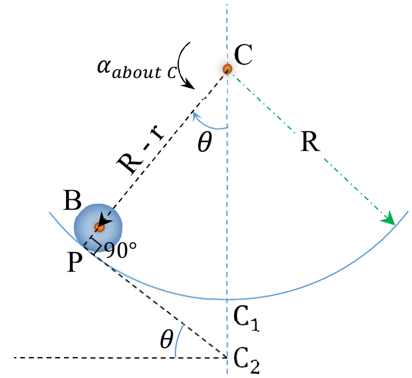
Q 39 Simple Harmonic Motion

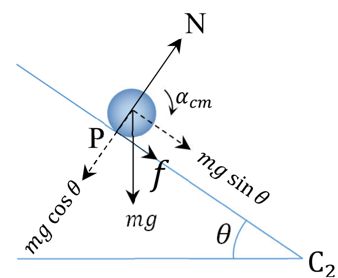
Question: A solid spherical ball of mass m and radius r rolls without slipping on a rough concave surface of large radius R. It makes small oscillations about the lowest point. Find the frictional force on the ball and the time period of the oscillation.



Solution: Method 1: Let at any point of time, the ball be at angle theta from the line joining the point C (centre of curvature of the concave surface) to point C_1.

Since we have to calculate the time period of the ball, and the ball is undergoing angular simple harmonic motion about an axis passing through the point C and perpendicular to the paper, therefore, we will calculate the torque about point C and then find angular acceleration, α about point C and that will give us the angular frequency ω about point C, and thereby time period.

Let B be the centre of the ball. P is the point of contact of ball with the surface. Line PC_2 is tangent to the surface at the point of contact P.



The forces acting on the ball is shown in the diagram beside (→). Since the surface is rough, there must be friction acting at the point of contact of the ball and the surface. Let the direction of friction be as shown in the pic. Since we have to find the angular acceleration about point C, therefore, we need to calculate torque about point C. The forces acting on the ball are gravity(mg), Normal reaction (N) and friction.

Out of these 3 forces, only friction and mgsinθ will create a torque about point C. So, now we need to calculate the frictional force acting on the ball.

Since the ball rolls without slipping, i.e. it's in pure rolling motion, therefore, the tangential acceleration of point of the ball which is instantaneously in contact with the surface, must be zero. Why the term tangential acceleration has been written here: Since the ball is actually undergoing a circular curve on a concave surface, therefore every point on the ball is in circular motion about point C, in addition to being in circular motion around the ball's centre of mass. Therefore, the acceleration at point P is perpendicular to the concave surface i.e. towards point C or towards point B (since points P, B and C are always colinear). However, tangential acceleration at point P would be always zero, due to the ball being in the state of pure rolling all throughout it's motion.

i.e. $a_{poc} = 0$ (in the tangential direction)

For the ball we have, $a_{cm} = \dfrac{mgsin\theta + f}{m}$. Let α_{cm} be the angular acceleration of the ball, about the ball's centre of mass. $\Rightarrow a_{poc} = a_{cm} - \alpha_{cm} r$ (in the tangential direction)

Now, torque about the centre of mass of the ball, $\tau_{cm} = -fr$. Minus sign has been taken here since the direction of torque is opposite to the shown direction of α_{cm}.

Moment of Inertia of the ball about the axis passing through it's centre, $I_{cm} = \frac{2}{5}mr^2$. Now, $\tau_{cm} = I_{cm}\alpha_{cm}$

$\Rightarrow -fr = \frac{2}{5}mr^2\alpha_{cm} \Rightarrow \alpha_{cm} = -\dfrac{5}{2}\dfrac{fr}{mr^2} = -\dfrac{5}{2}\dfrac{f}{mr}$

$\therefore\ a_{poc} = a_{cm} - \alpha_{cm} r = \dfrac{mgsin\theta + f}{m} + \dfrac{5}{2}\dfrac{f}{m} = g\sin\theta + \dfrac{7}{2}\dfrac{f}{m}$, and since $a_{poc} = 0$

$\Rightarrow g\sin\theta + \frac{7}{2}\frac{f}{m} = 0 \Rightarrow \frac{7}{2}\frac{f}{m} = -g\sin\theta \Rightarrow f = -\frac{2}{7}mg\sin\theta$. The minus sign here means that the direction of friction is up the incline. Friction is zero at the bottom most point.

Now that we have the magnitude of frictional force, we can calculate the magnitude of torque about the axis passing through point C and perpendicular to the paper.

$|\tau_{about\ C}| = mg\sin\theta\,(R-r) + fR = mg\sin\theta\,(R-r) - \frac{2}{7}mgR\sin\theta = \left(\frac{5}{7}mgR - mgr\right)\sin\theta$

This torque tries to bring the ball back towards $\theta = 0$. Therefore, we can write:

$$\tau_{about\ C} = -\left(\frac{5}{7}mgR - mgr\right)\sin\theta$$

Now, moment of inertia of the ball about point C = $\frac{2}{5}mr^2 + \mathrm{m}(\mathrm{R} - \mathrm{r})^2 = \frac{2}{5}mr^2 + \mathrm{m}(R^2 + r^2 - 2Rr) = \frac{7}{5}mr^2 + mR^2 - 2mRr$

Now, $\alpha_{about\ C} = \frac{\tau_{about\ C}}{I_{about\ C}} = \frac{-\left(\frac{5}{7}mgR - mgr\right)\sin\theta}{\frac{7}{5}mr^2 + mR^2 - 2mRr} = \frac{-\left(\frac{5}{7}R - r\right)\mathrm{g}\sin\theta}{\frac{7}{5}r^2 + R^2 - 2Rr} = -\frac{(5\mathrm{R} - 7r)\,\mathrm{g}\,\theta}{7\left(\frac{7}{5}r^2 + R^2 - 2Rr\right)}$ for small θ.

$$\alpha_{about\ C} = -\omega^2\theta$$

$$\Rightarrow \omega = \sqrt{\frac{(5\mathrm{R} - 7r)\mathrm{g}}{7\left(\frac{7}{5}r^2 + R^2 - 2Rr\right)}}$$

$$\Rightarrow \mathrm{T} = 2\pi\sqrt{\frac{7\left(\frac{7}{5}r^2 + R^2 - 2Rr\right)}{(5\mathrm{R} - 7r)\mathrm{g}}}$$