}\int_0^\infty x^2\exp(-\alpha x)\,dx &= -\left.\frac{x^2\exp(-\alpha x)}{\alpha}\right|_0^\infty + \frac{2}{\alpha}\int_0^\infty x\exp(-\alpha x)\,dx\\ &= -\left.\frac{x^2\exp(-\alpha x)}{\alpha}\right|_0^\infty\\ &+ \frac{2}{\alpha}\left(-\left.\frac{x}{\alpha}\exp(-\alpha x)\right|_0^\infty + \frac{1}{\alpha}\int_0^\infty \exp(-\alpha x)\,dx\right)\\ &= \frac{\exp(-\alpha x)}{\alpha}\left[-x^2 - \frac{2}{\alpha}x - \frac{2}{\alpha^2}\right]_0^\infty\\ &= \frac{2}{\alpha^3} - \lim_{x\to\infty}\exp(-\alpha x)\left(\frac{x^2}{\alpha}+\frac{2x}{\alpha^2}+\frac{2}{\alpha^3}\right)\\ &= \begin{cases}\frac{2}{\alpha^3} & \text{if } \alpha > 0\\ \text{not existent} & \text{if } \alpha \leq 0\end{cases}\end{aligned}$$

(c) With $f_n = \operatorname{Ind}[n, n+1] - \operatorname{Ind}[n+1, n+2]$ we have $\int_\mathbb{R} f_n(x)\,dx = 0$ and $\sum_\mathbb{N} f_n(x) = \operatorname{Ind}[1, 2]$ and so

$$0 = \sum_\mathbb{N}\int_\mathbb{R} f_n(x)\,dx \neq \int_\mathbb{R}\sum_\mathbb{N} f_n(x)\,dx = 1$$

(d) With $f_r(x) = \begin{cases}0 & \text{if } |x| \geq r\\ \frac{r-|x|}{r^2} & \text{if } |x| < r\end{cases}$ we have $\int_\mathbb{R} f_r(x)\,dx = 1$ for $r > 0$ and $\lim_{r\to 0} f_r(x) = f_0(x) = \begin{cases}\infty & \text{if } x = 0\\ 0 & \text{if } x \neq 0\end{cases}$ and so

$$1 = \lim_{r\to 0}\int_\mathbb{R} f_r(x)\,dx \neq \int_\mathbb{R}\lim_{r\to 0} f_r(x)\,dx = 0$$

(e) We first define $g(t) = \exp(-t^2)$ and show $I := \int_{\mathbb{R}} g(x)\,dx = \sqrt{\pi}\,.$

$$\begin{aligned} I^2 &= \left(\int_{\mathbb{R}} g(x)\,dx\right)\cdot\left(\int_{\mathbb{R}} g(y)\,dy\right) \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} g(x)g(y)\,dx\,dy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \exp(-x^2-y^2) \\ &= \int_0^\infty \int_0^{2\pi} r\exp(-r^2)\,d\varphi\,dr \\ &= 2\pi\int_0^\infty r\exp(-r^2)dr \\ &= -\pi\left[\exp(-r^2)\right]_0^\infty \\ &= \pi \end{aligned}$$

The function $h(x,y) = x\cdot g(xy)$ is continuous, however the parameter integral $H(x) = \int_{\mathbb{R}} h(x,y)\,dy$ is not:

$$\begin{aligned} H(x) &= \int_{\mathbb{R}} x\cdot g(xy)\,dy \\ &= \operatorname{sgn}(x)\int_{\mathbb{R}} |x|\cdot g(xy)\,dy \\ &= \operatorname{sgn}(x)\int_{\mathbb{R}} g(t)\,dt \\ &= \operatorname{sgn}(x)\cdot\sqrt{\pi} \end{aligned}$$

Now let $f : \mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ the continuously differentiable function $f(x,y) = x|x|g(xy)$ such that we for $F(x) = \int_{\mathbb{R}} f(x,y)\,dy$

$$\begin{aligned} F(x) &= \int_{\mathbb{R}} x|x|g(xy)\,dy \\ &= x\int_{\mathbb{R}} |x|g(xy)\,dy \\ &= x\int_{\mathbb{R}} g(t)dt \\ &= x\sqrt{\pi} \qquad \text{and} \\ F'(x) &= \sqrt{\pi} \end{aligned}$$

Differentiation under the integral gives us

$$\int_{\mathbb{R}} \frac{\partial}{\partial x} f(x,y)\,dy = \int_{\mathbb{R}} \left(2|x|g(xy) + x|x|\frac{d}{dx}g(xy)\right)\,dy$$

which vanishes at $x = 0$. For $x \neq 0$ we get

$$\begin{aligned}\int_{\mathbb{R}} \frac{\partial}{\partial x} f(x,y)\,dy &= 2\int_{\mathbb{R}} g(t)\,dt - 2\int_{\mathbb{R}} t^2 g(t)\,dt \\ &= 2\sqrt{\pi} - 2\left(\left[-\frac{t}{2}\exp(-t^2)\right]_{-\infty}^{\infty} + \frac{1}{2}\int_{\mathbb{R}} g(t)\,dt\right) \\ &= \sqrt{\pi}\end{aligned}$$

In the end we have

$$\sqrt{\pi} = \left.\frac{d}{dx}\right|_{x=0} F(x) = \left.\frac{d}{dx}\right|_{x=0} \int_{\mathbb{R}} f(x,y)\,dy \neq \int_{\mathbb{R}} \left.\frac{\partial}{\partial x}\right|_{x=0} f(x,y)\,dy = 0$$

and although both sides are identical for $x \neq 0$ they differ at $x = 0$ and thus cannot be the same.

3. (a) Show that $D_4 = \langle r, s \,|\, r^2 = s^2 = rsrs = 1\rangle$ is the smallest non-cyclic group.
   (b) Show that the converse of Lagrange's theorem is false, i.e. that there is a finite group with n elements which has no subgroup to one of the divisors of n .
   (c) Give an example of a non-Abelian finite and a non-Abelian infinite group.
   (d) Show that A_5 is simple, i.e. has only trivial normal subgroups.

Reason: Groups.

Solution:

(a) $D_4 = V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$
D_4 has four elements, but none is of order four, so it cannot by cyclic. Smaller groups have necessarily one ($\{1\}$), two ($\mathbb{Z}_2$) or three ($\mathbb{Z}_3$) elements and are all cylic.

(b) A_4 has no subgroup of order 6.
$6 \,|\, |A_4| = 12$. Since A_4 consists of eight $3-$cycles, three pairs of disjoint transpositions, and 1, a subgroup of order six has to contain at least one $3-$cycle, say (123). Now as a subgroup of index two, it would also be normal. By conjugation of (123) with all pairs of transpositions, and inversion, we would get all other $3-$cycles as elements of this subgroup. But a subgroup of at least eight elements cannot have six.

(c) $S_3 \cong D_6$ and $GL_n(\mathbb{R})$
$D_6 = \langle r,s \,|\, s^2 = r^3 = srsr = 1 \rangle$ and reflexion s and rotation r do not commute.
The center of $GL_n(\mathbb{R})$ is $Z(GL_n(\mathbb{R})) = \operatorname{Diag}_n(\mathbb{R}) \subsetneq GL_n(\mathbb{R})$.

(d) We use a similar argument as in part (b), namely:

If a normal subgroup $N \trianglelefteq A_n$, $n > 2$, *contains a* $3-$*cycle, then* $N = A_n$.

Proof: Say we have $(123) \in N$ and so are $(123)^{-1} = (132) \in N$ and $\sigma(132)\sigma^{-1} \in N$ for $\sigma \in A_n$. Thus

$$(12)(3k)(132)(3k)^{-1}m(12)^{-1} = (12k) \in N \ , \ k > 3$$

However, A_n is generated by all $3-$cycles of the form $(12k)$. (Ex.)

Now let us assume we have a normal subgroup $\{1\} \triangleleft N \triangleleft A_5$. For our proof which works for any group A_n , $n > 4$ we choose a permutation $1 \neq \tau \in N$ which leaves the maximal number of elements invariant, resp. permutes a minimal number of elements, and show, that τ has to be a $3-$cycle. The statement then follows by the above.

Assume τ moves four elements, which means it has to be w.l.o.g. $\tau = (12)(34)$. Then

$$N \ni \tau[(345)\tau(345)^{-1}] = (12)(34)(345)(12)(34)(354) = (345)$$

which permutes only three elements in contradiction to the minimality of τ.
Thus let us assume τ permutes more than four numbers. Then we can write w.l.o.g. $\tau = (12345)$ as only possibility to permute all five numbers, since all other possibilities involve odd permutations. (The general case $n > 5$ has to consider more possibilities. The trick then is to write τ by starting with the longest cycle on the left.) Again we get

$$N \ni (234)\tau(234)^{-1} = (234)(12345)(243) = (13425) \neq \tau$$

and $N \ni \tau^{-1}(13425) = (43215)(13425) = (124)$ leaving $3,5$ invariant in contradiction to the minimality of τ .

4. One tiny nocturnal and long-living beetle decided one night to climb a sequoia. The tree was exactly $100\,m$ high at this time. Every night the beetle made a distance of $10\,cm$. The tree grew every day evenly

$20\,cm$ along its entire length.
Did the beetle eventually reach the top of the tree? And if so, how many nights will he need at least?

Reason: Riddle.

Solution: On the first night the beetle manages 10/10000 of the tree trunk. On the second night he crawls 10/10020 of the tree trunk. On the third night he crawls 10/10040 of the tree trunk. He has reached the top when the sum of the track parts reaches 1.

$$\sum_{n=0}^{N} \frac{1}{1000+2n} \geq 1$$
$$\sum_{n=0}^{N} \frac{1}{500+n} \geq 2$$
$$\sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{499} \frac{1}{n} \geq 2$$
$$H_N - H_{499} \geq 2$$

A close estimation is $H_n = \gamma + \log(n) + \varepsilon$ with the Euler-Mascheroni constant γ and a small error ε. A numerical solution yields at least $3691 - 499 = 3192$ nights.

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1. (a) Let X be a set and $\mathcal{F} = \{\, \{\, x \,\} \mid x \in X \,\}$. Determine the $\sigma-$algebra $\sigma(\mathcal{F})$.
   (b) Let X be a set and $S \subseteq \mathcal{P}(X)$. Show that for $A \in \sigma(S)$ there is an $S_0 \subseteq S$ such that S_0 is countable and $A \in \sigma(S_0)$.

Reason: Analysis.

Solution: A $\sigma-$algebra $\sigma(\mathcal{B})$ over a set $\mathcal{B}$ is a subset of the power set $P(\mathcal{B})$, which contains $\mathcal{B}$ as an element or equivalently $\emptyset$, is closed under complements to $\mathcal{B}$, and closed under the union of countable many sets. For an arbitrary set X and a family of subsets $B \subseteq \mathcal{P}(X)$, $\sigma(B)$ denotes the intersection of all $\sigma-$algebras of subsets of X that contain B, i.e. $\sigma(B) = \cap\{\, \sigma(C) \mid C \subseteq B \,\}$ with complements are taken to X.

(a) We define $\mathcal{A} := \{\, A \subseteq X \mid A \text{ is countable or } X - A \text{ is countable } \}$ and show $\mathcal{A} = \sigma(\mathcal{F})$.

Every countable set $A = \{\, x_i \mid i \in \mathbb{N} \,\} = \bigcup_{i\in\mathbb{N}}\{\, x_i \,\} \in \sigma(\mathcal{F})$ and also each set with a countable complement. Thus $\mathcal{A} \subseteq \sigma(\mathcal{F})$. In order to show $\sigma(\mathcal{F}) \subseteq \mathcal{A}$, we show that $\mathcal{A}$ is a $\sigma-$algebra which contains $\mathcal{F}$.

i. $\emptyset \in \mathcal{A}$ since it is countable.

ii. If $A \in \mathcal{A}$ then $X - A \in \mathcal{A}$ since $A = X - (X - A)$.

iii. Let $A_i \in \mathcal{A}$, $i \in \mathbb{N}$. Then either all A_i are countable, and then $\cup_{i\in\mathbb{N}} A_i$ is countable, too, and in $\mathcal{A}$, or there is an index j with an uncountable set A_j. In this case is $X - A_j$ countable. Then we have that $X - \cup_{i\in\mathbb{N}} A_i = \cap_{i\in\mathbb{N}} X - A_i \subseteq X - A_j$ is countable, and again $\cup_{i\in\mathbb{N}} A_i \in \mathcal{A}$.

iv. If $A = \{\, x \,\} \in \mathcal{F}$, then it is countable and so is $A \in \mathcal{A}$, i.e. $\mathcal{F} \subseteq \mathcal{A}$.

(b) Let $\mathcal{A} = \bigcup\{\, \sigma(C) \mid C \subseteq S \text{ countable} \,\}$. We show that $\mathcal{A}$ is a $\sigma-$algebra over X which contains S.

i. $\emptyset \in \sigma(\emptyset) = \{\emptyset, X\}$ and $\emptyset$ is countable with $\emptyset \subseteq S$, so $\emptyset \in \mathcal{A}$.

ii. With $A \in \mathcal{A}$ we have a countable set $C \subseteq S$ with $A \in \sigma(C)$, and so is $X - A \in \mathcal{A}$.

iii. Let $A_i \in \mathcal{A}$ for $i \in \mathbb{N}$. Then there are countable sets $C_i \subseteq S$ with $A_i \in \sigma(C_i)$. The set $C = \bigcup_{i\in\mathbb{N}} C_i \subseteq S$ is also countable. Since $C_i \subseteq C$ and σ is monotone, we have $A_i \in \sigma(C)$ and so $\bigcup_{i\in\mathbb{N}} A_i \in \sigma(C) \in \mathcal{A}$.

If $A \in S$, then $\{\, A \,\} \subseteq S$ is countable and $A \in \sigma(\{A\}) = \{\, \emptyset, A, X - A, X \,\}$, i.e. $A \in \mathcal{A}$ and thus $S \subseteq \mathcal{A}$.

Since $\mathcal{A}$ is a $\sigma-$algebra which contains S, we have $\sigma(S) \subseteq \mathcal{A}$. This means that for all $A \in \sigma(S)$ there is a countable set $S_0 \subseteq S$ with $A \in \sigma(S_0)$.

2. (a) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and } x \leq y < x+1 \\ -1 & \text{if } x \geq 0 \text{ and } x+1 \leq y < x+2 \\ 0 & \text{elsewhere} \end{cases}$$

Now calculate $\int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y)\, d\lambda(x)\right] \, d\lambda(y)$ and $\int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(x,y)\, d\lambda(y)\right] \, d\lambda(x)$, and why isn't it a contradiction to Fubini's theorem.

(b) Show that the integral

$$\int_A \frac{1}{x^2+y}\, d\lambda(x,y)$$

with $A=(0,1)\times(0,1)\subseteq \mathbb{R}^2$ is finite.

Reason: Fubini.

Solution:

(a) For a certain $x>0$ we get

$$\int_{\mathbb{R}} f(x,y)\, d\lambda(y) = \int_x^{x+1} 1\, d\lambda(y) - \int_{x+1}^{x+2} 1\, d\lambda(y) = 1-1=0$$

and especially $\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\, d\lambda(y)\, d\lambda(x) = 0$

Now we fix a certain $y\in\mathbb{R}$. The condition $x\leq y<x+1$ means $y-1<x\leq y$ and $x+1\leq y<x+2$ means $y-2<x\leq y-1$. However, we also have the condition $x\geq 0$ so we have to distinguish $y\in[0,1)$, $y\in[1,2)$, $y\in[2,\infty)$. We therefore get

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\, d\lambda(x)\, d\lambda(y) = \\
&= \int_{[0,1)}\int_{\mathbb{R}} f(x,y)\, d\lambda(x)\, d\lambda(y) + \int_{[1,2)}\int_{\mathbb{R}} f(x,y)\, d\lambda(x)\, d\lambda(y) + \\
&+ \int_{[2,\infty)}\int_{\mathbb{R}} f(x,y)\, d\lambda(x)\, d\lambda(y) \\
&= \int_{[0,1)}\int_0^y 1\, dx\, dy + \int_{[1,2)} 1 - \int_0^{y-1} 1\, dx\, dy + \int_{[2,\infty)} 0\, dy \\
&= \int_{[0,1)} y\, dy + \int_{[1,2)} (1-(y-1))\, dy \\
&= \frac{1}{2} + 2 - \frac{3}{2} \\
&= 1
\end{aligned}$$

Fubini's theorem does not apply, because $f(x,y)$ isn't continuous, resp. not non-negative a.e.

(b) The function $f(x,y)=\dfrac{1}{x^2+y}$ is positive on $A=(0,1)\times(0,1)$ so we may apply the theorem of Fubini and get

$$\int_0^1\int_0^1 \frac{1}{x^2+y}\, dy\, dx = \int_0^1 \log(x^2+y)\Big|_{y=0}^{y=1} dx = \int_0^1 [\log(x^2+1)-\log(x^2)]\, dx$$

The logarithm function is monotone increasing and positive on $[1,2]$ so $\int_0^1 \log(x^2+1)\,dx \leq \log 2$. Furthermore we have

$$\int_0^1 \log(x^2)\,dx = 2\cdot [x\log(x) - x]_0^1 = -2$$

and $\int_A \dfrac{1}{x^2+2}\,d\lambda(x,y) \leq 2 + \log 2 < \infty$.

3. Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\,\infty\,\}$ with the usual Euclidean topology on $\mathbb{C}$ and

$$\hat{\mathcal{T}} = \{\,U \subseteq \hat{\mathbb{C}} \,|\, \infty \notin U \wedge U \subseteq \mathbb{C} \text{ open}\,\} \cup \{\,U \subseteq \hat{\mathbb{C}} \,|\, \infty \in U \wedge U^C \subseteq \mathbb{C} \text{ compact}\,\}$$

   (a) $\hat{\mathcal{T}}$ is a topology on $\hat{\mathbb{C}}$.
   (b) $(\hat{\mathbb{C}},\hat{\mathcal{T}})$ is Hausdorff.
   (c) $(\hat{\mathbb{C}},\hat{\mathcal{T}})$ is compact.

   Reason: Alexandroff Extension.

   Solution:

   (a) Since $\emptyset$ is open and compact, we have $\emptyset, \hat{\mathbb{C}} \in \hat{\mathcal{T}}$.

   Now let $(O_\alpha)_{\alpha\in I}$ be a family of sets in $\hat{\mathcal{T}}$ and $O := \cup_{\alpha\in I} O_\alpha$.

   If $\infty \notin O_\alpha$ for all $\alpha \in I$ then by definition the O_α are open in $\mathbb{C}$ and so is O with $\infty \notin O$. This means that $O \in \hat{\mathcal{T}}$.

   Now assume $\infty \in O_\beta$ for a $\beta \in I$. Then

$$O^C = O^C \cap \mathbb{C} = \bigcap_{\alpha\in I} (O_\alpha^C \cap \mathbb{C}) \subseteq O_\beta^C$$

   and each of the sets $O_\alpha^C \cap \mathbb{C}$ is closed in $\mathbb{C}$ because of $O_\alpha \in \hat{\mathcal{T}}$. But then O^C is a closed subset of a compact set in $\mathbb{C}$ and thus also compact. Since $\infty \in O$ we have $O \in \hat{\mathcal{T}}$.

   Now let $(O_i)_{i=1}^n$ be finitely many sets in $\hat{\mathcal{T}}$ and $O := \cap_{i=1}^n O_i$.

   Then $O_i \cap \mathbb{C} \subseteq \mathbb{C}$ is open for every $1 \leq i \leq n$. Is there a j with $\infty \notin O_j$ then $\infty \notin O$ and

$$O = O \cap \mathbb{C} = \cap_{i=1}^n (O_i \cap \mathbb{C})$$

   is open in $\mathbb{C}$ and thus $O \in \hat{\mathcal{T}}$.

   If $\infty \in O_i$ for all i, then $\infty \in O$ and $O^C = \cup_{i=1}^n O_i^C \subseteq \mathbb{C}$ is a finite union of compact sets compact in $\mathbb{C}$ again, i.e. $O \in \hat{\mathcal{T}}$.

(b) Let $x \neq y \in \hat{\mathbb{C}}$. If $x, y \in \mathbb{C}$ then we have disjoint open neighborhoods $U_x, U_y \subseteq \mathbb{C}$ by the Hausdorff property of $\mathbb{C}$, and they are also disjoint sets in $\hat{\mathcal{T}}$. Now let $y = \infty$, $x \in \mathbb{C}$. We set

$U_x = \{ z \in \mathbb{C} \,:\, |z-x| < 1 \}$ and $U_y = \{ z \in \mathbb{C} \,:\, |z-y| > 1 \} \cup \{ \infty \}$

and get two disjoint sets which contain x, y resp. As both are also in $\hat{\mathcal{T}}$, we have shown that $(\hat{\mathbb{C}}, \hat{\mathcal{T}})$ is Hausdorff.

(c) Let $(O_\alpha)_{\alpha \in I}$ be a family of open sets in $\hat{\mathcal{T}}$ and $\hat{\mathbb{C}} = \cup_{\alpha \in I} O_\alpha$. Let further be $\infty \in O_\beta$. Then $K = O_\beta^C \subseteq \mathbb{C}$ is compact and

$$K = K \cap \mathbb{C} \subseteq \hat{\mathbb{C}} \cap \mathbb{C} = \cup_{\alpha \in I}(O_\alpha \cap \mathbb{C})$$

an open covering of the compact set K. Therefore we have a finite covering $K \subseteq \cup_{i=1}^n O_{\alpha_i}$. This means

$$\hat{\mathbb{C}} = O_\beta \cup K \subseteq O_\beta \cup \bigcup_{i=1}^n O_{\alpha_i}$$

is a finite subcover and $\hat{\mathbb{C}}$ compact.

4. Calculate

$$\int_{-\infty}^{+\infty} \frac{4}{x^2 - x + 1}\, dx$$

Reason: Residue Theorem.

Solution: For a meromorphic function $f(x) = \dfrac{P(x)}{Q(x)}$ with polynomials $P, Q \in \mathbb{C}[x]$ with $Q^{-1}(0) \cap \mathbb{R} = \emptyset$ and $\deg Q > 1 + \deg P$ we have as a Corollary of the residue theorem (Ex.)

$$\int_{-\infty}^{+\infty} f(t)\, dt = 2\pi i \sum_{\operatorname{Im} z > 0} \operatorname{Res}_z(f)$$

which we can apply here as the poles are $z_{1,2} = \frac{1}{2} \pm \frac{1}{2} i \sqrt{3}$ and for $\operatorname{Im}(z_1) > 0$

$$Res_{z_1}(f) = \frac{4}{z_1 - z_2} = \frac{4}{2i \operatorname{Im}(z_1)} = -i \frac{4}{\sqrt{3}}$$

and we get for the integral $\int_{-\infty}^{+\infty} f(t)\, dt = (2\pi i) \cdot (-i \dfrac{4}{\sqrt{3}}) = \dfrac{8\pi}{\sqrt{3}}$

Part III
August, 2018

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1. Given a non-negative, monotone decreasing sequence $(a_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$. Prove that $\sum_{n\in\mathbb{N}} a_n$ converges if and only if $\sum_{n\in\mathbb{N}_0} 2^n a_{2^n}$ converges.

   Reason: Cauchy's Condensation criterion. (M)

   Solution: Let's assume $\sum_{n\in\mathbb{N}} a_n$ converges. We set $S_n = \sum_{k=1}^{n} a_k$ and calculate

   $$\begin{aligned} S_{2^n} &\geq a_1 + a_2 + 2a_4 + 4a_8 + \ldots + 2^{n-1}a_{2^n} \\ &\geq \frac{1}{2}\left(a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots + 2^n a_{2^n}\right) \\ &= \frac{1}{2}\sum_{k=0}^{2^n} 2^k a_k \end{aligned}$$

   Since $\sum_{k=1}^{\infty} a_k$ converges, so does the series $(S_n)_{n\in\mathbb{N}}$ of partial sums and thus twice the subsequence $2\cdot(S_{2^n})_{n\in\mathbb{N}}$. But this is the boundary from above for the non-negative sums $\sum_{k=1}^{n} 2^k a_k$, i.e. $\sum_{k=1}^{\infty} 2^k a_k$ converges.

   Let now $n < 2^{m+1} - 1$. Then

   $$\begin{aligned} \sum_{k=1}^{n} a_k &\leq \sum_{k=1}^{2^{m+1}-1} a_k \\ &\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + (a_8 + \ldots) + (a_{2^m} + \ldots) \\ &= \sum_{k=0}^{m} 2^k a_k \end{aligned}$$

   If $\sum_{k=0}^{\infty} 2^k a_k$ converges, then $\sum_{k=0}^{\infty} a_k$ is bounded and converges, too.

2. Calculate

   $$\sum_{k=0}^{\infty}\sum_{m=0}^{2k+1} \frac{\sqrt{5}^m}{m!}\cdot\left(\frac{(2k)!}{k!}\right)^2 \frac{2^{-6k-2}}{(2k-m+1)!}$$

   Reason: Easy if the series of arccos is given. (T)

Solution:

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{m=0}^{2k+1}\frac{\sqrt{5}^m}{m!}\cdot\left(\frac{(2k)!}{k!}\right)^2\frac{2^{-6k-2}}{(2k-m+1)!}\\
=&\sum_{k=0}^{\infty}\sum_{m=0}^{2k+1}\binom{2k}{k}\binom{2k+1}{m}\frac{\sqrt{5}^m}{2k+1}\cdot\frac{1}{4^k\cdot 4^{2k+1}}\\
=&\sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{1+\sqrt{5}}{4}\right)^{2k+1}\frac{1}{4^k\cdot(2k+1)}\\
=&\frac{\pi}{2}-\arccos\left(\frac{1+\sqrt{5}}{4}\right)\\
=&\frac{3}{10}\,\pi
\end{aligned}$$

3. Show that the product $P=xyz$ of a Pythagorean triple $x^2+y^2=z^2$ is always divisible by $60\,|\,P$. Since this is an easy problem, please make sure you won't forget to name an argument!

   Reason: Puzzle. (M)

   Solution: We have to show that $3,4,5\,|\,P$.

   (a) $3\,|\,P$: Squares only can have remainders $0,1$ by division by three, hence z^2. In case z^2 isn't divisible by three, the remainders of the sum must be 0 and 1 and thus one of x^2,y^2 is divisible by three and since 3 is prime, this is also true for x or y .

   (b) $4\,|\,P$: All Pythagorean triples are of the form $x=u^2-v^2$, $y=2uv$, $z=u^2+v^2$ so it remains to show, that $2\,|\,uv(u+v)(u-v)(u^2+v^2)$, but if both, u and v are odd, then $(u-v)$ and $(u+v)$ are even.

   (c) $5\,|\,P$: Squares only can have remainders $0,1,4$ by division by five, hence z^2. In case z^2 isn't divisible by five, the remainders of the sum must be 0 and 1 and thus one of x^2,y^2 is divisible by five and since 5 is prime, this is also true for x or y .

4. Given two vector fields $v,w\,:\,\mathbb{R}^2\longrightarrow\mathbb{R}^2$ by

$$v(x,y)=\begin{bmatrix}y\\x-y\end{bmatrix}\;,\;w(x,y)=\begin{bmatrix}y-x\\-y\end{bmatrix}$$

and two curves in $\mathbb{R}^2$ given as:
γ_1 is the half circle from $(0,-1)$ to $(0,1)$ with radius one and origin $(0,0)$, run anti-clockwise from bottom to top.
γ_2 is the straight line segments from $(0,-1)$ to $(1,0)$ and from $(1,0)$ to $(0,1)$, also run through from bottom to top.

(a) Compute all 4 path integrals of v and w with both paths γ_1, γ_2 .

(b) Determine whether v or w have potentials.

Reason: For engineers. (E)

Solution: In the first step we parameterize the two curves:

$$\begin{aligned} c_1 &: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow \gamma_1 \text{ with } c_1(t) = (\cos t, \sin t) \text{ for } \gamma_1 \text{ and} \\ c_{2,1} &: [0,1] \longrightarrow \gamma_2 \text{ with } c_{2,1}(t) = (t, t-1) \text{ and} \\ c_{2,2} &: [0,1] \longrightarrow \gamma_2 \text{ with } c_{2,2}(t) = (1-t, t) \text{ for } \gamma_2 \end{aligned}$$

and get $\dot c_1(t) = (-\sin t, \cos t)$, $\dot c_{2,1}(t) = (1,1)$, $\dot c_{2,2}(t) = (-1,1)$. Thus

$$\begin{aligned} \int_{\gamma_1} v\,ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v(c_1(t)) \cdot \dot c_1(t)\,dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \begin{bmatrix} \sin t \\ \cos t - \sin t \end{bmatrix} \cdot \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\sin^2 t + \cos^2 t - \sin t \cos t\,dt \\ &= \left[-\frac{1}{2}\cos^2 t + 2 \sin t \cos t \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \int_{\gamma_2} v\,ds &= \int_0^1 v(c_{2,1}(t)) \cdot \dot c_{2,1}(t)\,dt + \int_0^1 v(c_{2,2}(t)) \cdot \dot c_{2,2}(t)\,dt \\ &= \int_0^1 \begin{bmatrix} t-1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} t \\ 1-2t \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} dt \\ &= \int_0^1 1 - 2t\,dt \\ &= \left[t - t^2 \right]_0^1 \\ &= 0 \end{aligned}$$

$$\begin{aligned}\int_{\gamma_1} w\,ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w(c_1(t))\cdot \dot{c}_1(t)\,dt\\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \begin{bmatrix}\sin t-\cos t\\ -\sin t\end{bmatrix}\cdot\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}dt\\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\sin^2 t\,dt\\ &= \left[-\frac{1}{2}(t-\sin t\cos t)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\ &= -\frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{\gamma_2} w\,ds &= \int_0^1 w(c_{2,1}(t))\cdot \dot{c}_{2,1}(t)\,dt + \int_0^1 w(c_{2,2}(t))\cdot \dot{c}_{2,2}(t)\,dt\\ &= \int_0^1 \begin{bmatrix}-1\\ 1-t\end{bmatrix}\cdot\begin{bmatrix}1\\ 1\end{bmatrix} + \begin{bmatrix}2t-1\\ -t\end{bmatrix}\cdot\begin{bmatrix}-1\\ 1\end{bmatrix}dt\\ &= \int_0^1 1-4t\,dt\\ &= \left[t-2t^2\right]_0^1\\ &= -1\end{aligned}$$

So the vector field v is apparently path independent whereas w is not. We check this by the calculation of their curl.

$$\operatorname{rot}\vec{F}(x,y)=\operatorname{curl}\vec{F}(x,y)=\frac{\partial \vec{F}_y}{\partial x}-\frac{\partial \vec{F}_x}{\partial y}=\begin{cases}0 & \text{if } \vec{F}=v\\ -1 & \text{if } \vec{F}=w\end{cases}$$

So v has a potential and thus is path independent, and w has none.

5. Show that $\mathbb{Z}[x]/\langle x^2+2x+4\ ,\ 5\rangle \cong \mathbb{Z}_5[4+\sqrt{2}]$ are isomorphic rings.

Reason: Abstract algebra. (M)

Solution: For convenience let $\xi = 4+\sqrt{2}\,$.
Then $\xi \notin \mathbb{Z}_5$ because $2\in\mathbb{Z}_5$ has no root, and $\overline{m}(x) := x^2+2x+4$ is the minimal polynomial of ξ, since $\overline{m}(\xi)=0$. Hence

$$\mathbb{Z}_5[\xi]\cong \mathbb{Z}_5[x]/\langle x^2+2x+4\rangle$$

with the isomorphism $\varphi\;:\;\mathbb{Z}_5[\xi]\to\mathbb{Z}_5[x]/\langle x^2+2x+4\rangle\;,\;\varphi(\xi)=x\,$. The additivity is easy to verify and for the multiplication we have

$$\begin{aligned}\varphi((a\xi+b)(c\xi+d)) &= \varphi(ac\xi^2+(bc+ad)\xi+bd\\ &= acx^2+(bc+ad)x+bd\\ &= (3ac+bc+ad)x+(ac+bd)\\ &= 3acx+ac+adx+bcx+bd\\ &= acx^2+adx+bcx+bd\\ &= (ax+b)(cx+d)\\ &= \varphi(a\xi+b)\varphi(b\xi+c)\end{aligned}$$

Surjectivity is given by construction, and for an element $\varphi(a\xi+b)=ax+b\in\langle x^2+2x+4\rangle$ in the kernel, we get

$$\begin{aligned}ax+b &= (cx+d)\cdot(x^2+2x+4)\\ &= cx^3+(d+2c)x^2+(4c+2d)x+4d\\ &= cx(3x+1)+(d+2c)(3x+1)+(4c+2d)x+4d\\ &= 3cx^2+(c+3d+c+4c+2d)x+(d+2c+4d)\\ &= 3c(3x+1)+cx+2c\\ &= 0\end{aligned}$$

and hence $a\xi+b=0\,$.

It remains to show that $\mathbb{Z}[x]/\langle x^2+2x+4\;,\;5\rangle\cong\mathbb{Z}_5[x]/\langle x^2+2x+4\rangle\,$. Therefore we consider the ideals $5\mathbb{Z}[x]\subseteq\langle m(x)\;,\;5\rangle$ in $\mathbb{Z}[x]$ and apply the second isomorphism theorem for rings and get

$$\mathbb{Z}[x]/\langle m(x)\;,\;5\rangle\cong(\mathbb{Z}[x]/5\mathbb{Z}[x])/(\langle m(x)\;,\;5\rangle/5\mathbb{Z}[x])\cong\mathbb{Z}_5[x]/\langle\overline{m}(x)\rangle$$

where $m(x)=x^2+2x+4\in\mathbb{Z}[x]$

6. Calculate
$$\int_0^\pi\frac{\sin(\varphi)}{3\cos^2(\varphi)+2\cos(\varphi)+3}\,d\varphi$$

Reason: Weierstraß substitutions. (T)

Solution: To solve this integral we make use of the Weierstraß substitutions, resp. tangent half angle substitutions. We set $t:=\tan\left(\frac{1}{2}\varphi\right)$ and get

$$\sin(\varphi)=\frac{2t}{1+t^2}\,,\;\cos(\varphi)=\frac{1-t^2}{1+t^2}\,,\;d\varphi=\frac{2}{1+t^2}\,dt$$

and so

$$\begin{aligned}\frac{\sin(\varphi)\,d\varphi}{3\cos^2(\varphi)+2\cos(\varphi)+3} &= \frac{\dfrac{2t}{1+t^2}\cdot\dfrac{2}{1+t^2}}{3\dfrac{(1-t^2)^2}{(1+t^2)^2}+2\dfrac{1-t^2}{1+t^2}+3}\,dt\\ &= \frac{4t\;dt}{(3-6t^2+3t^4)+2(1-t^4)+(3+6t^2+3t^4)}\\ &= \frac{4t}{4t^4+8}\,dt\\ &= \frac{t}{t^4+2}\,dt\end{aligned}$$

With the substitution $u=\frac{1}{2}t^2$ we get

$$\begin{aligned}\int_0^\pi \frac{\sin(\varphi)}{3\cos^2(\varphi)+2\cos(\varphi)+3}\,d\varphi &= \int_0^\infty \frac{t}{t^4+2}\,dt\\ &= \frac{1}{2\sqrt{2}}\int_0^\infty \frac{du}{1+u^2}\\ &= \frac{1}{2\sqrt{2}}\left[\arctan\frac{1}{\sqrt{2}}\,t^2\right]_0^\infty\\ &= \lim_{t\to\infty}\frac{1}{2\sqrt{2}}\arctan(t)\\ &= \frac{\pi}{4\sqrt{2}}\end{aligned}$$

7. Integrate $\int_1^5 \frac{dx}{\sqrt{x^2+3x-4}}$

Reason: Euler Substitution $\sqrt{x^2+3x-4}=(x+4)t$ (T)

Solution: In order to solve this integral, we look at the zeros in the denominator which are $x=-4$ and $x=1$. We choose the former and proceed by an Euler substitution: $\sqrt{x^2+3x-4}=\sqrt{(x+4)(x-1)}=(x+4)t$. We then have

$$x=\frac{1+4t^2}{1-t^2}\ ,\ \sqrt{x^2+3x-4}=\frac{5t}{1-t^2}\ ,\ dx=\frac{10t}{(1-t^2)^2}\,dt$$

and

$$
\begin{aligned}
\int_1^5 \frac{dx}{\sqrt{x^2+3x-4}} &= 2\int_{\ldots}^{\ldots} \frac{dt}{1-t^2} \\
&= 2\int_{\ldots}^{\ldots} \left(\frac{1}{2}\cdot\frac{1}{1-t}+\frac{1}{2}\cdot\frac{1}{1+t}\right) dt \\
&= \left[\log\left|\frac{1+t}{1-t}\right|\right]_{\ldots}^{\ldots} \\
&= \left[\log\left|\left(1+\frac{\sqrt{x-1}}{\sqrt{x+4}}\right)\left(1-\frac{\sqrt{x-1}}{\sqrt{x+4}}\right)^{-1}\right|\right]_1^5 \\
&= \left[\log\left|\frac{\sqrt{x+4}+\sqrt{x-1}}{\sqrt{x+4}-\sqrt{x-1}}\right|\right]_1^5 \\
&= \log 5 \approx 1.61
\end{aligned}
$$

8. The random waiting time X on a telephone hotline is characterized by the distribution function F with

$$
F(x) = \begin{cases} 0 & \text{if } x < 0 \\ a - be^{-\lambda x} & \text{if } x \geq 0 \end{cases}
$$

for some parameters $a, b, \lambda \in \mathbb{R}$ with $\lambda > 0$. We further assume that $P(X = 0) = 0.5$, i.e. there is a 50 % chance not to wait at all, and $P(X > 1[min]) = 0.25$.

Reason: (E)

(a) Determine the parameters with the given information, such that F is actually a distribution function.

Solution: For a distribution function we need $\lim_{x\to\infty} F(x) = 1$ which means $a = 1$. Since $F(x) = 0$ for $x < 0$ and $P(X = 0) = \frac{1}{2}$ we have $F(0) = a - b = 1 - b = \frac{1}{2}$ and thus $b = \frac{1}{2}$. Finally we have

$$
F(1) = 1 - \frac{1}{2}e^{-\lambda} = P(X \leq 1) = 1 - P(X > 1) = \frac{3}{4}
$$

hence $\lambda = \log 2$ and $F(X) = 1 - \frac{1}{2}e^{-\log(2)\,X}$.

(b) Can the distribution be described by a density function? Why? If yes, calculate the density function.

Solution: Because the distribution function isn't continuous (at $x = 0$) and thus not differentiable, it cannot result from a density function.

9. A princess decided one day to go swimming in the circular lake far from the castle of her father. As soon as she got into the water, suddenly a witch appeared, who wanted to kidnap the girl. The princess swam quickly into the middle of the lake to think of an escape plan. She noticed three things:

- The witch can run four times as fast as I can swim.
- The witch always tries to stay close to me.
- On land, I'm faster than the witch.

Is there a way for the princess to escape, how? Why doesn't she have a chance to escape?

Reason: Riddle. (M)

Solution: The princess swims a bit towards the witch. Once there, she begins to swim in concentric circles. This allows her to move further and further away from the witch, as her angular velocity is higher than that of the witch. Once she has reached the maximum possible distance to the witch in this constellation (the princess and the witch are on a straight line that goes through the center of the pond), she floats on the shortest possible path to the shore. She reaches the shore in front of the witch.

So we have to determine what the maximal radius of the girl's circle is (being still faster), and whether this will be sufficient to reach the shore in time. We must have with the radius R of the lake

$$\omega_P > \omega_W \iff \frac{v_P}{R_P} > \frac{v_W}{R} = \frac{4v_P}{R} \iff R_P < \frac{1}{4}R$$

So we have to show that for the remaining time

$$\begin{aligned} T_W &= \frac{\text{half circle}}{\text{speed}} = \frac{\pi R}{v_W} \\ &> \frac{3R}{v_W} = \frac{\frac{3}{4}R}{\frac{1}{4}v_W} \\ &= \frac{R-\frac{1}{4}R}{v_P} = \frac{\text{remaining distance}}{v_P} \\ &= \frac{R-R_P}{v_P} = T_P \end{aligned}$$

10. Let $f\,:\,\mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as $f(x,y)=x(x-1)^2-2y^2$. Determine all critical points of f, decide whether there are extrema, and which, and at last consider, whether f has global extrema or not.

    Reason: Extrema. (E)

    Solution: We have $\nabla f(x,y)=((x-1)(3x-1),-4y)$ and the necessary condition $\nabla f=0$ yields the points $(1,0)$ and $(\frac{1}{3},0)$. The Hesse matrix of f is $\begin{bmatrix}6x-4 & 0\\ 0 & -4\end{bmatrix}$ and with the second partial derivative test we conclude that $(x,y)=(1,0)$ is a saddle point and $(x,y)=(\frac{1}{3},0)$ a local maximum. Since $\lim_{x\to\pm\infty} f(x,0)=\pm\infty$ there are no global extrema.

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1. Let R be a ring with identity element 1 and $r \in R$ an element without left inverse but at last one right inverse $r \cdot a_0 = 1$. Prove that there are infinitely many right inverses to r.

   Reason: Ring theory. (T)

   Solution: Let $a_0, \ldots, a_N$ be all distinct right inverse elements of r. Set

   $$c_k := a_0 + 1 - a_k \cdot r \ , \ 0 \leq k \leq N.$$

   Then $rc_k = r \cdot a_0 + r \cdot 1 - r \cdot a_k \cdot r = 1 + r - 1 \cdot r = 1$ and all c_k are right inverse to r. Next let us assume $c_i = c_j$. This means $a_i r = a_j r$ and multiplying by a_0 from the right yields $a_i = a_j$ which means $i = j$ since we assumed all a_k to be different. Thus all c_k are distinct, too. Now $a_0 \neq c_k$ for otherwise a_k would be a left inverse to r which doesn't exist. Hence $\{ a_0, c_0, \ldots, c_N \}$ are $N + 2$ distinct right inverse elements of r, so we have found one more than assumed, which means there are infinitely many of them.

2. Consider the Lie algebra of skew-Hermitian 2×2 matrices $\mathfrak{g} := \mathfrak{su}(2, \mathbb{C})$ and the Pauli matrices (note that Pauli matrices are not a basis!)

   $$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \ \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \ , \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   Now we define an operation on $V := \mathbb{C}_2[x, y]$, the vector space of all complex polynomials of degree less than three in the variables x, y by

   $$\begin{aligned} \varphi(\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3).(a_0 + a_1 x + a_2 x^2 + a_3 y + a_4 y^2 + a_5 xy) = \\ = x(-i\alpha_1 a_3 + \alpha_2 a_3 - \alpha_3 a_1) + \\ + x^2(2i\alpha_1 a_5 + 2\alpha_2 a_5 + 2\alpha_3 a_2) + \\ + y(-i\alpha_1 a_1 - \alpha_2 a_1 + \alpha_3 a_3) + \\ + y^2(2i\alpha_1 a_5 - 2\alpha_2 a_5 - 2\alpha_3 a_4) + \\ + xy(-i\alpha_1 a_2 - i\alpha_1 a_4 + \alpha_2 a_2 - \alpha_2 a_4) \end{aligned}$$

   Show that

   (a) an adjusted φ defines a representation of $\mathfrak{su}(2, \mathbb{C})$ on $\mathbb{C}_2[x, y]$

   (b) Determine its irreducible components.

(c) Compute a vector of maximal weight for each of the components.

Reason: Representation of $\mathfrak{su}(2,\mathbb{C})$. (M)

Solution: The solution might look a bit long due to the necessary matrix calculations, but I think it's worth having an explicit example of a $\mathfrak{su}(2,\mathbb{C})$ representation.

We start with the second point as we can get strong hints by inspection. The constant polynomials are obviously sent to zero, so $\mathbb{C}\cdot 1$ is the trivial representation. Next we observe, that $\mathbb{C}\cdot x \oplus \mathbb{C}\cdot y$ as well as $\mathbb{C}\cdot x^2 \oplus \mathbb{C}\cdot xy \oplus \mathbb{C}\cdot y^2$ are invariant subspaces which are our candidates for the irreducible components, and we will only have to check irreducibility, which we will do at the end. At first we choose $(1, x, y, x^2, xy, y^2)$ as the order of our basis vectors in V. Hence we get for the adjusted φ with $\sigma_k \mapsto i\sigma_k$

$$\begin{aligned}\varphi(\alpha_1(i\sigma_1),\alpha_2(i\sigma_2),\alpha_3(i\sigma_3)).(a_0+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2) =\\ = 1\cdot 0 +\\ + x\cdot(\alpha_1a_2+i\alpha_2a_2+i\alpha_3a_1)+\\ + y\cdot(\alpha_1a_1-i\alpha_2a_1-i\alpha_3a_2)+\\ + x^2\cdot(-2\alpha_1a_4+2i\alpha_2a_4+2i\alpha_3a_3)+\\ + xy\cdot(\alpha_1a_3+\alpha_1a_5+i\alpha_2a_3-i\alpha_2a_5)+\\ + y^2\cdot(-2\alpha_1a_4-2i\alpha_2a_4-2i\alpha_3a_5)\end{aligned}$$

and

$$\varphi(\alpha_1,\alpha_2,\alpha_3) \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i\alpha_3 & \alpha_1+i\alpha_2 & 0 & 0 & 0 \\ 0 & \alpha_1-i\alpha_2 & -i\alpha_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i\alpha_3 & -2\alpha_1+2i\alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_1+i\alpha_2 & 0 & \alpha_1-i\alpha_2 \\ 0 & 0 & 0 & 0 & -2\alpha_1-2i\alpha_2 & -2i\alpha_3 \end{bmatrix}$$

Since φ is linear, it remains to show that it defines a Lie algebra homomorphism, i.e. that $\varphi([A,B]) = [\varphi(A),\varphi(B)]$: The multiplication in $\mathfrak{g} = \mathfrak{su}(2,\mathbb{C})$ goes

$$[\alpha_1(i\sigma_1)+\alpha_2(i\sigma_2)+\alpha_2(i\sigma_2),\alpha_1'(i\sigma_1)+\alpha_2'(i\sigma_2)+\alpha_3'(i\sigma_3)]$$

$$= \left[\begin{bmatrix} i\alpha_3 & i\alpha_1+\alpha_2 \\ i\alpha_1-\alpha_2 & -i\alpha_3 \end{bmatrix}, \begin{bmatrix} i\alpha_3' & i\alpha_1'+\alpha_2' \\ i\alpha_1'-\alpha_2' & -i\alpha_3' \end{bmatrix}\right]$$

$$= 2\cdot \begin{bmatrix} -i(\alpha_1\alpha_2'-\alpha_2\alpha_1') & (\alpha_1\alpha_3'-\alpha_3\alpha_1')-i(\alpha_2\alpha_3'-\alpha_3\alpha_2') \\ -(\alpha_1\alpha_3'-\alpha_3\alpha_1')-i(\alpha_2\alpha_3'-\alpha_3\alpha_2') & i(\alpha_1\alpha_2'-\alpha_2\alpha_1') \end{bmatrix}$$

$$= -2(\alpha_2\alpha_3'-\alpha_3\alpha_2')(i\sigma_1)+2(\alpha_1\alpha_3'-\alpha_3\alpha_1')(i\sigma_2)-2(\alpha_1\alpha_2'-\alpha_2\alpha_1')(i\sigma_3)$$

We write $V = \underbrace{\mathbb{C}\cdot 1}_{V_0} \oplus \underbrace{\mathbb{C}x\oplus\mathbb{C}y}_{V_1} \oplus \underbrace{\mathbb{C}x^2\oplus\mathbb{C}xy\oplus\mathbb{C}y^2}_{V_2}$. Since the matrix of φ are diagonal blocks, and $\varphi|_{V_0}=0$ it is sufficient to show that $\varphi|_{V_1}$ and $\varphi|_{V_2}$ are homomorphisms. But we just calculated this property on V_1, which corresponds to the Lie algebra multiplication of the matrices in $\mathfrak{g}$ and it remains to show that

$$[\varphi|_{V_2}(\alpha_1,\alpha_2,\alpha_3),\varphi|_{V_2}(\alpha_1',\alpha_2',\alpha_3')]$$

$$= \left[\begin{bmatrix} 2i\alpha_3 & -2\alpha_1+2i\alpha_2 & 0 \\ \alpha_1+i\alpha_2 & 0 & \alpha_1-i\alpha_2 \\ 0 & -2\alpha_1-2i\alpha_2 & -2i\alpha_3 \end{bmatrix}, \begin{bmatrix} 2i\alpha_3' & -2\alpha_1'+2i\alpha_2' & 0 \\ \alpha_1'+i\alpha_2' & 0 & \alpha_1'-i\alpha_2' \\ 0 & -2\alpha_1'-2i\alpha_2' & -2i\alpha_3' \end{bmatrix}\right]$$

$$= 2i\begin{bmatrix} -2(\alpha_1\alpha_2'-\alpha_2\alpha_1') & 2(\alpha_1\alpha_3'-\alpha_3\alpha_1') & 0 \\ \alpha_1\alpha_3'-\alpha_3\alpha_1' & 0 & -(\alpha_1\alpha_3'-\alpha_3\alpha_1') \\ 0 & -2(\alpha_1\alpha_3'-\alpha_3\alpha_1') & 2(\alpha_1\alpha_2'-\alpha_2\alpha_1') \end{bmatrix}$$

$$+2\begin{bmatrix} 0 & 2(\alpha_2\alpha_3'-\alpha_3\alpha_2') & 0 \\ -(\alpha_2\alpha_3'-\alpha_3\alpha_2') & 0 & -(\alpha_2\alpha_3'-\alpha_3\alpha_2') \\ 0 & 2(\alpha_2\alpha_3'-\alpha_3\alpha_2') & 0 \end{bmatrix}$$

$$= \varphi|_{V_2}(-2(\alpha_2\alpha_3'-\alpha_3\alpha_2'),2(\alpha_1\alpha_3'-\alpha_3\alpha_1'),-2(\alpha_1\alpha_2'-\alpha_2\alpha_1'))$$

$$= \varphi|_{V_2}([(\alpha_1,\alpha_2,\alpha_3),(\alpha_1',\alpha_2',\alpha_3')])$$

We set $\mathfrak{h} = \mathbb{C}H$ with $H := (0,0,-i) = (-i)\cdot(i\sigma_3) = \sigma_3$ which is an Abelian and hence nilpotent subalgebra of $\mathfrak{g}$. By the formula for multiplication we have

$$[(0,0,\alpha_3),(\alpha_1',\alpha_2',\alpha_3')] = (2\alpha_3\alpha_2',-2\alpha_3\alpha_1',0)\in\mathfrak{h} \text{ only if } \alpha_1'=\alpha_2'=0$$

which means that $\mathfrak{h}$ is self-normalizing and hence the one-dimensional Cartan subalgebra of $\mathfrak{g}$.

Since $\varphi|_{V_0} = 0$ a vector of maximal height is $v_m = 0$.

Since $\varphi|_{V_1}(H).(a_1, a_2) = (a_1, -a_2)$ already is a basis of eigenvectors $x = (1, 0), y = (0, 1)$ to the eigenvalues ± 1 for the operation by H, the operation by the other two basis vectors of $\mathfrak{g}$ switch between those eigenspaces. We therefore change the basis for now and define

$$X := -\frac{1}{2}i(i\sigma_1) + \frac{1}{2}(i\sigma_2) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; Y := -\frac{1}{2}i(i\sigma_1) - \frac{1}{2}(i\sigma_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Thus

$$\varphi|_{V_1}(X).(a_1, a_2) = \varphi|_{V_1}\left(\left(-\frac{1}{2}i, +\frac{1}{2}, 0\right)\right)(a_1, a_2) = (a_2, 0)$$
$$\varphi|_{V_1}(Y).(a_1, a_2) = \varphi|_{V_1}\left(\left(-\frac{1}{2}i, -\frac{1}{2}, 0\right)\right)(a_1, a_2) = (0, a_1)$$
$$(0, 1) \xrightarrow{X} (1, 0) \xrightarrow{X} (0, 0)$$
$$(1, 0) \xrightarrow{Y} (0, 1) \xrightarrow{Y} (0, 0)$$

Let's take $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ as the "ladder up" operator. Then we have $[H, X] = 2X$, i.e. $\lambda = 2$ is our positive root, and $v_m = (1, 0)$ a vector of maximal weight $+1$, because - cp. section 7.1 in
https://www.physicsforums.com/insights/journey-manifold-su2-part-ii/

$$H.v_m = \varphi|_{V_1}(H)(1, 0) = +1 \cdot (1, 0) \text{ diagonal}$$
$$X.v_m = \varphi|_{V_1}(X)(1, 0) = (0, 0) \text{ "ladder up"}$$
$$Y.v_m = \varphi|_{V_1}(Y)(1, 0) = (0, 1) \text{ "ladder down"}$$

This also shows, that $V_1 = \mathbb{C}x \oplus \mathbb{C}y$ is an irreducible component.

It remains to show the same for V_2. With the same settings as above we get $\varphi|_{V_2}(H) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and directly see the weights $-2, 0, 2$ with the eigenvectors $v_m = x^2 = (1, 0, 0), xy = (0, 1, 0), y^2 = (0, 0, 1)$.

$$\varphi|_{V_2}(X).(a_3, a_4, a_5) = \varphi|_{V_2}\left(\left(-\frac{1}{2}i, +\frac{1}{2}, 0\right)\right)(a_3, a_4, a_5) = (2ia_4, -ia_5, 0)$$
$$\varphi|_{V_2}(Y).(a_3, a_4, a_5) = \varphi|_{V_2}\left(\left(-\frac{1}{2}i, -\frac{1}{2}, 0\right)\right)(a_3, a_4, a_5) = (0, -ia_3, 2ia_4)$$
$$(0, 0, 1) \xrightarrow{X} (0, -i, 0) \xrightarrow{X} (2, 0, 0) \xrightarrow{X} (0, 0, 0)$$
$$(1, 0, 0) \xrightarrow{Y} (0, -i, 0) \xrightarrow{Y} (0, 0, 2) \xrightarrow{Y} (0, 0, 0)$$

which again shows the irreducibility, since we can jump from vector to vector along the entire $V_2 = \mathbb{C}x^2 \oplus \mathbb{C}xy \oplus \mathbb{C}y^2$ by the operation φ.

3. Let $(X\ ,\ ||\,.\,||)$ be a normed vector space. Prove that X is complete if and only if for each sequence with $\sum_{n=1}^{\infty} ||x_n|| < \infty$ the series $\sum_{n=1}^{\infty} x_n$ converges as well in X.

   Reason: Completeness criterion. (M)

   Solution: Assume X is complete and the sum of the normed sequence is finite. Then we have

$$||\sum_{n=1}^{\infty} x_n|| \leq \sum_{n=1}^{\infty} ||x_n|| < \infty$$

   and as X is complete, it converges even in X. Let on the other hand $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a Cauchy sequence. Then we set $a_n := x_{N(n+1)} - x_{N(n)}$ where we found the indices $N(n)$ in such a way, that $||x_m - x_k|| < \varepsilon_n := 2^{-n}$ for all $m,k > N(n)$. Since we have now

$$\sum_{n=1}^{\infty} ||a_n|| \leq \sum_{n=1}^{\infty} 2^{-n} = 1$$

   the series $\sum_{n=1}^{\infty} a_n$ converges in X by our assumption. Hence

$$\sum_{n=1}^{\infty} a_n = \lim_{M\to\infty} \sum_{n<M} a_n = \lim_{M\to\infty} \left(-x_{N(1)} + x_{N(M)}\right) = -x_{N(1)} + \lim_{M\to\infty} x_{N(M)}$$

   and the limit of our sequence $(x_n)_{n\in\mathbb{N}}$ exists also in X and $(X\ ,\ ||\,.\,||)$ is complete.

4. Gauß' Divergence Theorem: $\iiint_V (\nabla F)\, dV = \iint_{\partial V} (F \cdot N) d(\partial V)$. See https://www.physicsforums.com/insights/pantheon-derivatives-part-v/

   (a) Let $B = B_1(0)$ the unit sphere in $\mathbb{R}^3$ and consider the vector field

$$F(x) = \begin{bmatrix} (x_2^4 + 2x_2^2x_3^2)x_1 \\ (x_3^4 + 2x_1^2x_3^2)x_2 \\ (x_1^4 + 2x_1^2x_2^2)x_3 \end{bmatrix}$$

   and calculate the integral $\int_{\partial B} F \cdot N\, d\mathbb{S}^2$

Reason: Gauß' Divergence Theorem. (E)

Solution: By Gauß' divergence theorem (the ball has a smooth boundary) we know that (N being the unit normal vector field)

$$\begin{aligned}\int_{\partial B} F\cdot N\, d\mathbb{S}^2 &= \int_B \operatorname{div} F\, dB \\ &= \int_B \sum_{i=1}^{3} \frac{\partial}{\partial x_i} F_i(x)\, dB \\ &= \int_B (x_1^2+x_2^2+x_3^2)^2\, dB \\ &= \int_B |x|^4\, dB \\ &= \int_0^1 \int_0^{2\pi} \int_0^{\pi} r^2 \sin(\theta) r^4\, d\theta\, d\phi\, dr \\ &= 2\pi \int_0^1 r^6\, dr \int_0^{\pi} \sin(\theta)\, d\theta \\ &= \frac{4}{7}\pi\end{aligned}$$

(b) Let $U \subseteq \mathbb{R}^n$ be open and $h \in C^1(U)$, $F \in C^1(U,\mathbb{R}^n)$. Show that on U we have

$$\operatorname{div}(hF) = h \operatorname{div} F + \nabla h \cdot F$$

Solution: This is basically the Leibniz rule, i.e. we have

$$\begin{aligned}\operatorname{div}(hF) &= \sum (\frac{\partial}{\partial x_i})(hF_i) \\ &= \sum (\frac{\partial}{\partial x_i})(h)\cdot F_i + h\cdot \sum (\frac{\partial}{\partial x_i})(F_i) \\ &= \nabla h \cdot F + h \cdot \operatorname{div} F\end{aligned}$$

(c) Let $B^n \subseteq \mathbb{R}^n$ be the closed unit ball and $f,g \in C^2(B^n)$. Show that with the unit normal vector field N

$$\int_{B^n} f\Delta g\, dB^n = -\int_{B^n} \nabla f \cdot \nabla g \ dB^n + \int_{\partial B^n} f\nabla g \cdot N\, d\mathbb{S}^{N-1}$$

Solution: We apply the previous formula and get

$$\begin{aligned}\int_{B^n} f\Delta g\,dB^n &= \int_{B^n} (\operatorname{div}(f\nabla g) - \nabla f\cdot\nabla g)\;dB^n\\ &= \int_{B^n} \operatorname{div}(f\nabla g)\,dB^n - \int_{B^n} \nabla f\cdot\nabla g\,dB^n\\ &= \int_{\partial B^n} f\nabla g\cdot N\,d\mathbb{S}^{n-1} - \int_{B^n} \nabla f\cdot\nabla g\,dB^n\end{aligned}$$

5. Let $f\,:\,(0,1)\longrightarrow\mathbb{R}$ be Lebesgue integrable and

$$Y := \{\,(x_1,x_2)\in\mathbb{R}^2\,|\,x_1,x_2\geq 0\,,\;x_1+x_2\leq 1\,\}$$

Show that for any $\alpha_1\,,\;\alpha_2>0$

$$\int_Y f(x_1+x_2)x_1^{\alpha_1}x_2^{\alpha_2}\,d\lambda(x_1,x_2) = \left[\int_0^1 f(u)u^{\alpha_1+\alpha_2+1}\,d\lambda(u)\right]\cdot\left[\int_0^1 v^{\alpha_1}(1-v)^{\alpha_2}\,d\lambda(v)\right]$$

Reason: Transformationtheorem. (T)

Solution: We define $\phi\,:\,(0,1)^2\longrightarrow\mathbb{R}^2$ by $\phi(u,v)=(vu,(1-v)u)$. Now $\operatorname{im}\phi=\{\,(x_1,x_2)\in\mathbb{R}^2\,|\,x_1,x_2>0\,,\;x_1+x_2<1\,\}=Y^\circ$ the inner of Y and ϕ is bijective with $\phi^{-1}(x_1,x_2)=\left(x_1+x_2,\dfrac{x_1}{x_1+x_2}\right)$. We further have $\det D\phi(u,v)=-u$ and note that $Y-Y^\circ$ is a nullset with respect to $\lambda(Y)$. Hence

$$\begin{aligned}&\int_Y f(x_1+x_2)x_1^{\alpha_1}x_2^{\alpha_2}\,d\lambda(x_1,x_2)\\ &=\int_{Y^\circ} f(x_1+x_2)x_1^{\alpha_1}x_2^{\alpha_2}\,d\lambda(x_1,x_2)\\ &=\int_{(0,1)^2} f(vu+(1-v)u)\,(vu)^{\alpha_1}\,((1-v)u)^{\alpha_2}\,u\;d\lambda(u,v)\\ &=\int_{(0,1)^2} f(u)\,u^{\alpha_1+\alpha_2+1}\,v^{\alpha_1}(1-v)^{\alpha_2}\;d\lambda(u,v)\\ &\stackrel{Fubini}{=}\left[\int_0^1 f(u)u^{\alpha_1+\alpha_2+1}\,d\lambda(u)\right]\cdot\left[\int_0^1 v^{\alpha_1}(1-v)^{\alpha_2}\,d\lambda(v)\right]\end{aligned}$$

6. Finite Groups. **Reason:** Basics. (E)

(a) Let $U \subsetneq G$ be a proper subgroup of a finite group.
Show that $\bigcup_{g\in G} gUg^{-1} \subsetneq G$ is a proper subset.

(b) Let $G \neq \{1\}$ be a finite group which operates transitive on X which has at least two elements $|X| > 1$. Transitive means all elements of X can be reached by the group operation from a single $x \in X$. Show that there is a group element $g \in G$ such that $g.x \neq x$ for all $x \in X$.

Solution:

(a) If U is a normal subgroup, we are done. So let us assume that $U \subsetneq G$ isn't normal. The conjugation in G gives us an operation on the powerset $\mathcal{P}G$ of G by:

$$\begin{aligned} G \times \mathcal{P}G &\longrightarrow \mathcal{P}G \\ (g, M) &\longmapsto g.M := gMg^{-1} = \{\, gmg^{-1} \mid m \in M \,\} \end{aligned}$$

The normalizer $N_G(U) = \{\, gUg^{-1} \mid gUg^{-1} \subseteq U \,\}$ of U in G is the largest subgroup of G which contains U as a normal subgroup. By the above operation we get

$$|G| = |N_G(U)| \cdot |G.U|$$

and therefore

$$\begin{aligned} |\bigcup_{g\in G} gUg^{-1}| = |\bigcup_{H\in G.U} H| &\overset{(*)}{<} \bigcup_{H\in G.U} |H| = \bigcup_{H\in G.U} |U| \\ &= |G.U| \cdot |U| = \frac{|G|}{|N_G(U)|} \cdot |U| \leq \frac{|G|}{|U|} \cdot |U| = |G| \end{aligned}$$

The inequality in $(*)$ is proper, because $1 \in H$ for all H and the union contains at least two sets, since U isn't normal in G.

(b) For a given $x \in X$ we consider its stabilizer

$$\operatorname{Stab}_G(x) = \{\, g \in G \mid g.x = x \,\}$$

Since G operates transitive on X and $|X| > 1$ the stabilizer of x is a proper subgroup of G:

$$\operatorname{Stab}_G(x) = G \Longrightarrow g.x = x \neq y \in X$$

so either G would not operate transitive or X would not contain two different elements. Furthermore are all stabilizers $\operatorname{Stab}_G(y)$ conjugates of $\operatorname{Stab}_G(x)$: Let $g.x = y$. Then $ghg^{-1}.y = gh.x = g.x = y$, i.e. conjugation by g maps $\operatorname{Stab}_G(x)$ isomorph on $\operatorname{Stab}_G(y)$. By the previous part we have

$$\bigcup_{g\in G} g\operatorname{Stab}_G(x)g^{-1} = \bigcup_{y\in X} \operatorname{Stab}_G(y) \neq G$$

Hence there is a $g \in G$ with $g \notin \operatorname{Stab}_G(y)$ for all $y \in X$, which means g doesn't fix any element of X, which we had to prove.

7. Let

$$\begin{aligned} O_n(\mathbb{R}) &= \{\, A \in \mathbb{M}(n,\mathbb{R}) \mid \langle Av, Aw\rangle = \langle v,w\rangle \text{ for all } v,w \in \mathbb{R}^n \,\} \\ &= \{\, A \in \mathbb{M}(n,\mathbb{R}) \mid A^\tau A = AA^\tau = 1 \,\} \end{aligned}$$

be the orthogonal group of $n \times n$ matrices which operate per matrix multiplication on $\mathbb{R}^n$ $(n \in N)$.

(a) Determine the orbit of $x \in \mathbb{R}^n$ under $O_n(\mathbb{R})$.

(b) Determine the stabilizer $\operatorname{Stab}_x(O_n(\mathbb{R})) = \{\, A \in O_n(\mathbb{R}) \mid A.x = x \,\}$ of $x = (0,0,\ldots,1) \in \mathbb{R}^n$ in $O_n(\mathbb{R})$.

(c) Determine a bijection $\mathbb{S}^{n-1} \stackrel{1:1}{\longleftrightarrow} O_n(\mathbb{R})/O_{n-1}(\mathbb{R})$ between the unit sphere in $\mathbb{R}^n$ and the factor of two consecutive orthogonal groups.

Reason: For physicists. (E)

Solution: From the defining relation we get

$$\langle Av, Aw\rangle = (Av)^\tau(Aw) = v^\tau A^\tau Aw = v^\tau(A^\tau A)w = v^\tau w = \langle v,w\rangle$$

that every $A \in O_n(\mathbb{R})$ is isometric and $||Ax|| = ||x||$. Thus the orbit of $x \in O_n(\mathbb{R})$ is

$$O_n(\mathbb{R}).x \subseteq \{\, v \in O_n(\mathbb{R}) \mid ||v|| = ||x|| \,\} = ||x|| \cdot \mathbb{S}^{n-1}$$

For $n = 1$ we are done. The other inclusion " $\supseteq$ " also holds, as $O_n(\mathbb{R})$ operates transitive on $\mathbb{S}^{n-1}$: For $v \neq w$ in $||x|| \cdot \mathbb{S}^{n-1}$, i.e. $||v|| = ||x|| = ||w||$ we consider the plane $\operatorname{span}_{\mathbb{R}}\{\, v,w \,\}$, $n > 1$. With a rotation axis in its origin and perpendicular to the plane, we can rotate v into w by an appropriate element of $O_n(\mathbb{R})$ and hence v, w are in the same orbit, which is the orbit of x - just select $w = x$.

In order to get a fixed point $A.x = x$ with $x = (0, 0, \dots, 1)$ the matrix A has to be of the form

$$A = \begin{bmatrix} A' & 0 \\ 0 & 1 \end{bmatrix}$$

Since matrix multiplication is blockwise, we get $(A')^\tau A' = A'(A')^\tau = 1$ and thus $A' \in O_{n-1}(\mathbb{R})$ and

$$\operatorname{Stab}_{(0,0,\dots,1)}(O_n(\mathbb{R})) = O_{n-1}(\mathbb{R}) \subseteq O_n(\mathbb{R})$$

by the embedding $A' \longmapsto A$ as above.

Now we know, that $O_n(\mathbb{R})$ operates transitive on $\mathbb{S}^{n-1}$ and the quotient of two consecutive orthogonal groups is the quotient of $O_n(\mathbb{R})$ by the stabilizer $\operatorname{Stab}_{(0,0,\dots,1)}(O_n(\mathbb{R}))$. By the formula for orbits

$$G/\operatorname{Stab}_x(G) \cong G.x$$

we get

$$\begin{aligned} O_n(\mathbb{R})/O_{n-1}(\mathbb{R}) &\cong O_n(\mathbb{R})/\operatorname{Stab}_{(0,0,\dots,1)}(O_n(\mathbb{R})) \\ &\cong O_n(\mathbb{R}).(0,0,\dots,1) \\ &= ||(0,0,\dots,1)|| \cdot \mathbb{S}^{n-1} \\ &= \mathbb{S}^{n-1} \end{aligned}$$

8. We define an equivalence relation on the topological two-dimensional unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ by $x \sim y \iff x \in \{\pm y\}$ and the projection $q : \mathbb{S}^2 \longrightarrow \mathbb{S}^2/\sim$. Furthermore we consider the homeomorphism $\tau : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ defined by $\tau(x) = -x$. Note that for $A \subseteq \mathbb{S}^2$ we have $q^{-1}(q(A)) = A \cup \tau(A)$. Show that

   (a) q is open and closed.

   (b) $\mathbb{S}^2/\sim$ is compact, i.e. Hausdorff and covering compact.

   (c) Let $U_x = \{ y \in \mathbb{S}^2 : ||y - x|| < 1 \}$ be an open neighborhood of $x \in \mathbb{S}^2$. Show that $U_x \cap U_{-x} = \emptyset$, $U_{-x} = \tau(U_x)$, $q(U_x) = q(U_{-x})$ and $q|_{U_x}$ is injective. Conclude that q is a covering.

Reason: Standard Sphere. (E)

Solution:

(a) Let $O \subseteq \mathbb{S}^2$ be open. Then $O \cup \tau(O) = q^{-1}(q(O))$ is open and by definition of the quotient topology $q(O) \subseteq \mathbb{S}^2/\sim$ is open and so is q. Let $B \subseteq \mathbb{S}^2$ be closed. Then by the same argument, $q(B)$ is closed and so is q.

(b) $\mathbb{S}^2/\sim$ is covering-compact as $\mathbb{S}^2$ is covering-compact, and the continuous function q is surjective. Furthermore q is closed by the previous part, so $\mathbb{S}^2/\sim$ is Hausdorff because $\mathbb{S}^2$ is.

(c) We would have $2 = ||x-(-x)|| \leq ||x-y|| + ||y-(-x)|| < 2$, so $U_x \cap U_{-x} = \emptyset$. We also have $-y \in U_x$ if and only if $y \in U_{-x}$ and so $U_{-x} = \tau(U_x)$. From $q \circ \tau = q$ we thus have $q(U_x) = q(\tau(U_x)) = q(U_{-x})$.

Now let $y, y' \in U_x$ such that $q(y) = q(y')$. Then $y' = \pm y$. As from $y' = -y \in U_x$ we would get $y \in U_{-x} \cap U_x = \emptyset$ which is impossible. Hence $y' = y$ and the restriction of q on U_x is injective.

By the previous we have that $q : \mathbb{S}^2 \longrightarrow \mathbb{S}^2/\sim$ is a continuous, open and closed, surjection. So we may set $U_{q(x)} := q(U_x)$ as open neighborhood of an element in $\mathbb{S}^2/\sim$ with $q^{-1}(U_{q(x)}) = U_x \dot\cup U_{-x}$, a disjoint union of two open sets in $\mathbb{S}^2$. We also know that q is injective on U_x, hence maps U_x homeomorph onto $U_{q(x)}$ and equally with U_{-x}. Thus all properties of a covering map are fulfilled.

9. A function $|\,.\,| : \mathbb{F} \longrightarrow \mathbb{R}_{\geq 0}$ on a field $\mathbb{F}$ is called a value function if

$$|x| = 0 \Longleftrightarrow x = 0$$
$$|xy| = |x|\,|y|$$
$$|x+y| \leq |x| + |y|$$

It is called Archimedean, if for any two elements a, b ($a \neq 0$) there is a natural number n such that $|na| > |b|$. We consider the rational numbers. The usual absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

is Archimedean, whereas the trivial value

$$|x|_0 = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

is not.
Determine all non-trivial and non-Archimedean value functions on $\mathbb{Q}$.

Reason: Ostrowski's Theorem. (D)

Solution: Since $|\,.\,|$ is non-Archimedean, there are elements a, b with $|n| < \frac{|b|}{|a|}$ for all $n \in \mathbb{N}$. If $|n| > 1$ for a natural number, then $|n^k| = |n|^k$ goes to infinity and cannot be bounded. Thus $|n| \leq 1$ for all $n \in \mathbb{N}$. Let's assume $|n| = 1$ for all $n \in \mathbb{N}$. Then for $a = \frac{m}{n}$ we get $1 = |m| = |an| = |a|\,|n| = |a|$ and the value function is trivial. Thus there is a smallest n_0 by its natural ordering with $|n_0| < 1$. Let's assume $n_0 = ab$. This means that either $|a| < 1$ or $|b| < 1$ and by minimality of n_0 we have $n_0 = a$ or $n_0 = b$ and thus $n_0 =: p$ is prime.

We next show that $|a{+}b| \leq \max\{\,|a|, |b|\,\}$. This is equivalent to $|a{+}1| \leq \max\{\,|x|, 1\,\}$ which can be seen by division by b and $|1| = 1$.

$$\begin{aligned}|a+1|^m &= \left|\sum_{k=0}^{m} \binom{m}{k} a^k\right| \leq \sum_{k=0}^{m} \left|\binom{m}{k}\right|\,|a|^k \leq \sum_{k=0}^{m} |a|^k \\ &\leq (m+1)\max\{\,|a|^m, 1\,\}\end{aligned}$$

hence $|a+1| \leq \sqrt[m]{m+1}\,\max\{\,|a|, 1\,\}$. Since $\lim_{m\to\infty} \sqrt[m]{m+1} = 1$ we have $|a+1| \leq \max\{\,|a|, 1\,\}$.

Let $m = kp + r$ with $p \nmid m$ and $r \in \{\,1, \ldots, p-1\,\}$. By minimality of p we have $|r| = 1$, and $|kp| = |k|\,|p| \leq |p| < 1$, so

$$|m| = |kp + r| \leq \max\{\,|kp|, |r|\,\} = 1$$

Now let $|a| < |b|$. Then we get

$$|a| < |b| = |(a{+}b){-}a| \leq \max\{\,|a{+}b|, |a|\,\} = |a{+}b| \leq \max\{\,|a|, |b|\,\} = |b|$$

and $|a + b| = \max\{\,|a|, |b|\,\}$.

So any natural number m which is coprime to p has $|m| = 1$ and all others are of the form $m = p^r m'$ with $|m| = |p|^r |m'| = |p|^r$. If we set $\alpha := \dfrac{\log p^{-1}}{|\log p|}$ we get $|p|^\alpha = p^{-1}$ and $|m|^\alpha = |p|^{r\alpha} = p^{-r}$.

For $m = p^r \cdot m'$, $n = p^s \cdot n'$ with $(m', p) = (n', p) = 1$ we therefore have

$$\left|\frac{m}{n}\right| = \begin{cases} 0 & \text{if } m = 0 \\ p^{-r+s} & \text{if } m \neq 0 \end{cases}$$

b which is the p-adic absolute value (norm) of $\mathbb{Q}$.

The topological completion of $\mathbb{Q}$ with respect to the p-adic norm is called the field of p-adic numbers $\mathbb{Q}_p$. It is a field with prime field $\mathbb{Q}$ and thus is of characteristic zero. Its algebraic closure is of infinite degree. So $\mathbb{Q}_p$ has infinitely many inequivalent algebraic extensions.

10. For a set X let

$$\mathcal{B}(X) = \{\, f \,:\, X \longrightarrow \mathbb{R} \,:\, \sup_{x\in X}\{\,|f(x)|\,\} =: ||f||_\infty < \infty\,\}$$

be the space of all bounded functions on X. We define a metric on $\mathcal{B}(X)$ by $d(f,g) = ||f-g||_\infty$.

(a) Show that $(\mathcal{B}(X)\,,\, d)$ is complete.

(b) If (X,d) is a metric space and $a \in X$. Prove that the function

$$\phi_a \,:\, X \longrightarrow \mathcal{B}(X)\,,\, \phi_a(x) = d(x,.) - d(a,.)$$

is an isometry of X in $\mathcal{B}(X)$.

(c) Show that the closure of $\operatorname{im}(\phi_a)$ is a completion of $X \sim \phi_a(X)$.

Reason: Functional Analysis Basics. (E)

Solution:

(a) Let (f_n) be a Cauchy sequence in $\mathcal{B}(X)$, that is for any $\varepsilon > 0$ there is an $N_\varepsilon \in \mathbb{N}$ such that for all $n,m > N_\varepsilon$

$$||f_n - f_m||_\infty < \varepsilon$$

Thus we also have $|f_n(x) - f_m(x)| < \varepsilon$ and $(f_n(x)) \subseteq \mathbb{R}$ is a Cauchy sequence, which converges to $\lim_{n\to\infty} f_n(x) =: f(x)$. For $\varepsilon = 1$ and $N := N_1 + 1$ we thus have

$$|f_n(x)| < |f_{N+1}(x)| + 1 \text{ for all } n > N\ ,\ x \in X$$

and thus $|f(x)| < |f_{N+1}(x)| + 1 < ||f_{N+1}||_\infty + 1$ which shows $f \in \mathcal{B}(X)$. Now for $m \to \infty$ we get from $|f_n(x) - f_m(x)| < \varepsilon$ that $|f_n(x) - f(x)| \leq \varepsilon$ and $||f_n - f||_\infty \leq \varepsilon$ so (f_n) converges to f in $(\mathcal{B}(X)\,,\, ||\,.\,||_\infty)$.

(b) Since $||\phi_a(x)||_\infty = \sup_{y\in X} |d(x,y) - d(a,y)| \leq d(x,a) < \infty$ we have $\phi_a(x) \in \mathcal{B}(X)$.
From the triangle inequality we get $|d(x,z) - d(y,z)| \leq d(x,y)$

where equality holds for $z = x$, and thus $\sup_{z\in X} |d(x,z)-d(y,z)| = d(x,y)$. Therefore is

$$\begin{aligned} ||\phi_a(x) - \phi_a(y)||_\infty &= \sup_{z\in X} |\phi_a(x)(z) - \phi_a(y)(z)| \\ &= \sup_{z\in X} |d(x,z) - d(a,z) - d(y,z) + d(a,z)| \\ &= \sup_{z\in X} |d(x,z) - d(y,z)| \\ &= d(x,y) \end{aligned}$$

and ϕ_a is an isometry for any (fixed) $a \in X$.

(c) A subset $N \subseteq M$ of a complete metric space is complete if and only if it is closed. So $\overline{N}$ is a completion of N .

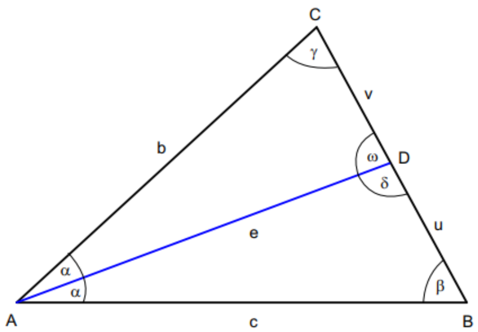

Figure 1: Triangle with bisector

Part IV

July, 2018

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1. The angle bisector in a given triangle $\bigtriangleup ABC$ of the angle $\alpha = \sphericalangle BAC$ at A intersects the side $\overline{BC}$ at the point D. We have the information:

$$\overline{BD} \cdot \overline{CD} = \overline{AD}^{\,2} \tag{14}$$
$$\sphericalangle ADB = 45° \tag{15}$$

   (a) Determine the inner angles of $\bigtriangleup ABC$

   (b) Determine the precise ratio at which D divides $\overline{BC}$

   Reason: Triangle practice.

   Solution: We are given $\delta = \sphericalangle ADB = 45°$ and $\omega = \sphericalangle ADC = 135°$, and asked for

$$\alpha' = 2\alpha = \sphericalangle BAC \ , \ \beta = \sphericalangle ABC = 135° - \alpha \ , \ \gamma = \sphericalangle BCA = 45° - \alpha$$

   The law of sines for $\bigtriangleup ADB$ and $\bigtriangleup ADC$ reads

$$\frac{u}{\sin \alpha} = \frac{e}{\sin \beta} \ , \ \frac{v}{\sin \alpha} = \frac{e}{\sin \gamma}$$

   and we get with the condition $u \cdot v = e^2$

$$e^2 = \frac{u \cdot \sin \beta}{\sin \alpha} \cdot \frac{v \cdot \sin \gamma}{\sin \alpha} = e^2 \cdot \frac{\sin \beta \sin \gamma}{\sin^2 \alpha}$$

or

$$\sin^2\alpha = \sin(135^\circ - \alpha)\sin(45^\circ - \alpha) = \frac{1}{2}(\cos^2\alpha - \sin^2\alpha) = \frac{1}{2} - \sin^2\alpha$$

and $\alpha = \arcsin\frac{1}{2} = 30^\circ$, $\alpha' = 60^\circ$, $\beta = 105^\circ$, $\gamma = 15^\circ$.

Next we consider the law of cosine for $\triangle ABD$ and $\triangle ADC$ and get with our condition $uv = e^2$

$$\begin{aligned} c^2 &= u^2 + e^2 - \sqrt{2}ue = u(u + v - \sqrt{2uv}) \\ b^2 &= v^2 + e^2 + \sqrt{2}ve = v(u + v + \sqrt{2uv}) \end{aligned}$$

and the theorem of angle bisectors says $cv = bu$. Thus

$$c^2v^2 = uv^2(u + v - \sqrt{2uv}) = u^2v(u + v + \sqrt{2uv}) = b^2u^2$$

from which we get with $r = \frac{u}{v}$

$$\begin{aligned} v(u + v - \sqrt{2uv}) &= u(u + v + \sqrt{2uv}) \\ r^{-1} - \sqrt{2r^{-1}} &= r + \sqrt{2r} \\ 1 - \sqrt{2r} &= r^2 + r\sqrt{2r} \\ (1 - r^2)^2 &= 2r(1 + r)^2 \\ 0 &= r^2 - 4r + 1 \\ \frac{u}{v} &= r \in \left\{ 2 + \sqrt{3} \, , \frac{1}{2 + \sqrt{3}} \right\} \end{aligned}$$

From the first part of the question we have

$$\frac{u}{v} = \frac{\sin\gamma}{\sin\beta} = \frac{\sin 15^\circ}{\sin 105^\circ} < 1, \text{ i.e. } \frac{u}{v} = \frac{1}{2 + \sqrt{3}}$$

2. At the cash desk of a shopping center five friends (Diana, Ike, Jessica, Stan, Valery) are standing in a row. They are all different in age (26, 27, 30, 33 and 35 years) and would like to buy all different tops (shirt, polo shirt, pullover, sweatshirt and T-shirt) for themselves. The tops are all different colors (blue, yellow, green, red and black) and different sizes (XS, S, M, L and XL).

   Find out who is where, how old and what top to buy in which color and size. The positions in the queue can be seen from the cashier, i.e. "front" or "the first person" is right at the cash register. There are no other people in the queue and the cashier is to be ignored.

(a) 1. Diana, who wants to buy a top in size XL, stands further ahead than the person who wants to buy a black top.

(b) 2. Jessica stands in front of the person who wants to buy a polo shirt.

(c) 3. The second person in the queue wants to buy a yellow top.

(d) 4. The t-shirt is not red.

(e) 5. Stan wants to buy a sweatshirt. The person standing in front of him is older than the person standing directly behind him.

(f) 6. Ike needs a top in size L.

(g) 7. The last person in the queue is 30 years old.

(h) 8. The oldest person wants to buy the top in the smallest size.

(i) 9. The person standing directly behind Valery wants to buy a red top that is larger than size S.

(j) 10. The youngest person wants to buy a yellow top.

(k) 11. Jessica wants to buy a shirt.

(l) 12. The third person in the queue wants to buy a top in size M.

(m) 13. The polo shirt is red, yellow or green.

Reason: Logic.

Solution:

- 3-7-12: (25),(42),(53)
- 10: +(22)
- 1-2-5-6-9: +(15),(55)
- 1-2-5-9: +(12),(32)
- 1-2-9: +(11),(51)
- 8: +(24),(52),(54)
- 5: +(21),(23)
- 9: +(14),(45)
- 2-11: +(13),(33),(34)
- 4: +(31),(35)
- 13: +(44)
- 1: +(41),(43)

Position	1	2	3	4	5
Person	Dana	Sören	Jessica	Valerie	Ingo
Alter	33	26	27	35	30
Oberteil	T-Shirt	Sweatshirt	Hemd	Poloshirt	Pullover
Farbe	Blau	Gelb	Schwarz	Grün	Rot
Größe	XL	S	M	XS	L

Figure 2: At the Cash Desk

Diana, Ike, Jessica, Stan, Valery

shirt, polo shirt, pullover, sweatshirt and T-shirt

3. Compute the arc length $\mathcal{L}$ of the cycloid

$$\gamma \,:\, \mathbb{R} \longrightarrow \mathbb{R}^2\,,\, \gamma(t) = (t - \sin(t), 1 - \cos(t))$$

between two neighboring singularities.

Reason: Training for physicists.

Solution: The singularities $\gamma'(t) = 0$ are $2\pi k$ $(k \in \mathbb{Z})$.

$$\begin{aligned}
\mathcal{L} &= \int_{2\pi k}^{2\pi(k+1)} ||\dot{\gamma}(t)||\,dt = \int_{2\pi k}^{2\pi(k+1)} \left|\left|\begin{pmatrix} 1-\cos(t) \\ \sin(t) \end{pmatrix}\right|\right| \,dt \\
&= \sqrt{2}\int_{2\pi k}^{2\pi(k+1)} \sqrt{1-\cos(t)}\,dt \\
&= \sqrt{2}\int_{2\pi k}^{2\pi(k+1)} \sqrt{1-\left(\cos^2\left(\frac{t}{2}\right) - \sin^2\left(\frac{t}{2}\right)\right)}\,dt \\
&= \sqrt{2}\int_{2\pi}^{2\pi(k+1)} \sqrt{2\sin^2\left(\frac{t}{2}\right)}\,dt = 2\int_{2\pi}^{2\pi(k+1)} \left|\sin\left(\frac{t}{2}\right)\right|\,dt \\
&= 4\int_0^{\pi} |\sin(x)|\,dx = 4\,(-\cos(\pi) + \cos(0)) \\
&= 8
\end{aligned}$$

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1. Let's consider complex functions in one variable and especially the involutions

$$\mathcal{I} = \{ z \overset{p}{\mapsto} z \ , \ z \overset{q}{\mapsto} -z \ , \ z \overset{r}{\mapsto} z^{-1} \ , \ z \overset{s}{\mapsto} -z^{-1} \}$$

We also consider the two functions

$$\mathcal{J} = \{ z \overset{u}{\longmapsto} \frac{1}{2}(-1+i\sqrt{3})z \ , \ z \overset{v}{\longmapsto} -\frac{1}{2}(1+i\sqrt{3})z \}$$

and the set $\mathcal{F}$ of functions which we get, if we combine any of them: $\mathcal{F} = \langle \mathcal{I}, \mathcal{J} \rangle$ by consecutive applications. We now define for $\mathcal{K} \in \{\mathcal{I}, \mathcal{J}\}$ a relation on $\mathcal{F}$ by

$$f(z) \sim_{\mathcal{K}} g(z) \ :\Longleftrightarrow \ (\forall\, h_1 \in \mathcal{K})\,(\exists\, h_2 \in \mathcal{K}) \ : f(h_1(z)) = g(h_2(z))$$

(a) Show that $\sim_{\mathcal{K}}$ defines an equivalence relation.

(b) Show that $\mathcal{F}/\sim_{\mathcal{J}}$ admits a group structure on its equivalence classes by consecutive application.

(c) Show that $\mathcal{F}/\sim_{\mathcal{I}}$ does not admit a group structure on its equivalence classes by consecutive applications.

Reason: Normal subgroups versus ordinary subgroups.

Solution: The relation is that of subgroups in a group and we have

$$D_6 \cong \mathcal{F} \cong \mathcal{I} \ltimes \mathcal{J} \cong V_4 \ltimes \mathbb{Z}_3 \cong D_3 \times \mathbb{Z}_2$$

with $\varphi \,:\, V_4 \longrightarrow \operatorname{Aut}(\mathbb{Z}_3)$, $\varphi(t)(w)(z) := (twt)(z) = t(w(t(z)))$. As $\mathcal{I} \ntriangleleft \mathcal{F}$ isn't a normal subgroup, $\mathcal{F}/\sim_{\mathcal{I}}$ doesn't carry a group structure for the same reason as $\sim_{\mathcal{J}}$ does.

$$D_6 = \langle (uq), r \,|\, (uq)^6 = r^2 = r(uq)r(uq) = 1 \rangle$$

2. We consider the vector field $X \,:\, \mathbb{R} \longrightarrow \mathbb{R}^2$ given by $X(p) := \left(p, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$.

(a) Compute the derivative $d\phi \,:\, T\mathbb{R}^2 \longrightarrow T\mathbb{R}^3$ of the stereographic projection to the north pole, i.e. plane to sphere with $\phi(0,0) = (0,0,-1)$, and describe the tangent bundle $T\mathbb{S}^2$ of $\mathbb{S}^2$. Show that position vectors and tangent vectors are orthogonal.

(b) Compute the vector field $d\phi(X)$ on $\mathbb{S}^2$. How is it related to the curves $\gamma(t) = \phi(t, y_0)$?

(c) Is $d\phi(X)$ a continuous vector field on $\mathbb{S}^2$ without zeros?

Reason: Training for physicists.

Solution:

(a) The stereographic projection to the north pole is given by

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\phi(x,y) = \frac{1}{x^2+y^2+1}\begin{bmatrix} 2x \\ 2y \\ x^2+y^2-1 \end{bmatrix}$$

from which we get

$$d_{(x,y)}\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}\right) = \frac{1}{(x^2+y^2+1)^2}\begin{bmatrix} 2(1-x^2+y^2) & -4xy \\ -4xy & 2(1+x^2-y^2) \\ 4x & 4y \end{bmatrix}$$

and $d\phi : T\mathbb{R}^2 \longrightarrow T\mathbb{R}^3$, $(p,v) \longmapsto (\phi(p), d_p\phi(v))$. The tangent bundle $T\mathbb{S}^2$ of $\mathbb{S}^2$ is given by the image of the $\phi-$parameterized set $\phi(\mathbb{S}^2)$ under $d\phi$. As expected is

$$\phi(p) \perp d_p(\phi)(v)$$

$$\begin{aligned}
&(1+r^2+s^2)^3 \cdot \left\langle \phi(r,s), d_{(r,s)}(\phi)(v_1,v_2) \right\rangle \\
&= \left\langle \begin{bmatrix} 2r \\ 2s \\ r^2+s^2-1 \end{bmatrix}, \begin{bmatrix} 2(1-r^2+s^2) & -4rs \\ -4rs & 2(1+r^2-s^2) \\ 4r & 4s \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\rangle \\
&= \left\langle \begin{bmatrix} 2r \\ 2s \\ r^2+s^2-1 \end{bmatrix}, \begin{bmatrix} 2(1-r^2+s^2)v_1 - 4rsv_2 \\ -4rsv_1 + 2(1+r^2-s^2)v_2 \\ 4rv_1+4sv_2 \end{bmatrix} \right\rangle \\
&= 4rv_1 - 4r^3v_1 + 4rs^2v_1 - 8r^2sv_2 - 8rs^2v_1 + 4sv_2 + 4r^2sv_2 \\
&- 4s^3v_2 + 4r^3v_1 + 4r^2sv_2 + 4rs^2v_1 + 4s^3v_2 - 4rv_1 - 4sv_2 \\
&= 0
\end{aligned}$$

(b) Since $X(p) = (1,0)$ we have

$$d_p\phi(X) = \left.\frac{\partial \phi}{\partial x}\right|_p (X) = \frac{1}{(r^2+s^2+1)^2}\begin{bmatrix} 2(1-r^2+s^2) \\ -4rs \\ 4r \end{bmatrix}$$

at position $p = (r, s)$, the unit tangent vector field along the curve $\gamma : \mathbb{R} \longrightarrow \mathbb{S}^2$, $\gamma(r) = \phi(r, s) = (r^2 + s^2 + 1)^{-1}(2r, 2s, r^2 + s^2 - 1)$ for a certain $s \in \mathbb{R}$.

(c) The vector field $d\phi(X) = (p, d_p\phi(X))$ is defined on the image of ϕ, namely $\mathbb{S}^2 - \{N\}$ where the pole N is missing. It's the point at infinity. $d\phi(X)$ is continuous on its domain without zeros, but it can be continuously extended over the north pole. However, here we get

$$\lim_{|p|\to\infty} d_p\phi(X) = 0$$

so this continuous extension has a zero at the pole.

3. (On the occasion of the centenary of Emmy Noether's theorem.)

The action on a classical particle is the integral of an orbit $\gamma : t \to \gamma(t)$

$$S(\gamma) = S(x(t)) = \int \mathcal{L}(t, x, \dot{x})\, dt$$

over the Lagrange function $\mathcal{L}$, which describes the system considered. Now we consider smooth coordinate transformations

$$x \longmapsto x^* := x + \varepsilon\psi(t, x, \dot{x}) + O(\varepsilon^2)$$
$$t \longmapsto t^* := t + \varepsilon\varphi(t, x, \dot{x}) + O(\varepsilon^2)$$

and we compare

$$S = S(x(t)) = \int \mathcal{L}(t, x, \dot{x})\, dt \text{ and } S^* = S(x^*(t^*)) = \int \mathcal{L}(t^*, x^*, \dot{x}^*)\, dt^*$$

Since the functional S determines the law of motion of the particle,

$$S = S^*$$

means, that the action on this particle is unchanged, i.e. invariant under these transformations, and especially

$$\frac{\partial S}{\partial \varepsilon} = 0 \quad \text{resp.} \quad \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(\mathcal{L}\left(t^*, x^*, \dot{x}^*\right) \cdot \frac{dt^*}{dt}\right) = 0 \qquad (16)$$

Emmy Noether showed exactly hundred years ago, that under these circumstances (invariance), there is a conserved quantity Q. Q is called the Noether charge.

$$S = S^* \Longrightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(\mathcal{L}\left(t^*, x^*, \dot{x}^*\right) \cdot \frac{dt^*}{dt}\right) = 0 \Longrightarrow \frac{d}{dt}Q(t, x, \dot{x}) = 0$$

with

$$Q = Q(t, x, \dot{x}) := \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \psi_i + \left(\mathcal{L} - \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{x}_i} \dot{x}_i \right) \varphi = \text{ constant}$$

The general way to proceed is:

(a) Determine the functions ψ, φ, i.e. the transformations, which are considered.
(b) Check the symmetry by equation (16).
(c) If the symmetry condition holds, then compute the conservation quantity Q with $\mathcal{L}, \psi, \varphi$.

Example: Given a particle of mass m in the potential $U(\vec{r}) = \dfrac{U_0}{\vec{r}^{\,2}}$ with a constant U_0. At time $t = 0$ the particle is at $\vec{r}_0$ with velocity $\dot{\vec{r}}_0$.

Hint: The Lagrange function with $\vec{r} = (x, y, z, t) = (x_1, x_2, x_3, t)$ of this problem is

$$\mathcal{L} = T - U = \frac{m}{2} \dot{\vec{r}}^{\,2} - \frac{U_0}{\vec{r}^{\,2}}$$

(a) Give a reason why the energy of the particle is conserved, and what is its energy?
(b) Consider the following transformations with infinitesimal ε

$$\vec{r} \longmapsto \vec{r}^{\,*} = (1 + \varepsilon)\, \vec{r}\ ,\ t \longmapsto t^* = (1 + \varepsilon)^2\, t$$

and verify the condition (16) to E. Noether's theorem.
(c) Compute the corresponding Noether charge Q and evaluate Q for $t = 0$.

Reason: For physicists. Centenary.

Solution:

(a) i. Energy is *homogeneous in time*, so we chose $\psi_i = 0, \varphi = 1$
ii. and check equation (16) by

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \left(\mathcal{L}^* \cdot \frac{d}{dt}(t + \varepsilon) \right) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (\mathcal{L}^* \cdot 1) = 0$$

since $\mathcal{L}^*$ doesn't depend on t^* and thus not on ε, and calculate

iii. the Noether charge as

$$\begin{aligned} Q(t,x,\dot{x}) &= \mathcal{L} - \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\, \dot{x}_i \\ &= T - U - \frac{m}{2}\left(\frac{\partial}{\partial \dot{x}_i}\left(\sum_{i=1}^{3} \dot{x}_i^2\right)\dot{x}_i\right) \\ &= \frac{m}{2}\,\dot{\vec{r}}^2 - U - m\,\dot{\vec{r}}^2 \\ &= -T - U \\ &= -E \\ &= -\frac{m}{2}\,\dot{\vec{r}}^2 - \frac{U}{\vec{r}^2} \\ &= -\frac{m}{2}\,\dot{\vec{r}}_0^2 - \frac{U}{\vec{r}_0^2} \end{aligned}$$

by time invariance.

(b) $\dot{\vec{r}}^* = \dfrac{d\vec{r}^*}{dt^*} = \dfrac{(1+\varepsilon)\,d\vec{r}}{(1+\varepsilon)^2\,dt} = \dfrac{1}{1+\varepsilon}\,\dot{\vec{r}}$ and thus $\mathcal{L}^* = \dfrac{1}{(1+\varepsilon)^2}\,\mathcal{L}$, i.e.

$$\begin{aligned} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left(\mathcal{L}\left(t^*,x^*,\dot{x}^*\right)\cdot \frac{dt^*}{dt}\right) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{L}^*\,\frac{dt^*}{dt} \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \frac{\mathcal{L}}{(1+\varepsilon)^2}\cdot(1+\varepsilon)^2 \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{L} \\ &= 0 \end{aligned}$$

and the condition (16) of Noether's theorem holds.

(c) For the given transformations we have

$$\begin{aligned} x &\longmapsto x^* = (1+\varepsilon)x && \Longrightarrow \quad \psi_x = x \\ y &\longmapsto y^* = (1+\varepsilon)y && \Longrightarrow \quad \psi_y = y \\ z &\longmapsto z^* = (1+\varepsilon)z && \Longrightarrow \quad \psi_z = z \\ t &\longmapsto t^* = (1+2\varepsilon)t && \Longrightarrow \quad \varphi = 2t \end{aligned}$$

and so the Noether charge is given by

$$\begin{aligned}
Q(t,x,\dot{x}) &= \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\,\psi_i + \left(\mathcal{L} - \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{x}_i}\,\dot{x}_i\right)\varphi \\
&= \sum_{i=1}^{3} \frac{\partial}{\partial \dot{x}_i}\left(\frac{m}{2}\,\dot{\vec{r}}^{2} - \frac{U_0}{\vec{r}^{2}}\right)\,\psi_i + \\
&+ \left(\frac{m}{2}\,\dot{\vec{r}}^{2} - \frac{U_0}{\vec{r}^{2}} - \sum_{i=1}^{3} \frac{\partial}{\partial \dot{x}_i}\,\left(\frac{m}{2}\,\dot{\vec{r}}^{2} - \frac{U_0}{\vec{r}^{2}}\right)\dot{x}_i\right)\varphi \\
&= m(\dot{x}x + \dot{y}y + \dot{z}z) + \\
&+ \left(\frac{m}{2}\dot{\vec{r}}^{2} - \frac{U_0}{\vec{r}^{2}} - m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right)2t \\
&= m\,\dot{\vec{r}}\,\vec{r} + \left(-\frac{m}{2}\dot{\vec{r}}^{2} - \frac{U_0}{\vec{r}^{2}}\right)2t \\
&= m\,\dot{\vec{r}}\,\vec{r} - (T+U)2t \\
&= m\,\dot{\vec{r}}\,\vec{r} - 2Et \\
&\overset{t=0}{=} m\,\dot{\vec{r}}_0\,\vec{r}_0
\end{aligned}$$

which shows that invariance under different transformations result in different conversation quantities.

4. Given the Heisenberg algebra

$$\mathcal{H} = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \right\} = \langle X, Y, Z \,:\, [X,Y] = Z \rangle$$

and

$$\mathfrak{A}(\mathcal{H}) = \{\, \alpha \,:\, \mathcal{H} \longrightarrow \mathcal{H} \,:\, [\alpha(X), Y] = [\alpha(Y), X]\, \forall\, X, Y \in \mathcal{H} \,\}$$

Since $\mathfrak{A}(\mathcal{H})$ is a Lie algebra and

$$[X, \alpha] = [\operatorname{ad}(X), \alpha] = \alpha(X) \circ \alpha - \alpha \circ \operatorname{ad(X)}$$

a Lie multiplication, we can define

$$\begin{aligned}
\mathcal{H}_0 &:= \mathcal{H} \\
\mathcal{H}_{n+1} &:= \mathcal{H}_n \ltimes \mathfrak{A}(\mathcal{H}_{\mathfrak{n}})
\end{aligned}$$

and get a series of subalgebras

$$\mathcal{H}_0 \leq \mathcal{H}_1 \leq \mathcal{H}_2 \leq \ldots$$

Show that

(a) $\mathfrak{sl}(2) < \mathcal{H}_n$ is a proper subalgebra for all $n \geq 1$

(b) $\dim \mathcal{H}_n \geq 3 \cdot (2^{n+1} - 1)$ for all $n \geq 0$, i.e. the series is infinite and doesn't get stationary

As a counterexample, if we started with $\mathcal{H} = \mathfrak{su}(2)$ or $\mathfrak{su}(3)$ we would get $\mathcal{H}_n = \mathcal{H}_0$ and we were stationary right from the start, which can easily be seen by solving the corresponding system of linear equations.

Reason: Linear algebra.

Solution: To show that $\mathfrak{sl}(2)$ is a subalgebra of all $\mathcal{H}_n$ it is sufficient to show that

$$\alpha = \begin{bmatrix} a & b & 0 \\ c & -a & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{A}(\mathcal{H})$$

which is true, because

$$\begin{aligned} &[\alpha(xX + yY + zZ)\ ,\ x'X + y'Y + z'Z] \\ &= [(ax + by)X + (cx - ay)Y\ ,\ x'X + y'Y + z'Z] \\ &= (axy' + ayx' + byy' - cxx')Z \\ &= [\alpha(x'X + y'Y + z'Z)\ ,\ xX + yY + zZ] \end{aligned}$$

In general we have

$$\mathfrak{A}(\mathcal{H}_0) = \left\{ \begin{bmatrix} a & b & 0 \\ c & -a & 0 \\ x & y & z \end{bmatrix} \right\}$$

so it is a proper subalgebra for $\mathfrak{A}(\mathcal{H}_0)$ and for $n > 0$ by the following.

Next we show, that Z is a central element in all $\mathcal{H}_n$. This is clear for $n = 0$, so we assume that $Z \in \mathfrak{Z}(\mathcal{H}_n)$ is in the center of $\mathcal{H}_n$. Then we have for all $\alpha \in \mathfrak{A}(\mathcal{H}_n)$ and all $H \in \mathcal{H}_n$

$$[Z, \alpha](H) = [Z, \alpha(H)] - \alpha([Z, H]) = 0$$

and $Z \in \mathfrak{Z}(\mathcal{H}_n \ltimes \mathfrak{A}(\mathcal{H}_n)) = \mathfrak{Z}(\mathcal{H}_{n+1})$. Let $\{ h_1, \ldots, h_m \}$ be a basis for $\mathcal{H}_n$, $n \geq 0$. Then

$$\alpha_i(a_1 h_1 + \ldots + a_m h_m) := a_i Z$$

define m linear independent transformations in $\mathfrak{A}(\mathcal{H}_n)$. For $m = 0$ we have $\dim \mathcal{H}_0 = \dim \mathcal{H} = 3 = 3 \cdot (2^1 - 1)$ and by induction

$$\begin{aligned}\dim \mathcal{H}_n &= \dim \mathcal{H}_{n-1} + \dim \mathfrak{A}(\mathcal{H}_{n-1}) \\ &\geq \dim \mathcal{H}_{n-1} + \dim \mathcal{H}_{n-1} + \dim \mathfrak{sl}(2) \\ &\geq 2 \cdot 3 \cdot (2^n - 1) + 3 \\ &= 3 \cdot (2^{n+1} - 1)\end{aligned}$$

because we have a copy of $\mathfrak{sl}(2)$ and a projection of every basis element on the central element Z.

5. A covering space $\tilde{X}$ of X is a topological space together with a continuous surjective map $p : \tilde{X} \longrightarrow X$, such that for every $x \in X$ there is an open neighborhood $U \subseteq X$ of x, such that $p^{-1}(U) \subseteq \tilde{X}$ is a union of pairwise disjoint open sets V_ι each of which is homeomorphically mapped onto U by p. A deck transformation with respect to p is a homeomorphism $h : \tilde{X} \longrightarrow \tilde{X}$ with $p \circ h = p$. Let $\mathcal{D}(p)$ be the set of all deck transformations with respect to p.

   (a) Show that $\mathcal{D}(p)$ is a group.

   (b) If $\tilde{X}$ is a connected Hausdorff space and $h \in \mathcal{D}(p)$ with $h(\tilde{x}) = \tilde{x}$ for some point $\tilde{x} \in \tilde{X}$. then $h = \operatorname{id}_{\tilde{X}}$.

   Reason: Coverings in set topology.

   Solution: (a) The homeomorphisms $\mathcal{H}(\tilde{X})$ build a group by successively applying the transformations. To an element $h : \tilde{X} \longrightarrow \tilde{X}$ we have the inverse $h^{-1} : \tilde{X} \longrightarrow \tilde{X}$ and the neutral element $\operatorname{id}_{\tilde{X}}$. Now the Deck transformations $\mathcal{D}(p) \leq \mathcal{H}(\tilde{X})$ is a subgroup, because

   - With $\operatorname{id}_{\tilde{X}} \in \mathcal{H}(\tilde{X})$ and $p(\operatorname{id}_{\tilde{X}}(\tilde{x})) = p(\tilde{x})$, we have $\operatorname{id}_{\tilde{X}} \in \mathcal{D}(p)$.
   - For $h, h' \in \mathcal{D}(p)$ we have $h \circ h' \in \tilde{\mathcal{X}}$ and

     $$p \circ (h^{-1} \circ h') = (p \circ h) \circ (h^{-1} \circ h') = p \circ (h \circ h^{-1}) \circ h' = p \circ h' = p$$

     and $D(p)$ is closed under inversion and multiplication.

   (b) Let $A := \{ \tilde{x} \in \tilde{X} : h(\tilde{x}) = \tilde{x} \} \neq \emptyset$. We have to show, that $A = \tilde{X}$ or equivalently due to connectedness, that A is open as well as closed.

   - If $h, h' : X \to Y$ are continuous functions and Y is a Hausdorff space, then $\{ x \in X : h(x) = h'(x) \}$ is closed. Since we can write $A = \{ \tilde{x} \in \tilde{X} : h(\tilde{x}) = \operatorname{id}_{\tilde{X}}(\tilde{x}) \}$ it is a closed set.

- Let $\tilde{x}_0 \in A$ and $x_0 := p(\tilde{x}_0)$. As p is a covering, there is an open neighborhood $x_0 \in U \subseteq X$ with equally many points in all fibers $p^{-1}(x)$, $x \in U$. We thus have $p^{-1}(U) = \coprod_{\iota \in I} V_\iota$ for a suited family $(V_\iota)_{\iota \in I}$ of open subsets of $\tilde{X}$, and the restrictions $p_\iota := p|_{V_\iota} : V_\iota \longrightarrow U$ are all homeomorphisms. Let $\iota_0 \in I$ be the index with $\tilde{x}_0 \in V_{\iota_0}$. Then $V := h^{-1}(V_{\iota_0}) \cap V_{\iota_0}$ is an open neighborhood of $\tilde{x}_0$ in $\tilde{X}$ since h is continuous. Since $\tilde{x}_0 \in A$, we have $\tilde{x}_0 = h(\tilde{x}_0) \in V_{\iota_0}$ and thus $\tilde{x}_0 \in h^{-1}(V_{\iota_0}) \cap V_{\iota_0} = V$. If $V \subseteq A$, then A is a neighborhood in $\tilde{X}$ of each of its points and therewith open.
- $V \subseteq A$.
  By definition of V we have $V \subseteq V_{\iota_0}$ and $h(V) \subseteq V_{\iota_0}$ and we can restrict h to a function $h_{\iota_0} : V \longrightarrow V_{\iota_0}$. From $p \circ h = p$ and $V \subseteq V_{\iota_0}$ we get for all $\tilde{x} \in V$

  $$p_{\iota_0}(\tilde{x}) = p(\tilde{x}) = p(h(\tilde{x})) = p(h_{\iota_0}(\tilde{x})) = p_{\iota_0}(h_{\iota_0}(\tilde{x}))$$

  Since $p_{\iota_0} : V_{\iota_0} \longrightarrow U$ is a homeomorphism and in particular injective, we have $h(\tilde{x}) = h_{\iota_0}(\tilde{x}) = \tilde{x}$ for all $\tilde{x} \in V$ and thus $V \subseteq A$.

Part V
June, 2018

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1. The general solution to $y^{(4)}(x) + 4y(x) = 0$ is given by

$$y(x) = \alpha e^{-x}\cos(x) + \beta e^{-x}\sin(x) + \gamma e^{x}\sin(x) + \delta e^{x}\cos(x)$$

   (a) How do the initial conditions at $x = 0$ have to be chosen in order to get $y(x) = e^{-x}\cos x$ as unique solution?

   (b) Which function do we get for the initial conditions $y'(0) = 1\ ,\ y''(0) = 0\ ,\ y'''(0) = 0\ ,\ y^{(4)}(0) = 0\,?$

   Reason: Initial conditions are essential.

   Solution:

$$\begin{aligned} y'(x) &= -\alpha e^{-x}(\sin(x) + \cos(x)) + \beta e^{-x}(\cos(x) - sin(x)) + \\ &\quad + \gamma e^{x}(\sin(x) + \cos(x)) + \delta e^{x}(\cos(x) - \sin(x)) \\ y''(x) &= 2\alpha e^{-x}\sin(x) - 2\beta e^{-x}\cos(x) + 2\gamma e^{x}\cos(x) - 2\delta e^{x}\sin(x) \\ y'''(x) &= -2\alpha e^{-x}(\sin(x) - \cos(x)) + 2\beta e^{-x}(\sin(x) + \cos(x)) + \\ &\quad - 2\gamma e^{x}(\sin(x) - \cos(x)) - 2\delta e^{x}(\sin(x) + \cos(x)) \\ y^{(4)}(x) &= -4\alpha e^{-x}\cos x - 4\beta e^{-x}\sin x - 4\gamma e^{x}\sin x - 4\delta e^{x}\cos x \end{aligned}$$

   As we are interested in initial conditions at $x = 0$ all terms $e^{\pm x} = 1\ ,\ \sin(x) = 0\ ,\ \cos(x) = 1$ and we get the following linear equation system

$$\begin{bmatrix} y'(0) \\ y''(0) \\ y'''(0) \\ y^{(4)}(0) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & -2 & 2 & 0 \\ 2 & 2 & 2 & -2 \\ -4 & 0 & 0 & -4 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$

that means in order to get $y(x) = 1 \cdot e^{-x}\cos(x)$ we have to chose $y'(0) = -1$, $y''(0) = 0$, $y'''(0) = 2$, $y^{(4)}(0) = -4$ and with

$$\begin{aligned}\begin{bmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{bmatrix} &= \begin{bmatrix}-1 & 1 & 1 & 1\\ 0 & -2 & 2 & 0\\ 2 & 2 & 2 & -2\\ -4 & 0 & 0 & -4\end{bmatrix}^{-1} \cdot \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\\ &= \frac{1}{8}\cdot\begin{bmatrix}-2 & 0 & 1 & -1\\ 2 & -2 & 1 & 0\\ 2 & 2 & 1 & 0\\ 2 & 0 & -1 & -1\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\\ &= \frac{1}{4}\cdot\begin{bmatrix}-1\\1\\1\\1\end{bmatrix}\end{aligned}$$

we get for the given initial conditions

$$y(x) = -\frac{1}{4}e^{-x}\cos(x) + \frac{1}{4}e^{-x}\sin(x) + \frac{1}{4}e^{x}\sin(x) + \frac{1}{4}e^{x}\cos(x)$$

2. Calculate
$$\int_{\pi^{-1}}^{\pi} \frac{1}{x}\sin^2\left(-x-\frac{1}{x}\right)\log x\,dx$$

Reason: Symmetries can be multiplicative, too.

Solution: We substitute $u = x^{-1}$, $du = -x^{-2}\,dx = -u^2\,dx$ and get

$$\begin{aligned}\mathcal{I} &= \int_{\pi^{-1}}^{\pi} \sin^2\left(-x-\frac{1}{x}\right)\cdot\frac{\log x}{x}\,dx\\ &= \int_{\pi}^{\pi^{-1}} \sin^2\left(-u-\frac{1}{u}\right)\cdot\frac{-\log u}{u^{-1}}\cdot(-u^{-2})\,du\\ &= \int_{\pi}^{\pi^{-1}} \sin^2\left(-u-\frac{1}{u}\right)\cdot\frac{\log u}{u}\,du\\ &= -\int_{\pi^{-1}}^{\pi} \sin^2\left(-u-\frac{1}{u}\right)\cdot\frac{\log u}{u}\,du\\ &= -\mathcal{I}\end{aligned}$$

3. Decryption:

   Reason: Fun (easy - diligence).

   (a) "ZC ULX QFFY L TBXCFSB FMFS XZYVF ZC RLX AZXVDMFSFA TDSF CULY KZKCB BFLSX LPD, LYA LOO PDDA CUFDS-FCZVLO IUBXZVZXCX IHC CUZX YHTQFS HI DY CUFZS RLOO LYA RDSSB LQDHC ZC"

   Solution: It is an easy Caesar code ($x \longmapsto 5x + 7$) and means

   "It has been a mystery ever since it was discovered more than fifty years ago, and all good theoretical physicists put this number up on their wall and worry about it" (R. Feynman)

   (b) "CO PXLIX BPX Y FPRDMCPO MP KPDOM, Y BDRR LIX-CAYMCPO PX VXPPB SDFM ZI WCAIO. YOFJIXF JCMU OP VXPPB JCRR ZI CWOPXIL. CM CF BCOI MP DFI OPOMX-CACYR XIFDRMF JCMUPDM VXPPB YF RPOW YF EPD KCMI MUIS YOL YF RPOW YF CM CF KPSSPO QOPJRILWI MP YRR SYMUISYMCKCYOF. JUIMUIX MUI RYMMIX CF FYMCFBCIL JCRR ZI LIKCLIL PO Y KYFIZEKYFI ZYFCF. CB EPD UYAI FIIO MUI VXPZRIS ZIBPXI YOL XISISZIX MUI FPRDMCPO, EPD KYOOPM VYXMCKCVYMI CO MUI FPRDMCPO MP MUYM VXPZRIS"

   Solution: It's a randomly chosen mapping of the alphabet onto itself.

A	1	25	Y
B	2	26	Z
C	3	11	K
D	4	12	L
E	5	9	I
F	6	2	B
G	7	23	W
H	8	21	U
I	9	3	C
J	10	20	T
K	11	17	Q
L	12	18	R
M	13	19	S
N	14	15	O
O	15	16	P
P	16	22	V
Q	17	8	H
R	18	24	X
S	19	6	F
T	20	13	M
U	21	4	D
V	22	1	A
W	23	10	J
X	24	14	N
Y	25	5	E
Z	26	7	G

"In order for a solution to count, a full derivation or proof must be given. Answers with no proof will be ignored. It is fine to use nontrivial results without proof as long as you cite them and as long as it is "common knowledge to all mathematicians". Whether the latter is satisfied will be decided on a case-by-case basis. If you have seen the problem before and remember the solution, you cannot participate in the solution to that problem."

4. Farmer Joe bought the blue area for \$10,000, the green for \$20,000 and the yellow for \$30,000. Assuming prices are proportional to the areas, what's the price for his entire field?

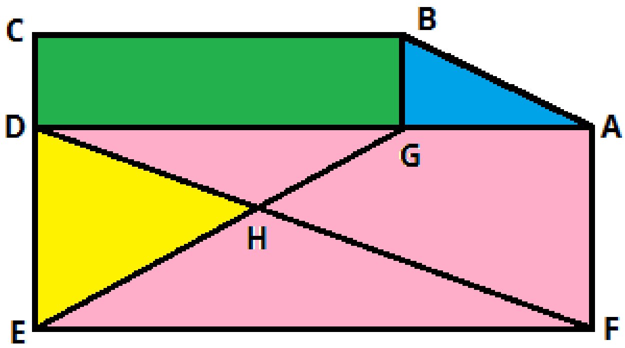

Figure 3:
Farmer Joe's Field

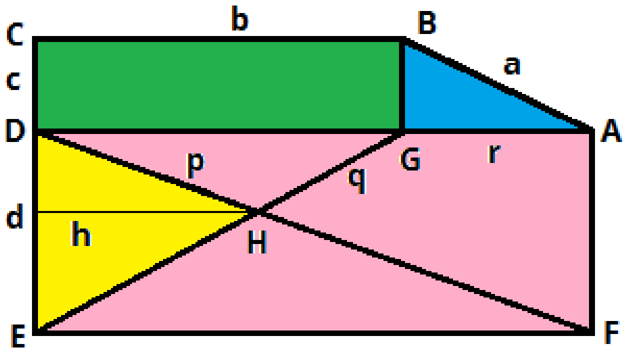

Figure 4:
Farmer Joe's Field

Reason: Geometry. Find the clue.

Solution: We will use the following notation: Thus we know that $bc = 2 \cdot \frac{1}{2} \cdot cr$, i.e. $r = b$. For the intersection of p and q we get the equation of straight lines by

$$\frac{d}{b} \cdot h = -\frac{d}{b+r} \cdot h + d$$

and therefore $h = \frac{2}{3}b$. Since the yellow area is three times the blue area, we have

$$\frac{1}{2}dh = \frac{1}{2} \cdot \frac{2}{3}db = 3 \cdot \frac{1}{2}bc$$

$d = \frac{9}{2}c$, and thus for the total area

$$A_{ABCEF} = (c+d) \cdot 2b - \frac{1}{2}bc = \frac{21}{2}bc = \frac{21}{2} \cdot \$20,000 = \$210,000$$

5. Inequalities.

Reason: Useful.

(a) Prove $(e+x)^{e-x} > (e-x)^{e+x}$ for $0 < x < e$.

Solution: We define

$$f\,:\,]0,e[\longrightarrow \mathbb{R}\,,\; f(x) := (e-x)\log(e+x) - (e+x)\log(e-x)$$

and observe $\lim_{x\searrow 0} f(x) = 0$. Then

$$\begin{aligned} f\,'(x) &= \frac{e-x}{e+x} + \frac{e+x}{e-x} - (\log(e+x) + \log(e-x)) \\ &= 2 \cdot \underbrace{\frac{e^2+x^2}{e^2-x^2}}_{>1} - \underbrace{\log(e^2-x^2)}_{<2} \\ &> 0 \end{aligned}$$

and $f(x) > 0$, resp. $(e-x)\log(e+x) > (e+x)\log(e-x)$ resp. $(e+x)^{e-x} > (e-x)^{e+x}$

(b) Show that for $0 < b < a$ we have

$$\frac{1}{a} < \frac{2}{a+b} < \frac{\log(a)-\log(b)}{a-b} < \frac{1}{\sqrt{ab}} < \frac{1}{b}$$

Reason: Napier's inequality.

Solution: For $x = \dfrac{a}{b} > 1$ we have

$$\begin{aligned} \log(x^2) &= 2\log(x) \\ &= 2\int_1^x \frac{1}{t}\,dt \\ &< \int_1^x \left(1 + \frac{1}{t^2}\right)\,dt \\ &= x - \frac{1}{x} \end{aligned}$$

or

$$\log(x) < \sqrt{x} - \frac{1}{\sqrt{x}}$$

With $\log(x) = \sum_{k=0}^{\infty} \frac{2}{2k+1}\left(\dfrac{x-1}{x+1}\right)^{2k+1} > 2 \cdot \dfrac{x-1}{x+1}$ we get

$$2\frac{\frac{a}{b}-1}{\frac{a}{b}+1} = 2\frac{a-b}{a+b} < \log(\frac{a}{b}) = \log(a) - \log(b) < \sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}} = \frac{a-b}{\sqrt{ab}}$$

(c) Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be two monotone integrable functions, either both increasing or both decreasing. Show that

$$\int_a^b f(x)g(x)\,dx \geq \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$$

Solution: We have $(f(x) - f(y))\,(g(x) - g(y)) > 0$ and therefore

$$\int_a^b f(x)g(x)\,dx + \int_a^b f(y)g(y)\,dy \geq$$
$$\int_a^b f(x)\,dx \int_a^b g(y)\,dy + \int_a^b f(y)\,dy \int_a^b g(x)\,dx$$

or

$$2\int_a^b f(x)g(x)\,dx \geq 2\int_a^b f(x)\,dx \int_a^b g(x)\,dx$$

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1. Determine with analytical methods, i.e. with a calculator only, the wavelengths of all local maximal radiation intensities of a black body of temperature T given the following function of radiation intensity up to three digits.

$$J(\lambda) = \frac{c^2 h}{\lambda^5 \cdot \left(\exp\left(\frac{ch}{\lambda\kappa T}\right) - 1\right)}$$

Reason: Physics.

Solution: With $x = \frac{\kappa T}{ch}\lambda$ we get for $x > 0$

$$F(x) = x^{-5}\left(e^{\frac{1}{x}} - 1\right)^{-1} = \frac{c^3 h^4}{\kappa^5 T^5}\,J(x)$$
$$F'(x) = \left(e^{\frac{1}{x}} - 5x\left(e^{\frac{1}{x}} - 1\right)\right) \cdot x^{-7} \cdot \left(e^{\frac{1}{x}} - 1\right)^{-2}$$

and $F'(x) = 0$ if and only if $e^{\frac{1}{x}} - 5x\left(e^{\frac{1}{x}} - 1\right) = 0$ or

$$f(t) := 5(1 - e^{-t}) = t \text{ with } t = x^{-1}$$

Because of $f'(t) > 1$ on $[0, \log 5]$ the function $f(t) - t$ is strictly monotone increasing there and $f(t) > t$ since $f(0) = 0$. For $t > \log 5$ we get that $f(t)-t$ is strictly monotone decreasing and thus has at most one zero. As $f(4)-4>0$ and $f(5)-5<0$ there is exactly one zero t^* in $[4,5]$ by the intermediate value theorem. Now

$$q := \sup\{\,|f'(t)|\,:\,t\in[4,5]\,\} = f'(4) = 5e^{-4} = 0.09157 < 1$$

and by the fixed-point theorem and a sequence $t_{n+1} := f(t_n)\ ,\ t_1 = 5$ we have

$$|t^*-t_n| < \frac{q}{q-1}\ |t_n - t_{n-1}| < 0.1008\ |t_n - t_{n-1}|$$

and thus $t^* = 4.965114 \pm 10^{-6}$ or $x^* = 0.2014052 \pm 10^{-7}$. Since $\lim_{x\searrow 0} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 0$ it is the only maximum and

$$\begin{aligned}\lambda^* &= \frac{ch}{\kappa T}\ x^* \\ &= 0.2014\ \frac{ch}{\kappa T} \\ &= 0.2014\ \cdot\ 1.4388\ \text{cm}\cdot T^{-1} \\ &= 0,28977\ \text{cm}\cdot T^{-1}\end{aligned}$$

2. Consider $\mathfrak{su}(3) = \operatorname{span}\{\,T_3, Y, T_\pm, U_\pm, V_\pm\,\}$ given by the basis elements

$$\begin{aligned}&T_3 = \frac{1}{2}\lambda_3\ ,\ Y = \frac{1}{\sqrt{3}}\lambda_8\ , \\ &T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda - 2)\ ,\ U_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7)\ ,\ V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5)\end{aligned}$$

(cp. https://www.physicsforums.com/insights/representations-precision-important) where the λ_i are the Gell-Mann matrices and its maximal solvable Borel-subalgebra

$$\mathfrak{B} := \langle T_3, Y, T_+, U_+, V_+\rangle$$

Now $\mathfrak{A}(\mathfrak{B}) = \{\,\alpha : \mathfrak{g}\to\mathfrak{g}\ :\ [X,\alpha(Y)] = [Y,\alpha(X)]\ \forall\, X,Y\ \in\mathfrak{B}\,\}$ is the one-dimensional Lie algebra spanned by $\operatorname{ad}(V_+)$ because $\mathbb{C}V_+$ is a one-dimensional ideal in $\mathfrak{B}$ (Proof?). Then $\mathfrak{g} := \mathfrak{B}\ltimes\mathfrak{A}(\mathfrak{B})$ is again a Lie algebra by the multiplication $[X,\alpha] = [\operatorname{ad}X,\alpha]$ for all $X\in\mathfrak{B}\ ,\ \alpha\in\mathfrak{A}(\mathfrak{B})$. (For a proof see problem 9 in https://www.physicsforums.com/threads/intermediate-math-challenge-may-2018.946386/)

(a) Determine the center of $\mathfrak{g}$, and whether it is semisimple, solvable, nilpotent or neither.

(b) Show that $(X,Y)\mapsto \alpha([X,Y])$ defines another Lie algebra structure on $\mathfrak{B}$, which one?

(c) Show that $\mathfrak{A}(\mathfrak{g})$ is at least two-dimensional.

Reason: One-dimensional representation of $\mathfrak{B}$.

Solution: We have the following multiplication table

$$[T_3,Y]=[T_+,Y]=[T_+,V_+]=[U_+,V_+]=0$$
$$[T_3,T_+]=T_+\ ,\ [T_3,U_+]=-\frac{1}{2}U_+\ ,\ [T_3,V_+]=\frac{1}{2}V_+$$
$$[U_+,T_+]=-V_+\ ,\ [Y,U_+]=U_+\ ,\ [Y,V_+]=V_+$$

With $\mathfrak{A}(\mathfrak{B})=\mathbb{C}\cdot\alpha\ ,\ \alpha(Z)=\operatorname{ad}V_+(Z)=[V_+,Z]$ we get

$$[X,\alpha]=X.\alpha=[\operatorname{ad}X,\operatorname{ad}V_+]=\operatorname{ad}[X,V_+]\sim\operatorname{ad}V_+\sim\alpha$$

and $\operatorname{span}\{T_+,U_+,V_+\}\subseteq\ker\alpha=\ker\operatorname{ad}V_+=\mathfrak{C}_{\mathfrak{B}}(V_+)$, so

$$\begin{aligned}
\mathfrak{g}^{(0)}&=\mathfrak{g}=\mathfrak{B}\oplus\mathfrak{A}(\mathfrak{B})\\
\mathfrak{g}^{(1)}&=[\mathfrak{g},\mathfrak{g}]=[\mathfrak{B},\mathfrak{B}]\oplus\mathfrak{A}(\mathfrak{B})=\langle T_+,U_+,V_+\rangle\oplus\mathfrak{A}(\mathfrak{B})\\
\mathfrak{g}^{(2)}&=[\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}]=\mathbb{C}V_+\oplus\{0\}\\
\mathfrak{g}^{(3)}&=[\mathfrak{g}^{(2)},\mathfrak{g}^{(2)}]=\{0\}
\end{aligned}$$

Therefore $\mathfrak{g}=\mathfrak{B}\ltimes\mathfrak{A}(\mathfrak{B})$ is solvable, and not semisimple. If we take a central element $Z=aT_3+bY+cT_++dU_++eV_++f\alpha\in\mathfrak{Z}(\mathfrak{g})$ and solve successively

$$[Z,U_+]=0\to[Z,V_+]=0\to[Z,Y]=0$$

then we get all coefficients have to be zero, i.e. $\mathfrak{Z}(\mathfrak{g})=\{0\}$ and $\mathfrak{g}$ cannot be nilpotent. It also shows, that $\alpha([X,Y])=0$ is an Abelian structure on $\mathfrak{B}$.

For a one-dimensional ideal $\mathfrak{I}=\langle V_0\rangle$ of any Lie algebra $\mathfrak{h}$ we have $[X,V_0]=\mu(X)V_0$ for all $X\in\mathfrak{h}$ and some linear form $\mu\in\mathfrak{h}^*$. With $\alpha(X):=\operatorname{ad}(V_0)(X)=-\mu(X)V_0$ we always get a non-trivial antisymmetric transformation of $\mathfrak{h}$. Therefore $\beta_1(B+b\alpha):=-\mu(X)V_+$ defines a non-trivial antisymmetric transformation of $\mathfrak{g}=\mathfrak{B}\ltimes\mathfrak{A}(\mathfrak{B})$, since $\mathfrak{I}=\mathbb{C}\cdot V_+\lhd\mathfrak{g}$ is a one-dimensional ideal. However, $\mathbb{C}\cdot\alpha=\mathfrak{A}(\mathfrak{B})$ is also a one-dimensional ideal of $\mathfrak{g}$, so $\beta_2(B+b\alpha):=\mu(X)\alpha$ is antisymmetric, too, and linear independent of β_1. Thus

$$\dim\mathfrak{A}(\mathfrak{g})=\mathfrak{A}(\mathfrak{B}\ltimes\mathfrak{A}(\mathfrak{B}))\geq 2$$

3. Consider the Hilbert space $\mathcal{H} = L_2([a,b])$ of Lebesgue square integrable functions on $[a,b]$, i.e.

$$\langle \psi, \chi \rangle = \int_a^b \psi(x)\chi(x)\,dx$$

The functions $\{\, \psi_n := x^n \,:\, n \in \mathbb{N}_0 \,\}$ build a system of linear independent functions which can be used to find an orthonormal basis by the Gram-Schmidt procedure.

Show that the Legendre polynomials

$$p_n(x) := \frac{1}{(b-a)^n\, n!}\sqrt{\frac{2n+1}{b-a}}\,\frac{d^n}{dx^n}[(x-a)(x-b)]^n \;,\; n \in \mathbb{N}_0$$

build an orthnormal system.

Reason: L_2 spaces. Educational.

Solution: We first show $\langle p_n, p_m \rangle = 0$ for $n < m$. As $\deg p_n(x) \leq n$ it is sufficient to show by successively integration by parts for $n < m$

$$\begin{aligned}
\int_a^b x^n p_m(x)\,dx &= C_m \int_a^b x^n \,\frac{d^m}{dx^m}[(x-a)(x-b)]^m\,dx \\
&= C_m \left[x^n \,\frac{d^{m-1}}{dx^{m-1}}[(x-a)(x-b)]^m \right]_a^b \\
&- \; C_m \int_a^b \frac{d}{dx}\,x^n \,\frac{d^{m-1}}{dx^{m-1}}\,[(x-a)(x-b)]^m\,dx \\
&= 0 - C_m \int_a^b \frac{d}{dx}\,x^n \,\frac{d^{m-1}}{dx^{m-1}}\,[(x-a)(x-b)]^m\,dx \\
&\;\vdots \\
&= C_m\,(-1)^{n+1} \int_a^b \frac{d^{n+1}}{dx^{n+1}}\,x^n \,\frac{d^{m-n-1}}{dx^{m-n-1}}\,[(x-a)(x-b)]^m\,dx \\
&= 0
\end{aligned}$$

Now we have to show that $||p_n|| = 1$, where we use again integration by parts and the fact that all boundary conditions vanish as long as terms $(x-a)(x-b)$ occur. Equivalently

$$\begin{aligned}
&\left\|\frac{d^n}{dx^n}[(x-a)(x-b)]^n\right\|^2\\
&= \int_a^b \left[\frac{d^n}{dx^n}[(x-a)(x-b)]^n\right]^2 dx\\
&= -\int_a^b \left[\frac{d^{n-1}}{dx^{n-1}}[(x-a)(x-b)]^n\right] \cdot \left[\frac{d^{n+1}}{dx^{n+1}}[(x-a)(x-b)]^n\right] dx\\
&\vdots\\
&= (-1)^n \int_a^b (x-a)^n(x-b)^n \frac{d^{2n}}{dx^{2n}}[(x-a)(x-b)]^n\\
&= (2n)!\,(-1)^n \int_a^b (x-a)^n(x-b)^n\,dx\\
&= (2n)!\,(-1)^n\,(-1)^1\,\frac{n}{n+1}\int_a^b (x-a)^{n+1}(x-b)^{n-1}\,dx\\
&\vdots\\
&= (2n)!\,(-1)^n\,(-1)^n\,\frac{n\cdot(n-1)\cdot\ldots\cdot 1}{(n+1)\cdot(n+2)\cdot\ldots\cdot(2n)}\int_a^b (x-a)^{2n}\,dx\\
&= (n!)^2\int_a^b (x-a)^{2n}\,dx\\
&= (n!)^2\frac{1}{2n+1}\left[(x-a)^{2n+1}\right]_a^b\\
&= \frac{(n!)^2}{2n+1}(b-a)^{2n+1}
\end{aligned}$$

4. Rings.
   (a) Give an example of an integral domain (no field), which has common divisors, but doesn't have greatest common divisors.
   (b) Show that there are infinitely many units (invertible elements) in $\mathbb{Z}[\sqrt{3}]$.
   (c) Determine the units of $\{\,\frac{1}{2}a+\frac{1}{2}b\sqrt{-3}\mid a+b \text{ even }\}$.
   (d) The ring R of integers in $\mathbb{Q}(\sqrt{-19})$ is the ring of all elements, which are roots of monic polynomials with integer coefficients. Show that R is built by all elements of the form $\frac{1}{2}a+\frac{1}{2}b\sqrt{-19}$ where $a,b\in\mathbb{Z}$ and both are either even or both are odd.

   Reason: World outside of $\mathbb{Z}$ and $\mathbb{C}$.

Solution:

(a) $R := \mathbb{Z}[\sqrt{-5}]$.

$$1, 3, 2+\sqrt{-5}, 2-\sqrt{-5} \mid 9 = 3 \cdot 3 = (2+\sqrt{-5}) \cdot (2-\sqrt{-5})$$

and there is no greatest common divisor in this set of divisors.

(b) All numbers $\pm(2 \pm \sqrt{3})^n$, $n \in \mathbb{N}_0$ are units in $\mathbb{Z}\sqrt{3}$.

(c) From $|\frac{1}{2}a + \frac{1}{2}b\sqrt{-3}|^2 = 1$ for units, we get $4 = a^2 + 3b^2$, i.e. $|a| = 2, b = 0$ or $|a| = |b| = 1$, which are the six elements $\left[\frac{1}{2}(1+\sqrt{-3})\right]^n$, $n = 0, \ldots, 5$.

(d) For $r = \frac{1}{2}(a+b\sqrt{-19}) \in R$ we have $\frac{1}{2}(a+b\sqrt{-19}) \cdot \frac{1}{2}(a-b\sqrt{-19}) = \frac{1}{4}(a^2 + 19b^2) \in \mathbb{Z}$, because $a + b \equiv 0(2)$. So

$$(x-\frac{1}{2}(a+b\sqrt{-19}))(x-\frac{1}{2}(a-b\sqrt{-19})) = x^2-ax+\frac{1}{4}(a^2+19b^2) \in \mathbb{Z}[x]$$

If we have $r \in \mathbb{Q}(\sqrt{-19})$ an integer, then $r^2 + ar + b = 0$ for $a, b \in \mathbb{Z}$, i.e. $2r = -a \pm \sqrt{a^2 - 4b}$. As $r \in \mathbb{Q}(\sqrt{-19})$ we have $\mathbb{Z} \ni a^2 - 4b = (\alpha + \beta\sqrt{-19})^2 = \alpha^2 - 19\beta^2 + 2\alpha\beta\sqrt{-19}$ and thus $\alpha\beta = 0$.

Case 1: $\beta = 0$. Then $b = \frac{a-\alpha}{2} \cdot \frac{a+\alpha}{2} \in \mathbb{Z}$ and $r_{1,2} = -\frac{1}{2}(a \pm \alpha)$ and $a \pm \alpha \equiv 0(2)$ which means $r_{1,2} \in R$.

Case 2: $\alpha = 0$. Then $b = \frac{1}{4}(a^2 + 19\beta^2) \in \mathbb{Z}$ and $r_{1,2} = -\frac{1}{2}(a \pm \beta\sqrt{-19})$ and we have to show that $-a \pm \beta \equiv 0(2)$. Let us assume this is not the case and $\beta^2 = (2k + a + 1)^2$. Then

$$4 \mid (a^2 + 19\beta^2) = 20a^2 + 76 \cdot (n^2 + an + n) + 38a + 19 \not\equiv 0(4)$$

which is a contradiction. Therefore we have again $r_{1,2} \in R$.

Part VI
May, 2018

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1. Finite Field $\mathbb{F}_8$.

   Reason: Do not always assume $\operatorname{char} \mathbb{F} = 0$.

   (a) Find a minimal polynomial to determine the factor ring which is isomorphic to $\mathbb{F}_8$.

   Solution: $\mathbb{F}_8$ is three dimensional as $\mathbb{F}_2$ vector space. Therefore the minimal polynomial is of degree three and $m(x;\mathbb{F}_2) = x^3+x+1$ is irreducible because neither element of $\mathbb{F}_2 = \{0,1\}$ is a zero. We then have $\mathbb{F}_8 \cong \mathbb{F}_2[x]/(x^3+x+1)$.

   (b) From there determine a basis of $\mathbb{F}_8$ over $\mathbb{F}_2$ and write down its multiplication and addition laws.

   Solution: Let $\xi^3+\xi+1=0$. Then

   $$m(x;\mathbb{F}_2) = (x+\xi)(x+\xi^2)(x+\xi+\xi^2)$$

   and $\{1,\xi,\xi^2\}$ is a $\mathbb{F}_2$ basis of $\mathbb{F}_8$. The elements are thus

   $$\{0,1,\xi,\xi+1=\xi^3,\xi^2,\xi^2+1=\xi^6,\xi^2+\xi=\xi^4,\xi^2+\xi+1=\xi^5\}$$

   which defines the multiplicative group generated by ξ as well as the addition table.

   (c) Why does the algebraic closure of a finite field have to be infinite?

   Solution: We get a field $\mathbb{F}_{p^n}$ for every natural number $n \in \mathbb{N}$ over the prime field $\mathbb{F}_p$ which is algebraic over $\mathbb{F}_p$ of dimension n. Since the algebraic closure has to contain all of them, it has to be infinite, although of (prime) characteristic $p \neq 0$.

2. Determine the open balls with radius 3 around $(2,1) \in \mathbb{R}^2$ w.r.t.

   Reason: Open neighborhoods can be very different.

(a) the French Railway metric with Paris at the origin P and Reims at $R = (2, 1)$.

Solution: The distance between two points A, B is defined by

$$d(A,B) = \begin{cases} ||A - B||_2 & \text{if } \overline{AP} = \overline{BP} \\ ||A||_2 - ||B||_2 & \text{in all other cases} \end{cases}$$

All distances between points have to include Paris, so we have $3 - \sqrt{5}$ left if we have to travel to Paris first and the full 3 if we travel on $R \cdot \lambda$ outbound. This results in

$$B_3\,(R; d) = B_{3-\sqrt{5}}\,(P; ||\,.\,||_2) \cup \left\{ R \cdot \lambda \,:\, 1 - \frac{3}{\sqrt{5}} < \lambda < 1 + \frac{3}{\sqrt{5}} \right\}$$

The open ball of radius three around Reims includes all the way to Luxembourg in one direction and to all points in the inner highway circle of Paris. The good news is, that we can't reach Saint-Quentin although it is nearer than Paris.

(b) the Manhattan metric

Solution: The distance between two points A, B is defined by

$$d(A,B) = \sum_i |a_i - b_i|$$

and thus we get an open rhombus with vertices $(-1, 1)$, $(2, 4)$ $(5, 1)$ and $(2, -2)$.

(c) the maximum metric

Solution: The distance between two points A, B is defined by

$$d(A,B) = \max\{|a_i - b_i| \,:\, i = 1, 2\}$$

and thus we get an open square with vertices $(-1, -2)$, $(-1, 4)$ $(5, 4)$ and $(5, -2)$.

3. Calculate the volume $\mu(A)$ of

$$A = \{(x, y, z) \in \mathbb{R}^3 \,:\, x, y, z \geq 0 \,,\, x + y + z \leq \sqrt{2} \,,\, x^2 + y^2 \leq 1\}$$

Reason: Practice volume integrals.

Solution: The volume is given by $\mu(A) = \int_A d\mu$. The first reduction of A by z gives us

$$A' = \{\{(x,y) \in \mathbb{R}^2 \,:\, x,y \geq 0 \,,\, x^2+y^2 \leq 1\,\} \,;\, A_{(x,y)} = [0, \sqrt{2}-x-y]$$

and the second by y

$$A'' = [0,1] \,;\, A'_x = [0, \sqrt{1-x^2}]$$

We thus have

$$\begin{aligned}
\mu(A) &= \int_A d\mu \\
&= \int_{A'} \left(\int_0^{\sqrt{2}-x-y} dz \right) d\mu_{A'} \\
&= \int_{A''} \left(\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{2}-x-y} dz \right) dy \right) d\mu_{A''} \\
&= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{2}-x-y} dz \right) dy \right) dx \\
&= \int_0^1 \left[(\sqrt{2}-x)\sqrt{1-x^2} - \frac{1}{2}(1-x^2) \right] dx \\
&= \sqrt{2} \int_0^1 \sqrt{1-x^2}\, dx - \int_0^1 x\sqrt{1-x^2}\, dx - \frac{1}{2}\int_0^1 dx + \frac{1}{2}\int_0^1 x^2\, dx \\
&= \sqrt{2} \cdot \frac{\pi}{4} - \frac{1}{3} - \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \\
&= \frac{\pi}{2\sqrt{2}} - \frac{2}{3}
\end{aligned}$$

4. Solve $\int_\Gamma \omega$ with the curve $\Gamma = \gamma([0,1])$ given by

$$\gamma : \mathbb{R} \longrightarrow \mathbb{R}^3 \,,\, \gamma(t) = (t^2, 2t, 1) \text{ and } \omega = z^2dx + 2ydy + xzdz$$

and compute the exterior derivative $\nu = d\omega$. As such, the result is an exact $2-$form. Is it also closed? Show this by calculation.

Reason: Differential forms.

Solution:

$$\begin{aligned}
\int_\Gamma \omega &= \int_{[0,1]} \gamma^*(\omega) \\
&= \int_{[0,1]} \omega(d\gamma) \\
&= \int_{[0,1]} (z^2dx + 2ydy + xzdz)d\gamma \\
&= \int_{[0,1]} (1^2\,dx + 2\cdot 2t\,dy + t^2\cdot 1\,dz)d\gamma \\
&= \int_{[0,1]} \left(1^2\frac{d(t^2)}{dt} + 2(2t)\frac{d(2t)}{dt} + t^2\cdot 1\frac{d(1)}{dt}\right)\,dt \\
&= \int_0^1 (2t + 8t + 0)\,dt \\
&= \int_0^1 10t\,dt \\
&= 5
\end{aligned}$$

$$\begin{aligned}
d\omega &= d(z^2dx + 2ydy + xzdz) \\
&= (\frac{\partial}{\partial x}z^2dx + \frac{\partial}{\partial y}z^2dy + \frac{\partial}{\partial z}z^2dz) \\
&\wedge dx + (\frac{\partial}{\partial x}2ydx + \frac{\partial}{\partial y}2ydy + \frac{\partial}{\partial z}2ydz) \\
&\wedge dy + (\frac{\partial}{\partial x}xzdx + \frac{\partial}{\partial y}xzdy + \frac{\partial}{\partial z}xzdz)?dz \\
&= 2zdz\wedge dx + 2dy\wedge dy + (zdx + xdz)\wedge dz \\
x &= 2zdz\wedge dx - zdz\wedge dx \\
&= zdz\wedge dx
\end{aligned}$$

$$\begin{aligned}
d\nu &= d(zdz\wedge dx) \\
&= d(-1dx\wedge zdz) \\
&= \left(\frac{\partial}{\partial x}(-1)\,dx + \frac{\partial}{\partial y}(0)\,dy + \frac{\partial}{\partial z}(z)\,dz\right)\wedge dx\wedge dz \\
&= dz\wedge dx\wedge dz \\
&= 0
\end{aligned}$$

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1. Solve $\mathcal{I} = \int_{-1}^{0} x \cdot \sqrt{x^2+x+1}\,dx$. Hint: Use $\cosh^2 x - \sinh^2 x = 1$.

   Reason: Practice integration.

   Solution: We have $x^2+x+1 = (x+\frac{1}{2})^2+\frac{3}{4}$ and we set $x+\frac{1}{2} = \frac{\sqrt{3}}{2}\sinh u$. Thus $dx = \frac{\sqrt{3}}{2}\cosh u\,du$ and $\sqrt{x^2+x+1} = \frac{\sqrt{3}}{2}\cosh u$ with the hint given. All together we get

$$\mathcal{I} = \int_{-1}^{0}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\sinh u\right)\left(\frac{\sqrt{3}}{2}\cosh u\right)\left(\frac{\sqrt{3}}{2}\cosh u\right)du$$

   Integration by parts and again using the hint results in

$$\mathcal{I} = \frac{\sqrt{3}}{8}\cosh^3 u - \frac{3}{16}\sinh u\,\cosh u - \frac{3}{16}u$$

   Now re-substitution with $\sinh u = \frac{2}{\sqrt{3}}(x+\frac{1}{2})$, $\cosh u = \frac{2}{\sqrt{3}}\sqrt{x^2+x+1}$ and with the inverse function $\operatorname{arsinh} v = \log(v+\sqrt{v^2+1})$ we have

$$u = \operatorname{arsinh}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right) = \log\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)\right)$$

   and for our integral (modulo constant terms)

$$\mathcal{I} = \frac{1}{3}\left(x^2+x+1\right)^{\frac{3}{2}} - \frac{1}{4}(x+\frac{1}{2})\sqrt{x^2+x+1} - \frac{3}{16}\log(x+\frac{1}{2}+\sqrt{x^2+x+1})$$

   which from $x=-1$ to $x=0$ results in

$$\mathcal{I} = -\frac{1}{4} - \frac{3}{16}\log 3 \approx -0.45598980\ldots \approx -0.456$$

2. Given the differential operators $D_n := x^n \cdot \dfrac{d}{dx}$ $(n \in \mathbb{Z})$ on smooth real valued functions $\mathcal{C}^\infty(\mathbb{R})$. Determine for which subsets $L \subseteq \mathbb{Z}$ the set $\{D_n \,|\, n \in L\}$ is a basis for a finite dimensional Lie algebra and which Lie algebra is it.

   Reason: Differential structures on the real line.

   Solution: For $f \in \mathcal{C}^\infty(\mathbb{R})$ we get

$$D_n(D_m.f) = D_n(x^m f\,') = x^{n+m} f\,'' + m x^{n+m-1} f\,'$$

and thus

$$[D_n, D_m].f = (m-n)x^{n+m-1}f\,' = (m-n)D_{n+m-1}.f$$

In order to get a finite dimensional Lie algebra, the set of all $L = \{n, m, n+m-1\}$ must be finite.

- The easiest examples are $n = m$ with $[D_n, D_n] = 0$, the one dimensional Abelian Lie algebra. It's the only possibility with commuting operators.
- We also observe, that $[D_1, D_n] = (n-1)D_n$. So for $n \neq 1$ this yields the two dimensional non Abelian Lie algebra. It is the maximal solvable subalgebra, a so called Borel subalgebra, of the simple Lie algebra of type A_1.
- The only remaining possibility, such that L does not contain infinite sequences of integers, is $m = \pm n + 1$, $n = 1$. Here we get the following three multiplications:

$$[D_1, D_{-n+1}] = -n \cdot D_{-n+1}$$
$$[D_1, D_{n+1}] = n \cdot D_{n+1}$$
$$[D_{n+1}, D_{-n+1}] = -2n \cdot D_1$$

  These are the multiplications in the three dimensional, simple Lie algebra of type A_1, i.e. $\mathfrak{sl}_2 \cong \mathfrak{su}_2$.

All other combinations lead to infinite dimesional Lie algebras.

3. Solve $\sum_{k=1}^{\infty} \dfrac{1}{k\binom{2k}{k}}$.

   Reason: Puzzle.

   Solution: The Taylor series for $\arcsin^2 x$ in $|x| < 1$ is given by

$$\arcsin^2 x = \frac{1}{2}\sum_{k=1}^{\infty} \frac{(2x)^{2k}}{k^2\binom{2k}{k}}$$

   Now applying the differential operator $x\frac{d}{dx}$ on both sides yields

$$2x \arcsin x \frac{1}{\sqrt{1-x^2}} = \sum_{k=1}^{\infty} \frac{(2x)^{2k}}{k\binom{2k}{k}}$$

   and thus for $x = \frac{1}{2}$:

$$\sum_{k=1}^{\infty} \frac{1}{k\binom{2k}{k}} = \frac{\pi}{6} \cdot \frac{2}{\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$$

4. For a given a real Lie algebra $\mathfrak{g}$, we define

$$\mathfrak{A}(\mathfrak{g}) = \{\, \alpha \,:\, \mathfrak{g} \longrightarrow \mathfrak{g} \,:\, [\alpha(X),Y] = -[X,\alpha(Y)] \text{ for all } X,Y \in \mathfrak{g} \,\} \quad (1)$$

the set of *antisymmetric transformations* of $\mathfrak{g}$. Remember that a real Lie algebra is a real vector space equipped with a multiplication for which holds

- (2) anti-commutativity: $[X,X]=0$
- (3) Jacobi-identity: $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$

Reason: Practice terminology.

(a) Show that $\mathfrak{A}(\mathfrak{g}) \subseteq \mathfrak{gl}(g)$ is a Lie subalgebra in the Lie algebra of all linear transformations of $\mathfrak{g}$ with the commutator as Lie product: $[\alpha,\beta]=\alpha\beta-\beta\alpha \quad (4)$.

Solution:

$$\begin{aligned}
[[\alpha,\beta]X,Y] &\stackrel{(4)}{=} [\alpha\beta X,Y]-[\beta\alpha X,Y]\\
&\stackrel{(1)}{=} [X,\beta\alpha Y]-[X,\alpha\beta Y]\\
&\stackrel{(4)}{=} [X,[\beta,\alpha]Y]\\
&\stackrel{(2)}{=} -[X,[\alpha,\beta]Y]
\end{aligned}$$

(b) Give an example of a non Abelian Lie algebra $\mathfrak{g}$ with trivial center, such that $\mathfrak{A}(\mathfrak{g}) \neq 0$.

Solution: $\mathfrak{g} = \langle X,Y \,:\, [X,Y]=Y\rangle$ is non Abelian with trivial center and $\mathfrak{A}(\mathfrak{g}) \cong \mathfrak{sl}(2,\mathbb{R})$. $\mathfrak{g} = \mathfrak{B}(\mathfrak{sl}(2,\mathbb{R}))$ is the maximal solvable subalgebra of $\mathfrak{sl}(2,\mathbb{R})$, a so called Borel subalgebra.

(c) Show that $\mathfrak{g} \rtimes \mathfrak{A}(\mathfrak{g})$ is a semidirect product given by

$$[X,\alpha] := [\operatorname{ad} X,\alpha] = \operatorname{ad} X\,\alpha - \alpha\,\operatorname{ad} X \quad (5)$$

Solution: We have to show that this multiplication makes $\mathfrak{A}(g)$

an ideal in $\mathfrak{g} \rtimes \mathfrak{A}(\mathfrak{g})$ and a $\mathfrak{g}-$module.

$$\begin{aligned}[[X,\alpha]Y,Z] &\stackrel{(5)}{=} [[X,\alpha Y],Z]-[\alpha[X,Y],Z]\\ &\stackrel{(3),(1)}{=} -[[\alpha Y,Z],X]-[[Z,X],\alpha Y]+[[X,Y],\alpha Z]\\ &\stackrel{(3),(1)}{=} [[Y,\alpha Z],X]+[\alpha[Z,X],Y]\\ &-[[Y,\alpha Z],X]-[[\alpha Z,X],Y]\\ &\stackrel{(2)}{=} [Y,\alpha[X,Z]]-[Y,[X,\alpha Z]]\\ &\stackrel{(5)}{=} -[Y,[X,\alpha Z]]\end{aligned}$$

and $\mathfrak{A}(\mathfrak{g})$ is an ideal in $\mathfrak{g} \rtimes \mathfrak{A}(\mathfrak{g})$. It is also a $\mathfrak{g}-$module, because ad is a Lie algebra homomorphism (6) and therefore

$$\begin{aligned}[[X,Y],\alpha] &\stackrel{(5)}{=} [\operatorname{ad}[X,Y],\alpha]\\ &\stackrel{(6)}{=} [[\operatorname{ad}X,\operatorname{ad}Y],\alpha]\\ &\stackrel{(3)}{=} -[[\operatorname{ad}Y,\alpha],\operatorname{ad}X]-[[\alpha,\operatorname{ad}X],\operatorname{ad}Y]\\ &\stackrel{(2)}{=} [\operatorname{ad}X,[\operatorname{ad}Y,\alpha]]-[\operatorname{ad}Y,[\operatorname{ad}X,\alpha]]\\ &\stackrel{(5)}{=} [X,[Y,\alpha]]-[Y,[X,\alpha]]\end{aligned}$$

(d) Show that for all $\alpha \in \mathfrak{A}(\mathfrak{g})$ and $X,Y,Z \in \mathfrak{g}$

$$[\alpha(X),[Y,Z]]+[\alpha(Y),[Z,X]]+[\alpha(Z),[X,Y]]=0 \quad (7)$$

Solution:

$$\begin{aligned}[\alpha(X),[Y,Z]] &\stackrel{(3)}{=} -[Y,[Z,\alpha(X)]]-[Z,[\alpha(X),Y]]\\ &\stackrel{(1)}{=} [Y,[\alpha(Z),X]]+[Z,[X,\alpha(Y)]]\\ &\stackrel{(3)}{=} -[\alpha(Z),[X,Y]]-[X,[Y,\alpha(Z)]]\\ &-[X,[\alpha(Y),Z]]-[\alpha(Y),[Z,X]]\\ &\stackrel{(1)}{=} -[\alpha(Y),[Z,X]]-[\alpha(Z),[X,Y]]\end{aligned}$$