Classical states, quantum field measurement

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The Hilbert spaces of the quantized electromagnetic field and of the quantized Dirac spinor field are constructed using classical random fields acting on a vacuum state, in an algebraic Koopman–von Neumann approach, so that the quantized and random fields act as different subalgebras acting on the same Hilbert space, allowing a classical interpretation of the states of the respective quantum theories. For the quantized electromagnetic field, there is a commutative subalgebra $\mathcal{R}$ that corresponds to a random field on Minkowski space, not just, as one might naïvely expect, on a space-like hyperplane. For the quantized Dirac spinor field, the Lie algebra $\mathcal{D}$ of global-U(1) invariant observables can be constructed as a subalgebra of a bosonic raising and lowering algebra, $\mathcal{D} \subset \mathcal{B}$, and the usual vacuum state over $\mathcal{D}$ can be extended (here, trivially) to act over $\mathcal{B}$, which contains a commutative subalgebra that corresponds to a random field on Minkowski space.

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I. INTRODUCTION

The vacuum state over an algebra of quantum field operators allows the construction of a Hilbert space of states over the algebra, however it is here shown that it is also possible —both for the quantized electromagnetic field and for the observables of the quantized Dirac spinor field— to construct an action of these algebras of quantum field operators on a Hilbert space of vectors that is constructed using everywhere mutually commutative random field operators acting on a vacuum vector.

It was shown in Ref. 1 that the Hilbert space of states over the complex Klein-Gordon quantum field is isomorphic to the Hilbert space of states over a real Klein-Gordon random field. [A similar construction can be found in Ref. 2, §5.] In Ref. 3 this was extended to show that the Hilbert space of states over the quantized electromagnetic field is isomorphic to the Hilbert space of states over an electromagnetic random field that has the same number of degrees of freedom. [A construction similar to this can be found in Ref. 4.] The construction for the electromagnetic field will be reproduced and extended here, in Section II, followed by a construction of an embedding of the algebra of global-$U(1)$ invariant observables of the quantized Dirac spinor field into an algebra of Dirac spinor random field operators in Section III.

We can construct a commuting subalgebra $\mathcal{R}$ for the quantized electromagnetic field that corresponds to a random field on Minkowski space. The Hilbert space of the quantized electromagnetic field is generated by the action of $\mathcal{R}$ on the vacuum state, then self-adjoint operators that act on the Hilbert space but do not commute with $\mathcal{R}$ generate canonical transformations, in parallel with the coordinate transformations that are generated by the Poisson bracket.

Appendix A shows that we can for ordinary Classical Mechanics construct a Hilbert space over $\mathbb{R}$ by a Koopman–von Neumann approach[5,6] (see Appendix A for recent references); the use of fourier transforms in field theory, however, provides a natural complex structure, allowing the construction of Hilbert spaces over $\mathbb{C}$, so that, paradoxically, random fields are somewhat closer to quantum fields than ordinary classical mechanics is to quantum mechanics. Some readers will feel that Appendix A should be in the main text of the paper, before Section II, because indeed Appendix A motivates and informs Section II, however the mathematics in the main text does not at all depend on Appendix A (which, to keep a relative simplicity, does not follow the manifest Lorentz covariance of the main text).

Hilbert spaces for random fields that are isomorphic to Hilbert spaces for quantum fields will be constructed here only for free field cases. Insofar as interacting QFT reduces to S-matrices[7] —which map from in– to out–free field Hilbert spaces—, the same S-matrix works equally as well between random field Hilbert spaces that are isomorphic to those quantum field Hilbert spaces. It's not *necessary* to construct an interacting dynamics for the random field case insofar as we already have a successful interacting dynamics for the quantum field case, after regularization and renormalization, although hopefully having a random field construction also available may suggest new avenues.

The interpretation suggested here is that we can consider the states of free field quantum electrodynamics to be classical, for which the classical dynamics is a group of canonical transformations acting on the states that is generated by a classical Hamiltonian, with a parallel unitary quantum dynamics acting on the observables that is generated by

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a quantized Hamiltonian operator. In the classical case, the action of the classical Hamiltonian requires the use of the Poisson bracket, whereas the quantized Hamiltonian has a direct adjoint action on observables. The construction in the main text, however, will preserve manifest Lorentz and translation covariance throughout, implicitly specifying a Poincaré invariant stochastic dynamics.

There is also an intention here to interpret quantum field theory as a stochastic signal analysis formalism, which in some empiricist sense quantum field theory has to be because experiments induce electrical and optical signals in cables, which are then statistically analyzed in hardware and software. A quantum field operator (i) allows us to modulate the vacuum state, and (ii) allows us to make local measurements of those modulated states, so it is often appropriate to call the test functions that parameterize these operations (taken from a Schwartz space of functions that are smooth both in real space and in wave number space) either (i) "modulation functions" or (ii) "window functions" depending on how they are used. General relativity can also be interpreted in terms of signals between places in space–time, so a signal analysis interpretation for quantum field theory introduces more possibilities for unifying quantum theory with general relativity.

The existence of such constructions does not mean that this is the way the world is. In particular, the constructions here are substantially nonunique, with there being many ways to construct the same system of states over the algebra of quantum field observables, so that it is absolutely necessary to be skeptical about any given model just as we are about Maxwell's vortices or about Bohmian trajectories. Nonetheless, the constructions here *are* manifestly Lorentz covariant, so they justify some further investigation of what advantages there might be in developing the classical dynamics of the random field states instead of developing the unitary dynamics of the quantum field states and observables. In any case, note that we are determinedly working with Hilbert spaces, for which noncommuting observables are natural, even if the way in which states are constructed can be (but does not have to be) construed as more–or–less classical.

For free quantum fields, expected values for quantum field operators depend linearly on the modulation that is applied to the vacuum. From a signal analysis perspective, that is a convenience more than a necessity, surely not definite enough for it to be enshrined in axioms (as it is directly by the Wightman axioms, by requiring quantum fields to be distributions, but only indirectly by the Haag-Kastler axioms, through additivity). Response to modulations is in general not linear in physics (except as a first approximation or because we engineer the response to be linear over as large a range as we can.) Thus, we might usefully introduce nonlinear dependence on test functions (which is well-defined) instead of introducing powers of distributions as interaction terms (which is not). If we take the renormalization scale that is required to construct interacting theories to be fixed by or at least to be correlated with parameters of the test functions used in detailed models of an experiment, then interacting theories are already weakly nonlinearly dependent on the test functions.

The notation used here may be offputtingly novel except for mathematical physicists, however it has a moderately principled motivation as an intrinsic vector formalism in the test function space (which for free fields are equipped with a pre–inner product).

II. THE ELECTROMAGNETIC FIELD

We can construct the quantized electromagnetic field $\hat{\mathsf{F}}_{\mathsf{f}} = a_{\mathsf{f}^*} + a^\dagger_{\mathsf{f}}$, $\hat{\mathsf{F}}^\dagger_{\mathsf{f}} = \hat{\mathsf{F}}_{\mathsf{f}^*}$, for bivector test functions $\mathsf{f}_{\mu\nu}(x)$, using a set of raising and lowering operators for which

$$\langle 0|\,\hat{\mathsf{F}}^\dagger_{\mathsf{f}}\hat{\mathsf{F}}_{\mathsf{g}}\,|0\rangle = [a_{\mathsf{f}}, a^\dagger_{\mathsf{g}}] = (\mathsf{f},\mathsf{g})_+, \qquad [a_{\mathsf{f}}, a_{\mathsf{g}}] = 0, \qquad \langle 0|a^\dagger_{\mathsf{f}} = 0 = a_{\mathsf{f}}|0\rangle,$$

where

$$\begin{aligned}(\mathsf{f},\mathsf{g})_\pm &= -\hbar\int \widetilde{\delta\mathsf{f}}^*(k)\cdot\widetilde{\delta\mathsf{g}}(k) 2\pi\delta(k\cdot k)\theta(\pm k_0)\frac{\mathrm{d}^4k}{(2\pi)^4}\\ &= -\hbar\int k^\alpha\tilde{\mathsf{f}}^*_{\alpha\mu}(k)g^{\mu\nu}k^\beta\tilde{\mathsf{g}}_{\beta\nu}(k)2\pi\delta(k\cdot k)\theta(\pm k_0)\frac{\mathrm{d}^4k}{(2\pi)^4},\end{aligned}$$

which are positive semi–definite sesquilinear forms on the vector space of test functions because $k^\alpha\tilde{\mathsf{f}}^*_{\alpha\mu}(k)$ and $k^\beta\tilde{\mathsf{g}}_{\beta\nu}(k)$ are space–like 4–vectors that are orthogonal to the light–like 4–vector k (the form of this *pre–inner product* is derived in Ref. 8, Eq. (3.27), for example). The commutation relation $[\hat{\mathsf{F}}_{\mathsf{f}}, \hat{\mathsf{F}}_{\mathsf{g}}] = (\mathsf{f}^*,\mathsf{g}) - (\mathsf{g}^*,\mathsf{f})$ is zero if the supports of f and g are space–like separated.

All of the above can be derived by constructing $\hat{\mathsf{F}}_{\mathsf{f}}$ as $\hat{\mathsf{F}}_{\mathsf{f}}=\int \mathsf{f}^{\mu\nu}(x)\hat{\mathsf{F}}_{\mu\nu}(x)\mathrm{d}^4x$, but a principal aim of the notation is to work intrinsically in test function space as far as possible. We can revert to a point–like quantum field at a point y by taking an improper test function $\mathsf{f}^{\mu\nu}(x)$ that is a multiple of a Dirac delta function $\delta(x-y)$ or to the fourier transform of the quantum field at wave–number k by taking an improper test function that is a multiple of $\mathrm{e}^{\mathrm{i}k\cdot x}$, however careful models typically use less singular test functions that include details such as line widths or pulse durations; indeed experiments may be intended to produce states, such as Bessel beams, that require very carefully shaped test functions as models. It is helpful that for the real Klein–Gordon quantum field an intrinsic vector in a test function space formalism has exactly the same presentation, except only that the pre–inner product for scalar test functions f and g is different,

$$(f,g)_+=\hbar\int \tilde{f}^*(k)\tilde{g}(k)2\pi\delta(k\cdot k-m^2)\theta(k_0)\frac{\mathrm{d}^4k}{(2\pi)^4},$$

and the same presentation is possible for the complex Klein–Gordon quantum field, given in Appendix B. It is also helpful that it is easy to make a connection to elementary discussions of quantum mechanics if we simply abbreviate $\hat{\mathsf{F}}_{\mathsf{f}_1}$, $\hat{\mathsf{F}}_{\mathsf{f}_2}$, ..., $\hat{\mathsf{F}}_{\mathsf{f}_n}$, as $\hat{\mathsf{F}}_1$, $\hat{\mathsf{F}}_2$, ..., $\hat{\mathsf{F}}_n$, and similarly for raising and lowering operators.

We introduce projection to left and right helicity, $\mathsf{f}\mapsto\frac{1}{2}(1\pm\mathrm{i}\star)\mathsf{f}$, using the Hodge dual $\star$, $[\star\mathsf{f}]_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu}{}^{\alpha\beta}\mathsf{f}_{\alpha\beta}$, $\star\star\mathsf{f}=-\mathsf{f}$ (when acting on 2–forms), so that we can construct raising and lowering operators for independent left and right helicity components of the quantized electromagnetic field,

$$\begin{aligned}
&\hat{\mathsf{l}}_{\mathsf{f}}=a_{\frac{1}{2}(1+\mathrm{i}\star)\mathsf{f}}, \qquad \hat{\mathsf{r}}_{\mathsf{f}}=a_{\frac{1}{2}(1-\mathrm{i}\star)\mathsf{f}}, \qquad [\hat{\mathsf{l}}_{\mathsf{f}},\hat{\mathsf{r}}^\dagger_{\mathsf{g}}]=0,\\
&[\hat{\mathsf{l}}_{\mathsf{f}},\hat{\mathsf{l}}^\dagger_{\mathsf{g}}]=(\mathsf{f},\tfrac{1}{2}(1+\mathrm{i}\star)\mathsf{g})_+, \quad [\hat{\mathsf{r}}_{\mathsf{f}},\hat{\mathsf{r}}^\dagger_{\mathsf{g}}]=(\mathsf{f},\tfrac{1}{2}(1-\mathrm{i}\star)\mathsf{g})_+,\\
&\hat{\mathsf{F}}_{\mathsf{f}}=\hat{\mathsf{l}}_{\mathsf{f}^*}+\hat{\mathsf{l}}^\dagger_{\mathsf{f}}+\hat{\mathsf{r}}_{\mathsf{f}^*}+\hat{\mathsf{r}}^\dagger_{\mathsf{f}}.
\end{aligned}$$

Using the left and right helicity components and adopting the reversal introduced in Ref. 1, $\mathsf{f}^-(x)=\mathsf{f}(-x)$, $\widetilde{\mathsf{f}^-}(k)=\tilde{\mathsf{f}}(-k)$, we construct one of the possible alternative field objects, $\hat{\mathbb{F}}_{\mathsf{f}}$, which will prove to be classical in the sense that the commutator is zero for all test functions, $[\hat{\mathbb{F}}_{\mathsf{f}},\hat{\mathbb{F}}_{\mathsf{g}}]=0$,

$$\begin{aligned}
\hat{\mathbb{F}}_{\mathsf{f}}&=\hat{\mathsf{l}}_{\mathsf{f}^*}+\hat{\mathsf{l}}^\dagger_{\mathsf{f}}+\hat{\mathsf{r}}_{\mathsf{f}^{-*}}+\hat{\mathsf{r}}^\dagger_{\mathsf{f}^-}, \qquad\qquad \hat{\mathbb{F}}^\dagger_{\mathsf{f}}=\hat{\mathbb{F}}_{\mathsf{f}^*},\\
&=a_{\frac{1}{2}(1+\mathrm{i}\star)\mathsf{f}^*+\frac{1}{2}(1-\mathrm{i}\star)\mathsf{f}^{-*}}+a^\dagger_{\frac{1}{2}(1+\mathrm{i}\star)\mathsf{f}+\frac{1}{2}(1-\mathrm{i}\star)\mathsf{f}^-}
\end{aligned}$$

so that, using the identities $(\mathsf{f}^*,\mathsf{g}^*)_\pm=(\mathsf{g},\mathsf{f})_\mp$ and $(\mathsf{f}^-,\mathsf{g}^-)_\pm=(\mathsf{f},\mathsf{g})_\mp$,

$$\begin{aligned}
\langle 0|\,\hat{\mathbb{F}}_{\mathsf{f}}\hat{\mathbb{F}}_{\mathsf{g}}\,|0\rangle&=\langle 0|\left[\hat{\mathsf{l}}_{\mathsf{f}^*}\hat{\mathsf{l}}^\dagger_{\mathsf{g}}+\hat{\mathsf{r}}_{\mathsf{f}^{-*}}\hat{\mathsf{r}}^\dagger_{\mathsf{g}^-}\right]|0\rangle\\
&=\Big(\tfrac{1}{2}(1{+}\mathrm{i}\star)\mathsf{f}^*,\tfrac{1}{2}(1{+}\mathrm{i}\star)\mathsf{g}\Big)_+ +\Big(\tfrac{1}{2}(1{-}\mathrm{i}\star)\mathsf{f}^{-*},\tfrac{1}{2}(1{-}\mathrm{i}\star)\mathsf{g}^-\Big)_+\\
&=\Big(\tfrac{1}{2}(1{+}\mathrm{i}\star)\mathsf{f}^*,\tfrac{1}{2}(1{+}\mathrm{i}\star)\mathsf{g}\Big)_+ +\Big(\tfrac{1}{2}(1{+}\mathrm{i}\star)\mathsf{g}^{-*},\tfrac{1}{2}(1{+}\mathrm{i}\star)\mathsf{f}^-\Big)_-\\
&=\Big(\tfrac{1}{2}(1{+}\mathrm{i}\star)\mathsf{f}^*,\tfrac{1}{2}(1{+}\mathrm{i}\star)\mathsf{g}\Big)_+ +\Big(\tfrac{1}{2}(1{+}\mathrm{i}\star)\mathsf{g}^*\ \ ,\tfrac{1}{2}(1{+}\mathrm{i}\star)\mathsf{f}\ \ \Big)_+\\
&=\langle 0|\,\hat{\mathbb{F}}_{\mathsf{g}}\hat{\mathbb{F}}_{\mathsf{f}}\,|0\rangle
\end{aligned}$$

is symmetric, so we have shown that $[\hat{\mathbb{F}}_{\mathsf{f}},\hat{\mathbb{F}}_{\mathsf{g}}]=0$. $\hat{\mathbb{F}}_{\mathsf{f}}$ is a classical Gaussian observable with variance $\langle 0|\,\hat{\mathbb{F}}_{\mathsf{f}}\hat{\mathbb{F}}_{\mathsf{f}}\,|0\rangle$ whenever $\mathsf{f}=\mathsf{f}^*$; however $\hat{\mathbb{F}}_{\mathsf{f}}$, as a normal operator, is an observable even when $\mathsf{f}\neq\mathsf{f}^*$ in the sense that all components of its real and imaginary parts are jointly measurable. Note that both the random field and the quantum field are translation invariant even though the transformation between them is not, because $(\mathsf{f},\mathsf{g})_\pm$ are both translation invariant.

For any state that we could construct by the action of a function of $\hat{\mathsf{F}}_{\mathsf{f}_1},...,\hat{\mathsf{F}}_{\mathsf{f}_n}$ on the vacuum vector $|0\rangle$, for some set of bivector test functions $\{\mathsf{f}_i(x)\}$, we can construct the same state by the action of some function of $\hat{\mathbb{F}}_{\mathsf{f}_1^\bullet},...,\hat{\mathbb{F}}_{\mathsf{f}_n^\bullet}$, because the linear map $\mathsf{f}\mapsto\mathsf{f}^\bullet=\frac{1}{2}(1+\mathrm{i}\star)\mathsf{f}+\frac{1}{2}(1-\mathrm{i}\star)\mathsf{f}^-$ is an involution, $\mathsf{f}^{\bullet\bullet}=\mathsf{f}$. Consequently, we can, if we wish, say that the states are classical even if we continue to say that the measurements are nontrivially quantum mechanical. The algebras that are generated by $\hat{\mathsf{F}}_{\mathsf{f}}$ and by $\hat{\mathbb{F}}_{\mathsf{f}}$ are both subalgebras of the commonplace raising and lowering algebra that is generated by a_{f} and $a^\dagger_{\mathsf{f}}$,

$$\hat{\mathsf{F}}_{\mathsf{f}}=a_{\mathsf{f}^*}+a^\dagger_{\mathsf{f}}, \qquad \hat{\mathbb{F}}_{\mathsf{f}}=a_{\mathsf{f}^{*\bullet}}+a^\dagger_{\mathsf{f}^\bullet},$$

which are different because $\mathsf{f}^{*\bullet}\neq\mathsf{f}^{\bullet *}$, so we can more–or–less say that the electromagnetic random field we have constructed here has been hiding in plain sight. With this presentation of $\hat{\mathsf{F}}_\mathsf{f}$ and of $\hat{\mathbb{F}}_\mathsf{f}$ and using the vacuum state, the isomorphism of the two Hilbert spaces they generate can be presented as an equality of normal–ordered expressions,

$$a^\dagger_{\mathsf{g}_1}\cdots a^\dagger_{\mathsf{g}_n}|0\rangle = {:}\hat{\mathsf{F}}_{\mathsf{g}_1}\cdots\hat{\mathsf{F}}_{\mathsf{g}_n}{:}|0\rangle = {:}\hat{\mathbb{F}}_{\mathsf{g}_1^\bullet}\cdots\hat{\mathbb{F}}_{\mathsf{g}_n^\bullet}{:}|0\rangle,$$

with the Hilbert space inner product determined by the pre–inner product

$$\langle 0|\, a_{\mathsf{f}_n}\cdots a_{\mathsf{f}_1} a^\dagger_{\mathsf{g}_1}\cdots a^\dagger_{\mathsf{g}_n}\,|0\rangle.$$

We can equally take the opposite perspective, however, that the quantized electromagnetic field was always hiding in plain sight in the full algebra of observables of classical electromagnetism, when we apply the constructions of Appendix A.

When considering how committed we should be to either a quantum or a random field perspective, it is instructive to consider how flexibly committed quantum field theory is to microcausality. In particular, quantum field models commonly introduce state transition probabilities as models of measurements, $|\langle S_1|S_2\rangle|^2=\langle S_1|S_2\rangle\langle S_2|S_1\rangle$, measurements of the projection–valued observable $|S_2\rangle\langle S_2|$ in the state $\langle S_1|\cdot|S_1\rangle$ (with both normalized); weighted sums of projection operators $|S_i\rangle\langle S_i|$ generate the space of normal operators on the Hilbert space of states. The algebra of observables generated by $|S_i\rangle\langle S_i|$ can alternatively be generated by adding just the vacuum projection operator $|0\rangle\langle 0|$ to the quantum field operators $\hat{\mathsf{F}}_\mathsf{f}$ (or to the random field operators $\hat{\mathbb{F}}_\mathsf{f}$), so we can construct operators of the form

$$\hat{\mathsf{O}}(\underline{\mathsf{g}};\underline{\mathsf{f}}) = a^\dagger_{\mathsf{g}_1}\cdots a^\dagger_{\mathsf{g}_n}|0\rangle\langle 0|a_{\mathsf{f}_n}\cdots a_{\mathsf{f}_1}$$

and sums of such operators, in which case we note that $|0\rangle\langle 0|$ and $\hat{\mathsf{O}}(\underline{\mathsf{g}};\underline{\mathsf{f}})$ are essentially global operators, $\big[|0\rangle\langle 0|,\hat{\mathsf{F}}_\mathsf{f}\big]\neq 0$, $\big[\hat{\mathsf{O}}(\underline{\mathsf{g}};\underline{\mathsf{f}}),\hat{\mathsf{F}}_\mathsf{f}\big]\neq 0\,\forall\mathsf{f}$ (and, for the random field $\hat{\mathbb{F}}_\mathsf{f}$, $\big[|0\rangle\langle 0|,\hat{\mathbb{F}}_\mathsf{f}\big]\neq 0$, $\big[\hat{\mathsf{O}}(\underline{\mathsf{g}};\underline{\mathsf{f}}),\hat{\mathbb{F}}_\mathsf{f}\big]\neq 0\ \forall\mathsf{f}$). Quantum field models commonly also discuss the measurement of number operators such as $\hat{\mathsf{N}}_\mathsf{f}{=}a^\dagger_\mathsf{f}a_\mathsf{f}$, which also do not satisfy microcausality. If we allow the same flexibility for random field models, as indeed is appropriate for even moderately sophisticated classical signal analysis (where time–frequency analysis for non-stochastic signals is well–known to require Wigner and other quasi-distributions[9] because of the ubiquitous use of the fourier transform), the algebras of observables as well as the Hilbert spaces are identical.

For the quantized electromagnetic field, the algebra of operators that is generated by $\hat{\mathsf{F}}_\mathsf{f}$ *and* the vacuum projection operator $|0\rangle\langle 0|$, is isomorphic to an algebra of operators that is generated by $\hat{\mathbb{F}}_f$ *and* the vacuum projection operator $|0\rangle\langle 0|$. The algebras of operators that are generated by $\hat{\mathsf{F}}_\mathsf{f}$ and by $\hat{\mathbb{F}}_f$ are both subalgebras of the algebra generated by the raising and lowering operators.

A significant difference between the random and quantum free field algebras is the *irreducibility*[10] (p.101) (also referred to as *completeness*[11] (§II.1.2)) of the algebra generated by the quantum field operators $\hat{\mathsf{F}}_\mathsf{f}$. Because of irreducibility, bounded commutative algebras generated by $\hat{\mathbb{F}}_\mathsf{f}$ are subalgebras of the algebra generated by $\hat{\mathsf{F}}_\mathsf{f}$, so that from this perspective it is possible to conclude that the algebra of quantum field operators is prior to the algebra of random field operators. The Poisson or Peierls bracket, however, generates an algebra of transformations of the space of states or trajectories, so that if we include canonical transformations between different commutative subalgebras as an intrinsic part of classical physics then there is no difference between the two constructions, because classical physics then does not give preference to any one maximally commutative subalgebra.

For any quantum field for which there is an involution on the test function space, ${}^\bullet:\mathcal{S}\to\mathcal{S}; f\mapsto f^\bullet, f^{\bullet\bullet}=f$ and for which the inner product satisfies $(f^{*\bullet},g^\bullet)=(g^{*\bullet},f^\bullet)$, whereas in general $(f^*,g)\neq(g^*,f)$, we can construct a random field that is equivalent in the sense we have seen for the quantized electromagnetic field (the construction for the complex Klein-Gordon quantum field is given in Appendix B). This can be thought of as introducing a new type of reflection positivity[12], which, however, preserves the 1+3–signature of space–time.

We have presented the construction above using raising and lowering operators. Avoiding their use, we can present the quantum and random field structures for the electromagnetic field as

<table>
<tr><td>$\hat{\mathsf{F}}^\dagger_\mathsf{f}=\hat{\mathsf{F}}_{\mathsf{f}^*}$
$[\hat{\mathsf{F}}_\mathsf{f},\hat{\mathsf{F}}_\mathsf{g}]=(\mathsf{f}^*,\mathsf{g})_+-(\mathsf{g}^*,\mathsf{f})_+$
$\langle 0|\,\hat{\mathsf{F}}_\mathsf{f}\hat{\mathsf{F}}_\mathsf{g}\,|0\rangle=(\mathsf{f}^*,\mathsf{g})_+$</td><td>$\hat{\mathbb{F}}^\dagger_\mathsf{f}=\hat{\mathbb{F}}_{\mathsf{f}^*}$
$[\hat{\mathbb{F}}_\mathsf{f},\hat{\mathbb{F}}_\mathsf{g}]=0$
$\langle 0|\,\hat{\mathbb{F}}_\mathsf{f}\hat{\mathbb{F}}_\mathsf{g}\,|0\rangle=(\mathsf{f}^{*\bullet},\mathsf{g}^\bullet)_+$</td></tr>
<tr><td colspan="2">${:}\hat{\mathsf{F}}_{\mathsf{g}_1}\cdots\hat{\mathsf{F}}_{\mathsf{g}_n}{:}|0\rangle={:}\hat{\mathbb{F}}_{\mathsf{g}_1^\bullet}\cdots\hat{\mathbb{F}}_{\mathsf{g}_n^\bullet}{:}|0\rangle$</td></tr>
</table>

with all other connected Wightman functions being zero. The final equality, which assumes appropriately different definitions of normal–orderings for the two cases, identifies the Hilbert spaces of the two theories, defining an action of the algebra generated by the $\hat{\mathsf{F}}_{\mathsf{f}}$ on the Hilbert space generated by $\hat{\mathbb{F}}_{\mathsf{f}}$. It is also worthwhile to present the construction above in a more unified Weyl–like coherent state formalism, using $\hat{\mathsf{W}}(\mathsf{f})=\mathrm{e}^{\mathrm{i}\hat{\mathsf{F}}_{\mathsf{f}}}$ and $\hat{\mathbb{W}}(\mathsf{f})=\mathrm{e}^{\mathrm{i}\hat{\mathbb{F}}_{\mathsf{f}}}$,

<table>
<tr>
<td>$\hat{\mathsf{W}}(\mathsf{f})^\dagger=\hat{\mathsf{W}}(-\mathsf{f}^*)$
$\hat{\mathsf{W}}(\mathsf{f})\hat{\mathsf{W}}(\mathsf{g})=\mathrm{e}^{-[(\mathsf{f}^*,\mathsf{g})_+-(\mathsf{g}^*,\mathsf{f})_+]/2}\hat{\mathsf{W}}(\mathsf{f}+\mathsf{g})$
$\langle 0|\,\hat{\mathsf{W}}(\mathsf{f})\,|0\rangle=\mathrm{e}^{-(\mathsf{f}^*,\mathsf{f})_+/2}$</td>
<td>$\hat{\mathbb{W}}(\mathsf{f})^\dagger=\hat{\mathbb{W}}(-\mathsf{f}^*)$
$\hat{\mathbb{W}}(\mathsf{f})\hat{\mathbb{W}}(\mathsf{g})=\hat{\mathbb{W}}(\mathsf{f}+\mathsf{g})$
$\langle 0|\,\hat{\mathbb{W}}(\mathsf{f})\,|0\rangle=\mathrm{e}^{-(\mathsf{f}^{*\bullet},\mathsf{f}^\bullet)_+/2}$</td>
</tr>
<tr>
<td colspan="2">$\mathrm{e}^{(\mathsf{f}^*,\mathsf{f})_+/2}\hat{\mathsf{W}}(\mathsf{f})|0\rangle=\mathrm{e}^{(\mathsf{f}^{\bullet*\bullet},\mathsf{f})_+/2}\hat{\mathbb{W}}(\mathsf{f}^\bullet)|0\rangle$</td>
</tr>
</table>

which can be seen to equate appropriately scaled coherent states.

So as not to delay the discussion of the quantized Dirac spinor field, a slightly complicated discussion of potentials for $\hat{\mathbb{F}}_{\mathsf{f}}$ is presented in Appendix C.

III. THE QUANTIZED DIRAC SPINOR FIELD

First a terse presentation of the quantized Dirac spinor field: we can construct the field operators $\hat{\psi}_U=d_{U^c}+b_U^\dagger$ for a Dirac spinor test function U using anticommuting raising and lowering operators for which

$$\begin{aligned}
&\{b_U,b_V^\dagger\}=\{d_U,d_V^\dagger\}=(U,V)_+,\\
&\{b_U,b_V\}=\{b_U,d_V\}=\{d_U,d_V\}=\{b_U,d_V^\dagger\}=0,\\
&\langle 0|b_V^\dagger=\langle 0|d_V^\dagger=0=d_V|0\rangle=b_V|0\rangle
\end{aligned}$$

where for the pre–inner product $(U,V)_+$ and its negative–frequency counterpart $(U,V)_-$, which is also positive semi–definite, we have

$$(U,V)_\pm=\pm\,\hbar\!\int\!\overline{\tilde{U}(k)}(k{\cdot}\gamma+m)\tilde{V}(k)\theta(\pm k_0)\mathrm{d}\mu_m(k);$$

the measure $\mathrm{d}\mu_m(k)=2\pi\delta(k{\cdot}k-m^2)\frac{\mathrm{d}^4k}{(2\pi)^4}$ on the wave–number space is zero except on the forward and backward mass shells for mass m. With this construction,

$$\begin{aligned}
\langle 0|\,\hat{\psi}_U^\dagger\hat{\psi}_V\,|0\rangle&=\langle 0|\,b_Ub_V^\dagger\,|0\rangle=(U,V)_+\\
\langle 0|\,\hat{\psi}_V\hat{\psi}_U^\dagger\,|0\rangle&=\langle 0|\,d_{V^c}d_{U^c}^\dagger\,|0\rangle=(V^c,U^c)_+=(U,V)_-\\
\{\hat{\psi}_U^\dagger,\hat{\psi}_V\}&=(U,V),\quad \{\hat{\psi}_U,\hat{\psi}_V\}=0,
\end{aligned}$$

where $(U,V)=(U,V)_++(U,V)_-$ [see below for a derivation of $(V^c,U^c)_\pm=(U,V)_\mp$]. The anti–commutator $\{\hat{\psi}_U^\dagger,\hat{\psi}_V\}$ is zero if the supports of the test functions U and V are space–like separated. Explicitly, the vital equalities $(U^c,V^c)_\pm=(V,U)_\mp$ depend on the identities $\overline{A^c}\gamma^\mu B^c=\overline{B}\gamma^\mu A$ and $\overline{A^c}\,B^c=-\overline{B}\,A$, because of which the arbitrary phase introduced by charge conjugation cancels,

$$\begin{aligned}
(U^c,V^c)_\pm&=\pm\hbar\!\int\!\overline{\widetilde{U^c}(k)}(k{\cdot}\gamma+m)\widetilde{V^c}(k)\theta(\pm k_0)\mathrm{d}\mu_m(k)\\
&=\pm\hbar\!\int\!\overline{\left[\widetilde{V^c}(k)\right]^c}(k{\cdot}\gamma-m)\left[\widetilde{U^c}(k)\right]^c\theta(\pm k_0)\mathrm{d}\mu_m(k)\\
&=\pm\hbar\!\int\!\overline{\tilde{V}(-k)}(k{\cdot}\gamma-m)\tilde{U}(-k)\theta(\pm k_0)\mathrm{d}\mu_m(k)\\
&=\mp\hbar\!\int\!\overline{\tilde{V}(k)}(k{\cdot}\gamma+m)\tilde{U}(k)\theta(\mp k_0)\mathrm{d}\mu_m(k)\\
&=(V,U)_\mp,\ \text{ so that, also, }(U^c,V^c)=(V,U).
\end{aligned}$$

All of the above can be derived by setting $\hat{\psi}_U=\int\overline{U^c(x)}\hat{\psi}(x)\mathrm{d}^4x$ (with charge conjugation introduced to ensure that the inessential but usual convention for quantum fields is followed, that $\hat{\psi}_U$ is linear in the test function U), but, again, it is a principal aim to work intrinsically in test function space as far as possible.

To proceed, we note that it is generally understood that for an operator to be an observable of the quantized Dirac spinor free field formalism it has to be invariant under global-$U(1)$ transformations ($U(1)$–gauge transformations are considered briefly in Section IV), for which the simplest case, $\hat{\Phi}_U = \hat{\psi}_U^\dagger \hat{\psi}_U = \hat{\Phi}_U^\dagger$, is a multiple of a projection operator, $\hat{\Phi}_U^{\,2} = \hat{\psi}_U^\dagger \hat{\psi}_U \hat{\psi}_U^\dagger \hat{\psi}_U = \hat{\psi}_U^\dagger \left((U,U) - \hat{\psi}_U^\dagger \hat{\psi}_U\right) \hat{\psi}_U = (U,U)\hat{\Phi}_U$, so that $\frac{\hat{\Phi}_U}{(U,U)}$ can be used to model yes/no measurement results, and is also local in that the commutator is

$$
\begin{aligned}
[\hat{\Phi}_U, \hat{\Phi}_V] &= [\hat{\psi}_U^\dagger \hat{\psi}_U, \hat{\psi}_V^\dagger \hat{\psi}_V] \\
&= \hat{\psi}_U^\dagger [\hat{\psi}_U, \hat{\psi}_V^\dagger \hat{\psi}_V] + [\hat{\psi}_U^\dagger, \hat{\psi}_V^\dagger \hat{\psi}_V] \hat{\psi}_U \\
&= (V,U) \hat{\psi}_U^\dagger \hat{\psi}_V - (U,V) \hat{\psi}_V^\dagger \hat{\psi}_U \\
&= \mathrm{i}\sqrt{(U,U)(V,V)} \Big[\hat{\Phi}_{Y(V,U)} - \hat{\Phi}_{Y(U,V)} \Big],
\end{aligned}
$$

where

$$
Y(U,V) = U\sqrt{\frac{\sqrt{(U,V)(V,U)}}{2(U,U)}} + \ \mathrm{i}\frac{V(V,U)}{\sqrt{2(V,V)\sqrt{(U,V)(V,U)}}},
$$

so that $[\hat{\Phi}_U, \hat{\Phi}_V]$ is zero when the test functions U and V have space–like separated supports and so that the commutator $[\hat{\Phi}_U, \hat{\Phi}_V]$ is closed in the algebra generated by $\hat{\Phi}_U$ (note that $Y(U,V)$ is defined only up to a phase, but the presentation here uses only $\sqrt{(U,V)(V,U)}$, avoiding the use of $\sqrt{(U,V)}$ alone or of $\sqrt{(V,U)}$). It is sufficient to consider *only* $\hat{\Phi}_U$ because any global-$U(1)$ invariant observable can be put into alternating $\hat{\psi}_U^\dagger$, $\hat{\psi}_V$ operator form, and, by polarization, using linearity and anti–linearity without using commutation relations,

$$
\hat{\psi}_U^\dagger \hat{\psi}_V = \frac{1}{4} \left[\hat{\Phi}_{U+V} - \hat{\Phi}_{U-V} - \mathrm{i}\hat{\Phi}_{U+\mathrm{i}V} + \mathrm{i}\hat{\Phi}_{U-\mathrm{i}V} \right],
$$

however it will often be convenient to use $\hat{\psi}_U^\dagger \hat{\psi}_V$, particularly noting the projection property $(\hat{\psi}_U^\dagger \hat{\psi}_V)^n = (U,V)^{n-1} \hat{\psi}_U^\dagger \hat{\psi}_V$, more–or–less as for $\hat{\Phi}_U$. The properties of $\hat{\Phi}_U$ ensure that it behaves rather like a decimation operator relative to lower–level observables $\hat{\Phi}_{U_i}$ (lower–level in the sense that $\mathrm{Supp}(U_i) \subset \mathrm{Supp}(U)$), with which it is in general incompatible. The vacuum vector allows us to construct the vacuum state $\langle 0| \cdot |0\rangle$ over the algebra generated by $\hat{\Phi}_U$ and non–vacuum states such as

$$
\frac{\langle 0| \hat{\Phi}_{V_1} \cdots \hat{\Phi}_{V_n} \quad \cdot \quad \hat{\Phi}_{V_n} \cdots \hat{\Phi}_{V_1} |0\rangle}{\langle 0| \hat{\Phi}_{V_1} \cdots \hat{\Phi}_{V_n} \hat{\Phi}_{V_n} \cdots \hat{\Phi}_{V_1} |0\rangle}.
$$

The intentional restriction to using only the observable operators $\hat{\Phi}_V$ to generate states, which can be characterized as zero charge states, is not operationally significant, because we can always construct operators $\hat{\psi}_U^\dagger \hat{\psi}_V$ for which either U or V is at arbitrarily large separation from the region of space–time that contains an experiment.

We now introduce a bosonic raising and lowering algebra that includes an observable that satisfies the same Lie algebra as is satisfied by $\hat{\Phi}_U$,

$$
[\mathsf{a}_U, \mathsf{a}_V^\dagger] = (U,V), \qquad [\mathsf{a}_U, \mathsf{a}_V] = 0,
$$

for which $\hat{\mathsf{X}}_U = \mathsf{a}_{U^c}^\dagger \mathsf{a}_{U^c}$ satisfies

$$
\begin{aligned}
[\hat{\mathsf{X}}_U, \hat{\mathsf{X}}_V] &= [\mathsf{a}_{U^c}^\dagger \mathsf{a}_{U^c}, \mathsf{a}_{V^c}^\dagger \mathsf{a}_{V^c}], \\
&= \mathsf{a}_{U^c}^\dagger [\mathsf{a}_{U^c}, \mathsf{a}_{V^c}^\dagger \mathsf{a}_{V^c}] + [\mathsf{a}_{U^c}^\dagger, \mathsf{a}_{V^c}^\dagger \mathsf{a}_{V^c}] \mathsf{a}_{U^c} \\
&= (U^c, V^c) \mathsf{a}_{U^c}^\dagger \mathsf{a}_{V^c} - (V^c, U^c) \mathsf{a}_{V^c}^\dagger \mathsf{a}_{U^c} \\
&= (V,U) \mathsf{a}_{U^c}^\dagger \mathsf{a}_{V^c} - (U,V) \mathsf{a}_{V^c}^\dagger \mathsf{a}_{U^c} \\
&= \mathrm{i}\sqrt{(U,U)(V,V)} \Big[\hat{\mathsf{X}}_{Y(V,U)} - \hat{\mathsf{X}}_{Y(U,V)} \Big]
\end{aligned}
$$

allowing the identification $\hat{\mathsf{X}}_U \equiv \hat{\Phi}_U$, or, equivalently, $\mathsf{a}_{U^c}^\dagger \mathsf{a}_{V^c} \equiv \hat{\psi}_U^\dagger \hat{\psi}_V$ (with care taken to ensure equivalence of complex linearity and anti–linearity in U and in V). The construction so far can be compared with the Jordan-Wigner transformation[13](§15.1),[14]. We can construct a real random field using a_U and $\mathsf{a}_U^\dagger$,

$$
\hat{\chi}_U = \mathsf{a}_{U^c} + \mathsf{a}_U^\dagger, \qquad \hat{\chi}_U^\dagger = \hat{\chi}_{U^c},
$$

with trivial commutator,

$$\begin{aligned}[\hat\chi_U,\hat\chi_V] &= [\mathsf{a}_{U^c},\mathsf{a}_V^\dagger]+[\mathsf{a}_U^\dagger,\mathsf{a}_{V^c}]\\ &= (U^c,V)-(V^c,U)=0.\end{aligned}$$

The next step is for us to extend the state we have over the algebra generated by the global-$U(1)$ invariant observables $\hat\Phi_U$, which is therefore also a state over the $\hat{\mathsf{X}}_U$, to be a state $\langle\mathcal{F}|\cdot|\mathcal{F}\rangle$ over the algebra generated by $\mathsf{a}_{U^c}^\dagger$ and a_{V^c}, which we will do here by the simplest possible prescription, that any term in an expanded expression that cannot be presented as a product of factors $\hat{\mathsf{X}}_U$ will be assigned the value 0. The notation $\langle\mathcal{F}|\cdot|\mathcal{F}\rangle$ intends to emphasize that the vacuum vector $|\mathcal{F}\rangle$ is not annihilated by a_{V^c}, just as $|0\rangle$ is not annihilated by $\hat\psi_V$.

The resulting state can be fixed by constructing a generating function for the random field $\hat\chi_U$,

(apply a Baker–Campbell–Hausdorff identity ...)

$$\langle\mathcal{F}|\,\mathrm{e}^{\mathrm{i}\lambda\hat\chi_U}\,|\mathcal{F}\rangle=\langle\mathcal{F}|\,\mathrm{e}^{\mathrm{i}\lambda\mathsf{a}_U^\dagger}\mathrm{e}^{\mathrm{i}\lambda\mathsf{a}_{U^c}}\,|\mathcal{F}\rangle\mathrm{e}^{-\lambda^2(U^c,U)/2}$$

(only include terms with equal numbers of $\mathsf{a}_U^\dagger$, a_{U^c} ...)

$$=\langle\mathcal{F}|\,1+\sum_{j=1}^\infty\frac{(-\lambda^2)^j}{j!^2}(\mathsf{a}_U^\dagger)^j(\mathsf{a}_{U^c})^j\,|\mathcal{F}\rangle\mathrm{e}^{-\lambda^2(U^c,U)/2}$$

(use the commutator $[\mathsf{a}_{U^c},\mathsf{a}_U^\dagger]=(U^c,U)$, giving $\mathsf{a}_U^{\dagger j}\mathsf{a}_{U^c}{=}\Big(\mathsf{a}_U^\dagger\mathsf{a}_{U^c}{-}(j{-}1)(U^c,U)\Big)\mathsf{a}_U^{\dagger(j-1)}$ ...)

$$=\langle\mathcal{F}|\,1+\sum_{j=1}^\infty\frac{(-\lambda^2)^j}{j!^2}\Big(\mathsf{a}_U^\dagger\mathsf{a}_{U^c}{-}(j{-}1)(U^c,U)\Big)\times\cdots\times\Big(\mathsf{a}_U^\dagger\mathsf{a}_{U^c}{-}(U^c,U)\Big)\mathsf{a}_U^\dagger\mathsf{a}_{U^c}\,|\mathcal{F}\rangle\mathrm{e}^{-\lambda^2(U^c,U)/2}$$

(only at this point do we map $\mathsf{a}_U^\dagger\mathsf{a}_{U^c}$ to $\hat\psi_{U^c}^\dagger\hat\psi_U$ and $\langle\mathcal{F}|\cdot|\mathcal{F}\rangle$ to $\langle0|\cdot|0\rangle$, ...)

$$=\langle0|1+\sum_{j=1}^\infty\frac{(-\lambda^2)^j}{j!^2}\Big(\hat\psi_{U^c}^\dagger\hat\psi_U{-}(j{-}1)(U^c,U)\Big)\times\;\cdots\;\times\Big(\hat\psi_{U^c}^\dagger\hat\psi_U{-}(U^c,U)\Big)\hat\psi_{U^c}^\dagger\hat\psi_U|0\rangle\mathrm{e}^{-\lambda^2(U^c,U)/2}$$

(only the constant and the $-\lambda^2$ terms survive ...)

$$=(1-\lambda^2(U^c,U)_+)\mathrm{e}^{-\lambda^2(U^c,U)/2}$$

$\Big[$more generally, we have $\langle\mathcal{F}|\,\mathrm{e}^{\mathrm{i}\lambda\mathsf{a}_{U'}+\mathrm{i}\mu\mathsf{a}_U^\dagger}\,|\mathcal{F}\rangle=(1-\lambda\mu(U',U)_+)\mathrm{e}^{-\lambda\mu(U',U)/2}$ as a generating function for the whole raising and lowering algebra$\Big]$. This should be compared with the generating function for the conventional Gaussian vacuum state $\langle\circledcirc|\cdot|\circledcirc\rangle$, which for $\hat\chi_U$ would be

$$\langle\circledcirc|\,\mathrm{e}^{\mathrm{i}\lambda\hat\chi_U}\,|\circledcirc\rangle=\mathrm{e}^{-\lambda^2(U^c,U)/2}.$$

For any test function V, we can construct a first degree raised state

$$\frac{\langle\circledcirc|\,\hat\chi_V^\dagger\mathrm{e}^{\mathrm{i}\lambda\hat\chi_U}\hat\chi_V\,|\circledcirc\rangle}{\langle\circledcirc|\,\hat\chi_V^\dagger\hat\chi_V\,|\circledcirc\rangle}=\left[1-\lambda^2\frac{(U^c,V)(V,U)}{(V,V)}\right]\mathrm{e}^{-\lambda^2(U^c,U)/2},$$

so we can consider $\langle\mathcal{F}|\cdot|\mathcal{F}\rangle$ to be an equally weighted convex mixture of this state for all test functions V for which $(V,V)_+=(V,V)=1$, so that $\langle\mathcal{F}|\cdot|\mathcal{F}\rangle$ is unitarily inequivalent to $\langle\circledcirc|\cdot|\circledcirc\rangle$. When $U^c=U$, $\hat\chi_U$ is Hermitian, so we obtain a probability density for single measurements, by inverse fourier transform,

$$\langle\mathcal{F}|\,\delta(\hat\chi_U-v)\,|\mathcal{F}\rangle=\left[\frac{(U,U)_-}{(U,U)}+\frac{(U,U)_+}{(U,U)}v^2\right]\frac{\mathrm{e}^{-\frac{v^2}{2(U,U)}}}{\sqrt{2\pi(U,U)}},$$

varying continuously between a Gaussian probability density and the globally raised second degree probability density, depending on the ratio $(U,U)_+/(U,U)_-$.

Despite this continuous probability density, for $\hat{\mathsf{X}}_U=\hat{\mathsf{X}}_U^\dagger$ we obtain a generating function

$$\begin{aligned}\langle\mathcal{F}|\,\mathrm{e}^{\mathrm{i}\lambda\hat{\mathsf{X}}_U}\,|\mathcal{F}\rangle=\langle0|\,\mathrm{e}^{\mathrm{i}\lambda\hat\Phi_U}\,|0\rangle&=\langle0|\left(1-\frac{\hat\Phi_U}{(U,U)}+\mathrm{e}^{\mathrm{i}\lambda(U,U)}\frac{\hat\Phi_U}{(U,U)}\right)|0\rangle\\ &=\frac{(U,U)_-}{(U,U)}+\frac{(U,U)_+}{(U,U)}\mathrm{e}^{\mathrm{i}\lambda(U,U)},\end{aligned}$$

from which we obtain a discrete two–valued probability density, isolated at 0 and at (U,U),

$$\langle\mathcal{F}|\,\delta(\hat{\mathsf{X}}_U - v)\,|\mathcal{F}\rangle = \frac{(U,U)_-}{(U,U)}\delta(v) + \frac{(U,U)_+}{(U,U)}\delta(v - (U,U)).$$

For any state generated by the action of $\hat{\Phi}_V$ on $\langle 0|\cdot|0\rangle$ we have equivalences between generating functions such as

$$\frac{\langle 0|\,\hat{\Phi}_{V_1}\cdots\hat{\Phi}_{V_n}\mathrm{e}^{\mathrm{i}\lambda\hat{\Phi}_U}\hat{\Phi}_{V_n}\cdots\hat{\Phi}_{V_1}\,|0\rangle}{\langle 0|\,\hat{\Phi}_{V_1}\cdots\hat{\Phi}_{V_n}\hat{\Phi}_{V_n}\cdots\hat{\Phi}_{V_1}\,|0\rangle} = \frac{\langle\mathcal{F}|\,\hat{\mathsf{X}}_{V_1}\cdots\hat{\mathsf{X}}_{V_n}\mathrm{e}^{\mathrm{i}\lambda\hat{\mathsf{X}}_U}\hat{\mathsf{X}}_{V_n}\cdots\hat{\mathsf{X}}_{V_1}\,|\mathcal{F}\rangle}{\langle\mathcal{F}|\,\hat{\mathsf{X}}_{V_1}\cdots\hat{\mathsf{X}}_{V_n}\hat{\mathsf{X}}_{V_n}\cdots\hat{\mathsf{X}}_{V_1}\,|\mathcal{F}\rangle},$$

the latter of which can be extended to a generating function for $\hat{\chi}_U$,

$$\frac{\langle\mathcal{F}|\,\hat{\mathsf{X}}_{V_1}\cdots\hat{\mathsf{X}}_{V_n}\mathrm{e}^{\mathrm{i}\lambda\hat{\chi}_U}\hat{\mathsf{X}}_{V_n}\cdots\hat{\mathsf{X}}_{V_1}\,|\mathcal{F}\rangle}{\langle\mathcal{F}|\,\hat{\mathsf{X}}_{V_1}\cdots\hat{\mathsf{X}}_{V_n}\hat{\mathsf{X}}_{V_n}\cdots\hat{\mathsf{X}}_{V_1}\,|\mathcal{F}\rangle} = \left[1 - \lambda^2\frac{\langle 0|\,\hat{\Phi}_{V_1}\cdots\hat{\Phi}_{V_n}\hat{\psi}^\dagger_{U^c}\hat{\psi}_U\hat{\Phi}_{V_n}\cdots\hat{\Phi}_{V_1}\,|0\rangle}{\langle 0|\,\hat{\Phi}_{V_1}\cdots\hat{\Phi}_{V_n}\hat{\Phi}_{V_n}\cdots\hat{\Phi}_{V_1}\,|0\rangle}\right]\mathrm{e}^{-\lambda^2(U^c,U)/2}.$$

For a state generated by the action of $\hat{\Phi}_V$, for example, we obtain a generating function

$$\begin{aligned}\frac{\langle\mathcal{F}|\,\hat{\mathsf{X}}_V\mathrm{e}^{\mathrm{i}\lambda\hat{\chi}_U}\hat{\mathsf{X}}_V\,|\mathcal{F}\rangle}{\langle\mathcal{F}|\,\hat{\mathsf{X}}_V\hat{\mathsf{X}}_V\,|\mathcal{F}\rangle} &= \left[1 - \lambda^2\frac{\langle 0|\,\hat{\Phi}_V\hat{\psi}^\dagger_{U^c}\hat{\psi}_U\hat{\Phi}_V\,|0\rangle}{\langle 0|\,\hat{\Phi}_V\hat{\Phi}_V\,|0\rangle}\right]\mathrm{e}^{-\lambda^2(U^c,U)/2}\\ &= \left[1 - \lambda^2\left((U^c\bot V, U\bot V)_+ + \frac{(U^c,V)(V,U)}{(V,V)}\right)\right]\mathrm{e}^{-\lambda^2(U^c,U)/2},\end{aligned}$$

where $U\bot V = U - \frac{(V,U)}{(V,V)}V$ (see Appendix D for this calculation in more detail); for a state generated by the action of $\hat{\psi}^\dagger_V\hat{\psi}_W$ (with V and W orthogonal, $(V,W)=0$, to simplify the expression, which in particular will be satisfied whenever V and W have space–like separated supports),

$$\begin{aligned}\frac{\langle\mathcal{F}|\,\mathsf{a}^\dagger_{W^c}\mathsf{a}_{V^c}\mathrm{e}^{\mathrm{i}\lambda\hat{\chi}_U}\mathsf{a}^\dagger_{V^c}\mathsf{a}_{W^c}\,|\mathcal{F}\rangle}{\langle\mathcal{F}|\,\mathsf{a}^\dagger_{W^c}\mathsf{a}_{V^c}\mathsf{a}^\dagger_{V^c}\mathsf{a}_{W^c}\,|\mathcal{F}\rangle} &= \left[1 - \lambda^2\frac{\langle 0|\,\hat{\psi}^\dagger_W\hat{\psi}_V\hat{\psi}^\dagger_{U^c}\hat{\psi}_U\hat{\psi}^\dagger_V\hat{\psi}_W\,|0\rangle}{\langle 0|\,\hat{\psi}^\dagger_W\hat{\psi}_V\hat{\psi}^\dagger_V\hat{\psi}_W\,|0\rangle}\right]\mathrm{e}^{-\lambda^2(U^c,U)/2}\\ &= \left[1 - \lambda^2\left(\left(U^c\bot V\bot W, U\bot V\bot W\right)_+ + \frac{(U^c,V)(V,U)}{(V,V)}\right)\right]\mathrm{e}^{-\lambda^2(U^c,U)/2}\end{aligned}$$

[where we can omit brackets when $(V,W)=0$, so that $U\bot V\bot W = (U\bot V)\bot W = (U\bot W)\bot V$];

and for a state generated by the action of $\hat{\Phi}_V\hat{\Phi}_W$ (again with V and W orthogonal),

$$\frac{\langle\mathcal{F}|\,\hat{\mathsf{X}}_W\hat{\mathsf{X}}_V\mathrm{e}^{\mathrm{i}\lambda\hat{\chi}_U}\hat{\mathsf{X}}_V\hat{\mathsf{X}}_W\,|\mathcal{F}\rangle}{\langle\mathcal{F}|\,\hat{\mathsf{X}}_W\hat{\mathsf{X}}_V\hat{\mathsf{X}}_V\hat{\mathsf{X}}_W\,|\mathcal{F}\rangle} = \left[1 - \lambda^2\left((U^c\bot V\bot W, U\bot V\bot W)_+ + \frac{(U^c,V)(V,U)}{(V,V)} + \frac{(U^c,W)(W,U)}{(W,W)}\right)\right]\mathrm{e}^{-\lambda^2(U^c,U)/2};$$

so that $\hat{\Phi}_V$, $\hat{\psi}^\dagger_V\hat{\psi}_W$, $\hat{\Phi}_V\hat{\Phi}_W$, and higher degree operators act to modulate the vacuum state in a limited way, always with a factor $1 - \lambda^2\mathsf{M}(U^c,U)$, linear in U^c and in U, which can be reproduced by an appropriately chosen mixture of actions of $\hat{\chi}_V$ on the vacuum state $\langle\circledcirc|\cdot|\circledcirc\rangle$; we could equally well say that such states are generated by a constrained action of the algebra generated by $\hat{\chi}_V$ and call them classical.

The fermionic Hilbert space does not contain all the states we can construct using $\hat{\chi}_U$ acting on $|\mathcal{F}\rangle$ or on $|\circledcirc\rangle$ (just as superselection prohibits many vectors that we could construct using the unconstrained action of $\hat{\psi}_U$ and $\hat{\psi}^\dagger_U$ on $|0\rangle$, allowing superpositions only of vectors of equal global-$U(1)$ charge), instead being restricted only to the states we can construct using many different $\hat{\mathsf{X}}_V$ acting on $|\mathcal{F}\rangle$. In effect, we are only able to construct states — from the vacuum state we are given as a starting point— using the quantum mechanical measurements we can actually perform. We can construct all the necessary states using the random field χ_U, but the constraint is perhaps not at this point classically well–motivated enough for us to say that such states are truly "classical".

IV. $U(1)$–GAUGE INVARIANT OBSERVABLES

We here only indicate a possible approach to $U(1)$–gauge invariance. Under $U(1)$–gauge transformations, the Dirac spinor wave function transforms as $\hat{\psi}_\xi(x) \mapsto \mathrm{e}^{\mathrm{i}\theta(x)}\hat{\psi}_\xi(x)$[15] (Eq. 2-63), so that the vacuum expectation values $\mathsf{iS}_{+\xi\xi'}(x,x') = \langle 0|\,\hat{\psi}_\xi(x)\overline{\hat{\psi}_{\xi'}(x')}\,|0\rangle$ and $\mathsf{iS}_{-\xi\xi'}(x,x') = \langle 0|\,\overline{\hat{\psi}_{\xi'}(x')}\hat{\psi}_\xi(x)\,|0\rangle$ transform as parallel transports,

$$\mathsf{iS}_{\pm\xi\xi'}(x,x') \mapsto \mathrm{e}^{\mathrm{i}(\theta(x)-\theta(x'))}\mathsf{iS}_{\pm\xi\xi'}(x,x').$$

We can therefore construct two–point $U(1)$–gauge invariant operators (for both free and interacting fields, supposing interacting fields exist), using these parallel transports and two Dirac matrix–valued test functions, $P_{\xi'\eta'}(x')$ and $Q_{\eta\xi}(x)$,

$$\sum_{\xi'\eta'\eta\xi}\int \overline{\hat{\psi}_{\xi'}(x')}P_{\xi'\eta'}(x')\mathrm{i}\mathsf{S}_{\pm\eta'\eta}(x',x)Q_{\eta\xi}(x)\hat{\psi}_{\xi}(x)\mathrm{d}^4x\mathrm{d}^4x',$$

which we will write without indices as $\int \overline{\hat{\psi}(x')}P(x')\mathrm{i}\mathsf{S}_{\pm}(x',x)Q(x)\hat{\psi}(x)\mathrm{d}^4x\mathrm{d}^4x'$. Of the possible linear combinations of these two constructions, $\int \overline{\hat{\psi}(x')}P(x')\mathrm{i}\mathsf{S}_{+\!/\!-}(x',x)Q(x)\hat{\psi}(x)\mathrm{d}^4x\mathrm{d}^4x'$, where $\mathrm{i}\mathsf{S}_{+\!/\!-}(x',x)=\mathrm{i}\mathsf{S}_{+}(x',x)-\mathrm{i}\mathsf{S}_{-}(x',x)$, is notably less singular on the light–cone in the free field case, indeed this is the least singular $U(1)$–gauge invariant construction known to the author; in particular, it is less singular than constructions that use the electromagnetic potential operator. The appearance of Dirac matrix–valued test functions $P(x')$ and $Q(x)$ in this $U(1)$–gauge invariant construction lessens the significance of the double cover of the Lorentz group. The construction $\int \overline{\hat{\psi}(x')}P(x')\mathrm{i}\mathsf{S}_{+\!/\!-}(x',x)Q(x)\hat{\psi}(x)\mathrm{d}^4x\mathrm{d}^4x'$ cannot be written as a simple product $\hat{\psi}_U^\dagger\hat{\psi}_U$, nonetheless it can be written in terms of bosonic raising and lowering operators.

V. DISCUSSION

The constructions above, for free quantum fields, which only apply where interactions are taken to be insignificant —that is, to the in– and out–states of the S–matrix and to simple quantum optics— do not touch on how we might discuss interactions in classical canonical terms as well as or instead of in quantum unitary terms. The introduction of negative frequency components by the constructions here of random fields makes it impossible to preserve the Correspondence Principle, which includes correspondence between 4–momentum and wave–number, $p=\hbar k$, with the energy component p_0 required to be positive semi–definite. The Correspondence Principle, however, can be regarded as a property of the quantization process, not of the Hilbert space that is created by quantization of a classical dynamics, whereas the Koopman–von Neumann approach described in Appendix A is a different process for constructing the same Hilbert space from a different classical dynamics, for which frequency is not related in the same way to energy.

As noted in Ref. 3, there has been discussion of the similarities and differences between "random electrodynamics" and quantum electrodynamics at least since the 1960s[16,17], however the algebraic formalism used here makes the comparison and the establishment of empirical equivalence much more direct. The negative frequencies that appear explicitly in the algebraic approach here appear implicitly in random electrodynamics (which has come to be called "Stochastic Electrodynamics" or "SED"[18]) as a factor $\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)$ in the 2–point correlation functions of the random electromagnetic field[16] (Eqs. (10), (13), and (14)), instead of a factor $\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ in the 2–point vacuum expectation values of the quantized electromagnetic field[16] (Eqs. (15), (16), and (17)). There is also a quantum optics literature that works with stochastic classical electrodynamics using a Hilbert space–motivated formalism[19].

Although the main text has used a manifestly Lorentz invariant 4–dimensional block world formalism, if we choose a time–like 4–vector we can construct a 3–dimensional formalism by reducing

$$2\pi\delta(k\cdot k-m^2)\frac{\mathrm{d}^4k}{(2\pi)^4}\quad\text{to}\quad 2\pi\frac{\delta(k_0-\sqrt{\mathbf{k}\cdot\mathbf{k}+m^2})+\delta(k_0+\sqrt{\mathbf{k}\cdot\mathbf{k}+m^2})}{2\sqrt{\mathbf{k}\cdot\mathbf{k}+m^2}}\frac{\mathrm{d}^4k}{(2\pi)^4},$$

and hence reduce the 4–dimensional formalism to a 3–dimensional formalism, so we do not have to interpret the formalism as requiring a 4–dimensional block world ontology.

Given classical states, the properties of which are determined as far as possible by using quantum field measurements, it is of course possible to ask what the results of classical measurements would be, as indeed we did above for the Dirac spinor–valued random field, if only we could make those classical measurements or if we could correct for whatever a classical theory takes to be, from its perspective, the inadequacies of the quantum measurements. From a classical perspective, we can take a presentation of what the results of such classical measurements would be to constitute a different way to work with the Poincaré invariant noise of the vacuum state. To address only one of very many contemporary issues, a classical perspective takes quantum computation and other exploitation of the quantum-Hilbert space formalism as a consequence of the measurement process, the reduction of elaborate noisy classical states by the use of local and nonlocal observables that have discrete spectra and incompatible eigenspaces, however this is only a matter of interpretation, because the mathematical landscape provided by bounded operators acting on the vacuum sector Hilbert space is completely unchanged.

Particularly for the quantized Dirac spinor field, however, the projective quality of many measurements very much limits how closely we might determine a putative underlying classical state, so that we may well be best to discuss quantum measurements as about quantum states, with only a background acknowledgment that perhaps there are classical random fields underpinning all this, or perhaps there are not, with no prejudicial determination either way. From a practical point of view, the Correspondence Principle is so embedded in physicists' thinking that it will be best to keep thinking in terms of both quantum fields and random fields.

Finally, the focus on the space of test functions and its pre–inner product structure is very much aimed towards future consideration of interacting quantum or random fields in test function algebraic terms, insofar as products and derivatives of test functions are always well–defined, steps towards which may be found in[20], in contrast to using renormalization to fix the problems introduced by defining a Lagrangian evolution using products of distributions.

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Appendix A: Classical mechanics to quantum mechanics: Koopman–von Neumann

We can present Classical Mechanics as a commutative, associative algebra $\mathcal{A}$ of observables over a phase space, $A:\mathcal{P}\to\mathbb{R}; P\mapsto A(P)$, with a multiplication $\cdot{:}\mathcal{A}{\times}\mathcal{A}{\to}\mathcal{A}; A,B\mapsto A\cdot B$, *together with* a Poisson bracket, $\{\bullet,\bullet\}{:}\mathcal{A}{\times}\mathcal{A}{\to}\mathcal{A}: A,B\mapsto\{A,B\}$, and a Hamiltonian $H(P)$ that generates, for an observable $A(P,t)$, an evolution $\frac{dA}{dt}=\{H,A\}+\frac{\partial A}{\partial t}$. [We can present Classical Mechanics equivalently, with considerably more elaborate machinery but Lorentz covariantly, using the Peierls bracket over the solution space instead of using the Poisson bracket over phase space[23] (§4.4.1), however this Appendix will use the simpler machinery of phase space.]

We can use the multiplication $\cdot$ to construct an action

$$\hat{Y}_A:\mathcal{A}\to\mathcal{A};\bullet\mapsto\hat{Y}_A(\bullet)=A\cdot\bullet,$$

where we can identify the algebra generated by the $\hat{Y}_A$ with $\mathcal{A}$, and we can similarly use the Poisson bracket to construct what can be called *generators of transformations*,

$$\hat{Z}_A:\mathcal{A}\to\mathcal{A};\bullet\mapsto\hat{Z}_A(\bullet)=\{A,\bullet\},$$

which act non-commutatively but associatively on $\mathcal{A}$ [5,6,13](§§2.1.1, 5.5.1),[24](§§1.5-6),[25–29]. These two actions allow us to construct a non-commutative, associative algebra $\mathcal{A}_+$ that is generated by the $\hat{Y}_A$ *and* the $\hat{Z}_A$, satisfying the commutation relations $[\hat{Y}_A,\hat{Y}_B]=0$, $[\hat{Z}_A,\hat{Y}_B]=\hat{Y}_{\{A,B\}}$, and $[\hat{Z}_A,\hat{Z}_B]=\hat{Z}_{\{A,B\}}$. Note that this algebra of operators has always been implicitly part of a Hamiltonian presentation of classical mechanics, even though it has been explicitly presented by the Poisson bracket. If the phase space is elementary, without constraints, $\mathcal{A}_+$ is generated by q_i, p_i, $\partial/\partial q_i$, and $\partial/\partial p_i$.

The above is all just Classical Mechanics. To move towards Quantum Mechanics, we introduce raising and lowering operators for the elementary case,

$$a_i^\dagger=\frac{1}{\sqrt{2}}\left(q_i-\frac{\partial}{\partial q_i}\right),\qquad b_i^\dagger=\frac{1}{\sqrt{2}}\left(p_i-\frac{\partial}{\partial p_i}\right),$$
$$a_i=\frac{1}{\sqrt{2}}\left(q_i+\frac{\partial}{\partial q_i}\right),\qquad b_i=\frac{1}{\sqrt{2}}\left(p_i+\frac{\partial}{\partial p_i}\right),$$

which, when taken with $(\hat{W}\hat{X})^\dagger{=}\hat{X}^\dagger\hat{W}^\dagger$, define an involution $\hat{X}{\mapsto}\hat{X}^\dagger$, making $\mathcal{A}_+$ a $*$–algebra over $\mathbb{R}$ (that is, we allow only real scalar multiples). The raising and lowering operators satisfy the commutation relations $[a_i,a_j^\dagger]=\delta_{ij}$, $[b_i,b_j^\dagger]=\delta_{ij}$, and $[a_i,a_j]{=}[a_i,b_j]{=}[b_i,b_j]{=}[a_i,b_j^\dagger]{=}0$. If we set $\rho(1)=1$ and

$$\rho(\hat{X}a_i)=\rho(\hat{X}b_i)=\rho(a_i^\dagger\hat{X})=\rho(b_i^\dagger\hat{X})=0\quad\forall\hat{X}\in\mathcal{A}_+,$$

then we obtain by the usual manipulations a Gaussian statistical state[11] (§III.2.2) over $\mathcal{A}_+$, $\rho:\mathcal{A}_+\to\mathbb{R}$, which is a linear form that satisfies $\rho(\hat{X}^\dagger\hat{X})\geq 0$, $\rho(1)=1$, and $\rho(\hat{X}^\dagger)=\overline{\rho(\hat{X})}$ (where for the real case the conjugate $\overline{}:\mathbb{R}\to\mathbb{R}$

is trivial), which allows a probability interpretation for those operators for which $\hat{X}^\dagger = \hat{X}$ (which we can therefore call “observables”), and which allows the GNS-construction[11] (§III.2.2) of a real Hilbert space $\mathcal{H}_+$.

$\mathcal{H}_+$ as a vector space can be generated by a basis that contains a vacuum vector $|0\rangle$ and *real*–valued multiples of $a_1^{\dagger j_1} b_1^{\dagger k_1} \cdots a_m^{\dagger j_m} b_m^{\dagger k_m} |0\rangle$. Classical mechanical systems that have a natural complex structure —which for a random field can be provided, as in Section II, by the cosine and sine components of the fourier transform relative to space–time coordinates— are equivalent to a system of quantized simple harmonic oscillators, which can be generated by a basis that contains a vacuum vector $|0\rangle$ and *complex*–valued multiples of $a_1^{\dagger j_1} b_1^{\dagger k_1} \cdots a_m^{\dagger j_m} b_m^{\dagger k_m} |0\rangle$. Note that superposition and entanglement do not require a complex structure for their definition, so they are as natural for the real Hilbert space $\mathcal{H}_+$ as they are for the complex Hilbert spaces of quantum mechanics. If we introduce an engineering imaginary j for the purposes of signal processing (though it is not necessary if we restrict ourselves to using fourier sine and cosine transforms) then each $q_i, \mathsf{j}\partial/\partial q_i$ and $p_i, \mathsf{j}\partial/\partial p_i$ pair allows the construction of a Wigner or other quasi–probability distribution, using the mathematics and classical measurement theory associated with time–frequency distributions[9] as q_i–$q_i$frequency distributions.

The positive definite Hamiltonian function for a collection of non-interacting simple harmonic oscillators, in a vector notation, is $H(\underline{q},\underline{p}) = \frac{1}{2}(\underline{q}\cdot\underline{q} + \underline{p}\cdot\underline{p})$, from which we obtain two operators,

$$\hat{Y}_H = \tfrac{1}{4}\left[(\underline{a}+\underline{a}^\dagger)\cdot(\underline{a}+\underline{a}^\dagger) + (\underline{b}+\underline{b}^\dagger)\cdot(\underline{b}+\underline{b}^\dagger)\right], \qquad \hat{Z}_H = \underline{a}\cdot\underline{b}^\dagger - \underline{a}^\dagger\cdot\underline{b}.$$

which are Hermitian and anti–Hermitian respectively. Classical physics requires only that $\hat{Y}_H$ is bounded below, not that $\hat{Z}_H$ is positive. $\hat{Z}_H$ generates time–like translations; given a complex structure j, we can transform to the basis $\underline{c} = (\underline{a} + \mathsf{j}\underline{b})/\sqrt{2}$, $\underline{d} = (\underline{a} - \mathsf{j}\underline{b})/\sqrt{2}$, to obtain $\hat{Z}_H = \mathsf{j}\hat{H}_c$, where $\hat{H}_c = \underline{c}^\dagger\cdot\underline{c} - \underline{d}^\dagger\cdot\underline{d}$, so that from the perspective of quantum mechanics the Hamiltonian operator $\hat{H}_c$ is not positive–definite (however we see in the main text that we can transform negative frequency components into positive frequency components for at least some random field constructions). The construction of the involution and state above is notable, however, for fixing statistics directly instead of assuming or requiring that a Hamiltonian or a stochastic dynamics is available to define, for example, a Gibbs state. We have become accustomed to presenting classical physics using a Lagrangian or Hamiltonian, however we can in a stochastic context present classical physics using a state over an abstract $*$–algebra of observables.

The whole process here is an algebraic form of a Koopman–von Neumann approach, having four steps: (1) use the injection $\mathcal{P} : \mathcal{A} \hookrightarrow \mathcal{A}_+$; (2) introduce an involution $\hat{X}\mapsto\hat{X}^\dagger$, making $\mathcal{A}_+$ a $*$–algebra; (3) introduce a statistical state over $\mathcal{A}_+$; (4) use the GNS-construction of the Hilbert space $\mathcal{H}_+$. Only (2) and (3) need the introduction of new structure, an involution and a state, for either of which there may be obstructions for more elaborate phase spaces; (1) and (4) use structure that's already there. It is clear that this Koopman–von Neumann approach is quite different from canonical quantization, a map that is not an algebra morphism that for elementary cases can be presented as $\mathcal{Q} : \mathcal{A} \to \mathcal{A}' \subset \mathcal{A}_+; (q_i, p_i) \mapsto (q_i, -\mathrm{i}\partial/\partial q_i)$. At the level of quantum field theory, this should cause little concern, insofar as the choice of a classical field theory to quantize is rather instrumental: we choose a classical field to quantize that gives by quantization a quantum field that is empirically successful. As an instrumental process, if there is an empirically successful quantum field that is the result of a Koopman–von Neumann treatment of some classical field, we can choose to use that classical field; Ref. 1 (and Appendix B) and Section II effectively show that there is such a classical field for the quantized complex Klein-Gordon field and for the quantized electromagnetic field, respectively.

Appendix B: The complex Klein-Gordon field

To emphasize the similarity between the quantized electromagnetic field and the complex Klein-Gordon quantum field, instead of relying on the rather dissimilar construction in Ref. 1, we can present the complex Klein-Gordon quantum field as $\hat{\mathsf{F}}_f = a_{f^*} + a_f^\dagger = \hat{\phi}_{f_1} + \hat{\phi}_{f_2^*}^\dagger$, where $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is a two component test function, $f^* = \begin{pmatrix} f_2^* \\ f_1^* \end{pmatrix}$ ensures that $\hat{\mathsf{F}}_f^\dagger = \hat{\mathsf{F}}_{f^*}$, and the raising and lowering operators satisfy $[a_f, a_g^\dagger] = (f,g)_+$,

$$(f,g)_\pm = \int \left(\widetilde{f_1}^*(k)\widetilde{g_1}(k) + \widetilde{f_2}^*(k)\widetilde{g_2}(k)\right) 2\pi\delta(k\cdot k - m^2)\theta(\pm k_0)\frac{\mathrm{d}^4k}{(2\pi)^4}.$$

For this quantum field, we can use the matrix $I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the same way as the Hodge dual was used for the quantized electromagnetic field in Section II to construct an involution ${}^\bullet : \mathcal{S} \to \mathcal{S}; f \mapsto f^\bullet = \frac{1}{2}(1+\mathrm{i}I)f + \frac{1}{2}(1-\mathrm{i}I)f^-$, $f^{\bullet\bullet} = f$, so that (because as for the quantized electromagnetic field we have the identities $(f^*, g^*)_\pm = (g,f)_\mp$ and $(f^-, g^-)_\pm = (f,g)_\mp$) we can derive $(f^{*\bullet}, g^\bullet)_\pm = (g^{*\bullet}, f^\bullet)_\pm$, and hence we can construct a random field $\hat{\mathbb{F}}_f = a_{f^{*\bullet}} + a_{f^\bullet}^\dagger$ for which $[\hat{\mathbb{F}}_f, \hat{\mathbb{F}}_g] = 0$.

Appendix C: The electromagnetic potential

The electromagnetic potential does not "play nice" with the helicity projection used in the main text.

For the quantized electromagnetic field, we have $\langle 0|\,\hat{\mathsf{F}}_{\mathsf{f}}^{\dagger}\hat{\mathsf{F}}_{\mathsf{g}}\,|0\rangle = (\mathsf{f},\mathsf{g})_+ = (\!(\delta\mathsf{f},\delta\mathsf{g})\!)_+$, where for 1–forms u and v we have the sesquilinear forms

$$(\!(u,v)\!)_{\pm} = -\hbar\int \tilde{u}^*(k)\cdot\tilde{v}(k)2\pi\delta(k\cdot k)\theta(\pm k_0)\frac{\mathrm{d}^4k}{(2\pi)^4},$$

which are not positive semi–definite in general but are positive semi–definite for $(\!(\delta\mathsf{f},\delta\mathsf{g})\!)_{\pm}$. We have, therefore, for arbitrary 3–forms $u_{(3)}$ and 1–forms $u_{(1)}$,

$$\begin{aligned}
&\langle 0|\,\hat{\mathsf{F}}_{\mathsf{f}}\hat{\mathsf{F}}_{\delta u_{(3)}}\,|0\rangle = (\!(\delta\mathsf{f}^*,\delta\delta u_{(3)})\!)_+ = 0 &&\Rightarrow\quad \hat{\mathsf{F}}_{\delta u_{(3)}}\equiv 0 &&\Rightarrow\quad \mathrm{d}\hat{\mathsf{F}}\equiv 0;\\
&\langle 0|\,\hat{\mathsf{F}}_{\mathsf{f}}\hat{\mathsf{F}}_{\mathrm{d}u_{(1)}}\,|0\rangle = (\!(\delta\mathsf{f}^*,\delta\mathrm{d}u_{(1)})\!)_+ = (\!(\delta\mathsf{f}^*,(\delta\mathrm{d}+\mathrm{d}\delta)u_{(1)})\!)_+ = 0 &&\Rightarrow\quad \hat{\mathsf{F}}_{\mathrm{d}u_{(1)}}\equiv 0 &&\Rightarrow\quad \delta\hat{\mathsf{F}}\equiv 0
\end{aligned}$$

(because $(\!(\delta\mathsf{f}^*,\mathrm{d}v)\!)_{\pm} = 0\ \forall v$, and for the other term because of projection to the light-cone);

and $\langle 0|\,\hat{\mathsf{F}}_{\mathsf{f}}\hat{\mathsf{F}}_{\mathsf{g}}\,|0\rangle = \langle 0|\,\hat{\mathsf{F}}_{\mathsf{f}}\hat{\mathsf{F}}_{\mathsf{g}^{(+)}}\,|0\rangle \quad\Rightarrow\quad \hat{\mathsf{F}}_{\mathsf{g}}\equiv\hat{\mathsf{F}}_{\mathsf{g}^{(+)}},\qquad$ where $\widetilde{g^{(\pm)}}(k) = \tilde{g}(k)\theta(\pm k)$,

so we can write $\hat{\mathsf{A}}$ as a potential for $\hat{\mathsf{F}}$, $\hat{\mathsf{F}} = \mathrm{d}\hat{\mathsf{A}}$, $\hat{\mathsf{F}}_{\mathsf{f}} = \hat{\mathsf{A}}_{\delta\mathsf{f}}$, and we have $\langle 0|\,\hat{\mathsf{A}}_u\hat{\mathsf{A}}_v\,|0\rangle = (\!(u^*,v)\!)_+$, albeit problematically because $(\!(u,v)\!)_+$ is not a pre–inner product.

For a random field operator $\hat{\mathbb{F}}_{\mathsf{f}}$, we have, using $P_{\pm} = \frac{1}{2}(1\pm\mathrm{i}\star)$ for brevity and clarity,

$$\langle 0|\,\hat{\mathbb{F}}_{\mathsf{f}}\hat{\mathbb{F}}_{\mathsf{g}}\,|0\rangle = (\!(\delta P_+\mathsf{f}^*,\delta P_+\mathsf{g})\!)_+ + (\!(\delta P_-\mathsf{f}^*,\delta P_-\mathsf{g})\!)_-,$$

then, using $\delta P_{\pm}\delta = \pm\frac{1}{2}\mathrm{i}\delta\star\delta = \pm\frac{1}{2}\mathrm{i}\delta\mathrm{d}\star$ (when acting on 3–forms) and $\delta P_{\pm}\mathrm{d} = \frac{1}{2}\delta\mathrm{d}$,

$$\begin{aligned}
\langle 0|\,\hat{\mathbb{F}}_{\mathsf{f}}\hat{\mathbb{F}}_{\delta u_{(3)}}\,|0\rangle &= \tfrac{\mathrm{i}}{2}(\!(\delta P_+\mathsf{f}^*,\delta\mathrm{d}\star u_{(3)})\!)_+ - \tfrac{\mathrm{i}}{2}(\!(\delta P_-\mathsf{f}^*,\delta\mathrm{d}\star u_{(3)})\!)_- = 0 &&\Rightarrow\quad \mathrm{d}\hat{\mathbb{F}}\equiv 0;\\
\langle 0|\,\hat{\mathbb{F}}_{\mathsf{f}}\hat{\mathbb{F}}_{\mathrm{d}u_{(1)}}\,|0\rangle &= \tfrac{1}{2}(\!(\delta P_+\mathsf{f}^*,\delta\mathrm{d}u_{(1)})\!)_+ + \tfrac{1}{2}(\!(\delta P_-\mathsf{f}^*,\delta\mathrm{d}u_{(1)})\!)_- = 0 &&\Rightarrow\quad \delta\hat{\mathbb{F}}\equiv 0;
\end{aligned}$$

so $\hat{\mathbb{F}}$ still satisfies the Maxwell equations, however the projection to positive frequency becomes

$$\langle 0|\,\hat{\mathbb{F}}_{\mathsf{f}}\hat{\mathbb{F}}_{\mathsf{g}}\,|0\rangle = \langle 0|\,\hat{\mathbb{F}}_{\mathsf{f}}\hat{\mathbb{F}}_{P_+\mathsf{g}^{(+)}+P_-\mathsf{g}^{(-)}}\,|0\rangle \qquad\Rightarrow\qquad \hat{\mathbb{F}}_{\mathsf{g}}\equiv\hat{\mathbb{F}}_{P_+\mathsf{g}^{(+)}+P_-\mathsf{g}^{(-)}}.$$

Because of the Hodge dual in this construction, to construct a potential for $\hat{\mathbb{F}}$ we have to introduce both a 1–form $\hat{\mathbb{A}}$ and a 3–form $\hat{\mathbb{B}}$, setting $\hat{\mathbb{F}} = \mathrm{d}\hat{\mathbb{A}}+\delta\hat{\mathbb{B}}$ so that $\hat{\mathbb{F}}_{\mathsf{f}} = \hat{\mathbb{A}}_{\delta\mathsf{f}}+\hat{\mathbb{B}}_{\mathrm{d}\mathsf{f}}$. Defining $\hat{\mathbb{X}}_{u_{(1)}\oplus u_{(3)}} = \hat{\mathbb{A}}_{u_{(1)}}+\hat{\mathbb{B}}_{u_{(3)}}$, we have $\hat{\mathbb{F}}_{\mathsf{f}} = \hat{\mathbb{X}}_{\delta\mathsf{f}\oplus\mathrm{d}\mathsf{f}}$. If we write

$$\langle 0|\,\hat{\mathbb{X}}_{u_{(1)}\oplus u_{(3)}}\hat{\mathbb{X}}_{v_{(1)}\oplus v_{(3)}}\,|0\rangle = (\!(\tfrac{1}{2}(u^*_{(1)}-\mathrm{i}\star u^*_{(3)}),\tfrac{1}{2}(v_{(1)}-\mathrm{i}\star v_{(3)}))\!)_+ + (\!(\tfrac{1}{2}(u^*_{(1)}+\mathrm{i}\star u^*_{(3)}),\tfrac{1}{2}(v_{(1)}+\mathrm{i}\star v_{(3)}))\!)_-,$$

then, using that $\mathrm{d}\star = -\star\delta$ when acting on 2–forms,

$$\begin{aligned}
\langle 0|\,\hat{\mathbb{F}}_{\mathsf{f}}\hat{\mathbb{F}}_{\mathsf{g}}\,|0\rangle &= (\!(\tfrac{1}{2}(\delta\mathsf{f}^*-\mathrm{i}\star\mathrm{d}\mathsf{f}^*),\tfrac{1}{2}(\delta\mathsf{f}^*-\mathrm{i}\star\mathrm{d}\mathsf{f}^*))\!)_+ + (\!(\tfrac{1}{2}(\delta\mathsf{f}^*+\mathrm{i}\star\mathrm{d}\mathsf{f}^*),\tfrac{1}{2}(\delta\mathsf{f}^*+\mathrm{i}\star\mathrm{d}\mathsf{f}^*))\!)_-\\
&= (\!(\tfrac{1}{2}(\delta\mathsf{f}^*+\mathrm{i}\delta\star\mathsf{f}^*),\tfrac{1}{2}(\delta\mathsf{f}^*+\mathrm{i}\delta\star\mathsf{f}^*))\!)_+ + (\!(\tfrac{1}{2}(\delta\mathsf{f}^*-\mathrm{i}\delta\star\mathsf{f}^*),\tfrac{1}{2}(\delta\mathsf{f}^*-\mathrm{i}\delta\star\mathsf{f}^*))\!)_-\\
&= (\!(\delta P_+\mathsf{f}^*,\delta P_+\mathsf{g})\!)_+ + (\!(\delta P_-\mathsf{f}^*,\delta P_-\mathsf{g})\!)_-.
\end{aligned}$$

We could, for quantized Electromagnetism, have set $\hat{\mathsf{F}} = \mathrm{d}\hat{\mathsf{A}}+\delta\hat{\mathsf{B}}$, so that $\hat{\mathsf{F}} = \hat{\mathsf{A}}_{\delta\mathsf{f}}+\hat{\mathsf{B}}_{\mathrm{d}\mathsf{f}} = \hat{\mathsf{X}}_{\delta\mathsf{f}\oplus\mathrm{d}\mathsf{f}}$, in which case to ensure that $\langle 0|\,\hat{\mathsf{F}}_{\mathsf{f}}^{\dagger}\hat{\mathsf{F}}_{\mathsf{g}}\,|0\rangle = (\!(\delta\mathsf{f},\delta\mathsf{g})\!)_+$ we would have to set $\langle 0|\,\hat{\mathsf{X}}_{u_{(1)}\oplus u_{(3)}}\hat{\mathsf{X}}_{v_{(1)}\oplus v_{(3)}}\,|0\rangle = (\!(u^*_{(1)},v_{(1)})\!)_+$, which has the effect that $\hat{\mathsf{B}}\equiv 0$. The constructions of potentials for $\hat{\mathsf{F}}$ and $\hat{\mathbb{F}}$, as for the constructions in Section II, have the same number of effective degrees of freedom, however with projections to different linear subspaces.

Appendix D: Calculating fermionic VEVs

It is worth showing briefly how the calculation of fermionic VEVs proceeds efficiently. We first choose (part of) an orthogonal basis, then manipulate objects that are either orthogonal or parallel. For $\langle 0|\,\hat{\psi}^{\dagger}_W\hat{\psi}_V\hat{\psi}^{\dagger}_A\hat{\psi}_B\hat{\psi}^{\dagger}_V\hat{\psi}_W\,|0\rangle$, with V and W orthogonal, we choose V and W as a partial basis (if they were not orthogonal, we would first Gram-Schmidt orthogonalize), then write $A = A\perp V\perp W + A\|V + A\|W$, where $A\|V = A - A\perp V$ is the component of A parallel to V, and similarly for B. Only three and then two terms of the nine terms in the expansion of $\hat{\psi}_A^\dagger \hat{\psi}_B$ survive,

$$\begin{aligned}\langle 0|\,\hat{\psi}_W^\dagger \hat{\psi}_V \hat{\psi}_A^\dagger \hat{\psi}_B \hat{\psi}_V^\dagger \hat{\psi}_W\,|0\rangle &= \langle 0|\,\hat{\psi}_W^\dagger \hat{\psi}_V \hat{\psi}_{A\perp V\perp W}^\dagger \hat{\psi}_{B\perp V\perp W} \hat{\psi}_V^\dagger \hat{\psi}_W\,|0\rangle + \langle 0|\,\hat{\psi}_W^\dagger \hat{\psi}_V \hat{\psi}_{A\|V}^\dagger\, \hat{\psi}_{B\|V}\, \hat{\psi}_V^\dagger \hat{\psi}_W\,|0\rangle \\ &\quad + \langle 0|\,\hat{\psi}_W^\dagger \hat{\psi}_V \hat{\psi}_{A\|W}^\dagger \hat{\psi}_{B\|W} \hat{\psi}_V^\dagger \hat{\psi}_W\,|0\rangle \\ &= \left[(A\perp V\perp W, B\perp V\perp W)_+ + (A\|V, B\|V)\right] \langle 0|\,\hat{\psi}_W^\dagger \hat{\psi}_V \hat{\psi}_V^\dagger \hat{\psi}_W\,|0\rangle,\end{aligned}$$

where the $\hat{\psi}_{A\|W}^\dagger \hat{\psi}_{B\|W}$ term vanishes because $\{\hat{\psi}_V, \hat{\psi}_{A\|W}^\dagger\} = 0$ and $\hat{\psi}_W^\dagger \hat{\psi}_{A\|W}^\dagger = 0$. Note especially that factors move outside the vacuum state evaluation with or without a projection to positive frequency, $(\cdot,\cdot)_+$ or $(\cdot,\cdot)$, depending respectively on whether they are orthogonal to all other factors or a linear multiple of some other factor. The two pre–inner products on (the infinite–dimensional) test function space are, as always, crucial.

[1] P. Morgan, EPL **87**, 31002(2009).
[2] J. Cohn, American Journal of Physics **48**, 379(1980).
[3] P. Morgan, arXiv:0908.2439 [quant-ph].
[4] A. K. Rajagopal and P. Ghose, Pramana – J. Phys. **86**, 1161(2016).
[5] B. O. Koopman, Proc. Nat. Acad. Sci. **17**, 315(1931).
[6] J. von Neumann, Ann. Math. **33**, 587(1932).
[7] A. S. Blum, Studies in History and Philosophy of Modern Physics **60**, 46(2017).
[8] R. Menikoff and D. H. Sharp, J. Math. Phys. **18**, 471(1977).
[9] L. Cohen, Proc. IEEE **77**, 941(1989).
[10] R. F. Streater and A. S. Wightman, “PCT, spin & statistics, and all that”, (W. A. Benjamin, New York, 1964).
[11] R. Haag, “Local Quantum Physics”, 2nd Edition (Springer, Berlin, 1996).
[12] K.-H. Neeb and G. Ólafsson, arXiv:1802.09037 [math.RT].
[13] G. ’t Hooft, “The Cellular Automaton Interpretation of Quantum Mechanics” (Springer, Heidelberg, 2016).
[14] P. Jordan and E. Wigner, Z. Phys. **47**, 631(1928).
[15] C. Itzykson and J.-B. Zuber, “Quantum Field Theory” (McGraw-Hill, New York, 1980).
[16] T. H. Boyer, Phys. Rev. D **11**, 809(1975).
[17] T. H. Boyer, Phys. Rev. D **11**, 790(1975).
[18] L. de la Peña and A. M. Cetto, “The Quantum Dice: An Introduction to Stochastic Electrodynamics” (Kluwer, Dordrecht, 1996).
[19] X.-F. Qian, A. N. Vamivakas, and J. H. Eberly, Optics and Photonics News **28** (10), 34(2017).
[20] P. Morgan, arXiv:1507.08299 [hep-th].
[21] W. H. Zurek, Phys. Rev. A **76**, 052110(2007)
[22] J. Riedel, http://blog.jessriedel.com/2016/11/12/how-to-think-about-quantum-mechanics-part-0-measurements-are-about-bases/ (accessed February 10th, 2018).
[23] K. Rejzner,“Perturbative Algebraic Quantum Field Theory” (Springer, Heidelberg, 2016).
[24] N. M. J. Woodhouse, “Geometric Quantization”, 2nd Edition (Oxford University Press, Oxford, 1991).
[25] F. Zalamea, arXiv:1711.06914 [physics.hist-ph].
[26] V. Kisil, Geometry, Integrability and Quantization, **18**, 11 (2017); arXiv:1611.05650v1 [math-ph].
[27] D. Mauro, Phys. Lett. **A 315**, 28(2003).
[28] M. de Gosson, “Symplectic Geometry and Quantum Mechanics” (Birkhäuser, Basel, 2006).
[29] M. A. de Gosson and B. J. Hiley, Found. Phys. **41**, 1415(2011)