To Prove:

$$\forall x[\forall y(y \in x \leftrightarrow y \in x) \wedge \forall y(x \in y \leftrightarrow x \in y)] \tag{1}$$

First we show that the following is provable:

$$\forall x(\forall y[x \in y \leftrightarrow x \in y \wedge y \in x \leftrightarrow y \in x]) \tag{2}$$

Proof

1. $x \in y \leftrightarrow x \in y \wedge y \in x \leftrightarrow y \in x$ Tautology.
2. $\forall x(\forall y[x \in y \leftrightarrow x \in y \wedge y \in x \leftrightarrow y \in x])$ By universal closure of propositional tautologies.

Now, if we could show the following:

$$\forall x[\forall y(\phi \wedge \psi)] \rightarrow \forall x(\forall y\phi \wedge \forall y\psi) \tag{3}$$

we could, by making the relevant substitutions for ϕ and ψ, use 2 to derive 1.

To do this, first consider proving the easier (but still fiddly):

$$\forall y(\phi y \wedge \psi y) \rightarrow (\forall y\phi y \wedge \forall y\psi y) \tag{4}$$

Proof sketch:

1. $\forall y\phi y \rightarrow [\forall y\psi y \rightarrow ([\forall y\phi y \wedge \forall y\psi y])]$ Taut
2. $(\phi y \wedge \psi y) \rightarrow \phi y$ Taut
3. $\forall y[(\phi y \wedge \psi y) \rightarrow \phi y] \rightarrow [\forall y(\phi y \wedge \psi y) \rightarrow \forall y\phi y]$ Axiom 3
4. $\forall y[(\phi y \wedge \psi y) \rightarrow \phi y]$ closure of 2 (which is a taut)
5. $\forall y(\phi y \wedge \psi y) \rightarrow \forall y\phi y)$ 4 3 MP
6. $(A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)]$ (where A is the antecedant of 5, B is the consequent of 5, and C is the consequent of 1) Tautology.
7. $(B \rightarrow C) \rightarrow (A \rightarrow C)$ 5 6 MP
8. $A \rightarrow C$ 1 7 MP
9. $\forall y(\phi y \wedge \psi y) \rightarrow (\forall y\psi y \rightarrow [\forall y\phi y \wedge \forall y\psi y])$ Above line with A and C written out in full.

10. $\phi y \wedge \psi y \to \psi y$ Taut.

11. then basically repeat the steps above to eliminate the middle term in 9 to get the result:

12. $\forall y(\phi y \wedge \psi y) \to (\forall y \phi y \wedge \forall y \psi y)$

Using the ideas in this proof, we can show:

$$\forall x(\forall y(\phi y \wedge \psi y) \to (\forall y \phi y \wedge \forall y \psi y)) \tag{5}$$

We do this by piggybacking on the above proof: we effectively follow it, but repeatedly performing closure on the tautologies above, and axiom 3 which allows us to repeatedly distribute $\forall$ across conditionals.

Proof sketch.

1. $\forall y \phi y \to [\forall y \psi y \to ([\forall y \phi y \wedge \forall y \psi y])]$ Taut

2. $\forall z[\forall y \phi y \to [\forall y \psi y \to ([\forall y \phi y \wedge \forall y \psi y])]$ closure of tautology at 1.

3. $(\phi y \wedge \psi y) \to \phi y$ Taut

4. $\forall z(\forall y[(\phi y \wedge \psi y) \to \phi y])$ a closure of above tautology

5. $\forall y[(\phi y \wedge \psi y) \to \phi y] \to [\forall y(\phi y \wedge \psi y) \to \forall y \phi y]$ Axiom 3

6. $\forall z(\forall y[(\phi y \wedge \psi y) \to \phi y] \to [\forall y(\phi y \wedge \psi y) \to \forall y \phi y])$ closure of above Axiom

7. $\forall z(\forall y[(\phi y \wedge \psi y) \to \phi y]) \to \forall z[\forall y(\phi y \wedge \psi y) \to \forall y \phi y]$ Use 6, axiom 3, and Modus Ponens.

8. $\forall z[\forall y(\phi y \wedge \psi y) \to \forall y \phi y]$ 4 7 MP.

9. So, with effort, we have managed to get the closure of 5 of the previous proof, which we couldn't do directly.

10. Keep taking closures of tautologies and distributing $\forall z$ as in the move from 6 to 7, to mimic the earlier proof - it's a lot of steps, but one should end up with the theorem.