Proof about Ordered Sets

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1 Statement

Let R be a relation on a set X such that there is no finite sequence of elements $x_1, x_2, \ldots, x_k$ of X satisfying $x_1 R x_2, x_2 R x_3, \ldots, x_{k-1} R x_k, x_k R x_1$ (we say that such an R is acyclic). Prove that there exists an ordering $\preceq$ on X such that $R \subseteq \preceq$. You may assume that X is finite if this helps.

This problem was taken from the second edition of Invitation to Discrete Mathematics.

2 Answer

Take any ordering $\preceq$ on X.

It is true that $x_a \preceq x_b \wedge x_b \preceq x_c \to x_a \preceq x_c$.

It is also true that $x_a \preceq x_b \to x_b \npreceq x_a$.

Assume there is a finite sequence of elements $x_1, x_2, \ldots, x_k$ of X such that $x_1 \preceq x_2 \preceq x_3 \preceq \ldots \preceq x_{k-1}$ and $x_k \preceq x_1$.

However, $x_1 \preceq x_2 \preceq x_3 \preceq \ldots \preceq x_{k-1} \preceq x_k \to x_1 \preceq x_k \to x_k \npreceq x_1$, which demonstrates that the assumption is absurd and, consequently, that $\exists\, (X, \preceq)\, R \subseteq \preceq$.