Baby problems in Quantum Field Theory

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1 The $i - \epsilon$ prescription in Quantum Field Theory

When I first encountered Quantum Field Theory, I was really puzzled by the so-called $i-\epsilon$ prescription which is used in propagators of quantum fields. Only much later I encountered a simple example which clarified many of my confusions. So if you are also confused by this topic, I hope these notes will take your confusion away.

1.1 Some complex analysis and Fourier analysis

Whenever we integrate a complex function about a closed contour C in the complex plane, something remarkable happens. Under certain conditions we can write the function as a power series in the complex variable z, and then a very powerful result can be used: the integral

$$\int_C z^n dz \tag{1}$$

is only non-zero for $n = -1$. Namely, writing $z = Re^{i\theta}$ and $dz = iRe^{i\theta}d\theta$, we obtain

$$\int_C z^n dz = \int_0^{2\pi} \left(Re^{i\theta}\right)^n iRe^{i\theta}d\theta = iR^{n+1}\int_0^{2\pi} e^{i\theta(n+1)}d\theta\,. \tag{2}$$

It follows that this integral is zero (after all, $e^{2n\pi} = e^0$ for integer n), unless $n = -1$:

$$\int_C z^{-1}dz = \int_0^{2\pi}\left(Re^{i\theta}\right)^{-1} iRe^{i\theta}d\theta = i\int_0^{2\pi} d\theta = 2\pi i\,. \tag{3}$$

However, in general it is important that the contour encircles the so-called singularity $z = 0$, because otherwise the integral would still yield zero. This brings us to the following important result: if one integrates a complex function $f(z)$ along a contour C containing a singularity, then the value of this integral is given by $2\pi i$ times the a_{-1} (the so-called "residue") coefficient in the series expansion of the function. Often we will choose the contour as a semi-circle in the complex plane. If the function then has a certain asymptotic behaviour, we can apply Jordan's lemma which states that the half-circle of the contour doesn't contribute to the integral if we make the radius of the semi-circle infinitely big. Our integral then effectively becomes an integral over the real axis from minus infinity to plus infinity, and such integrals are often encountered in physics.

We define for any function $f(t)$ the Fourier transform and its inverse as follows:

$$\tilde{f}(\omega) = \int_t f(t)e^{-i\omega t}dt\,, \qquad f(t) = \frac{1}{2\pi}\int_\omega \tilde{f}(\omega)e^{+i\omega t}dt\,. \tag{4}$$

Now we can discuss Jordan's lemma. Often we take as a contour the real line $[-R, +R]$ and close the contour by a semi-circle C_R, after which we take the limit $R \to \infty$. Jordan's lemma states the following:

$$\lim_{R\to\infty} \int_{C_R} f(z) e^{itz} dz = 0 \tag{5}$$

for certain functions $f(z)$, where $t > 0$ is real. These functions satisfy $f(z) \to 0$ for $|z| \to \infty$. In physics these functions often have the form $f(z) = \frac{1}{P(z)}$ where $P(z)$ is a polynomial of degree 1 or higher. As such there are only a finite number of poles inside the contour. Note: here we close the contour in the upper half plane, see figure 1. The intuition behind this is that the imaginary part is positive and becomes very large, giving a damping factor.

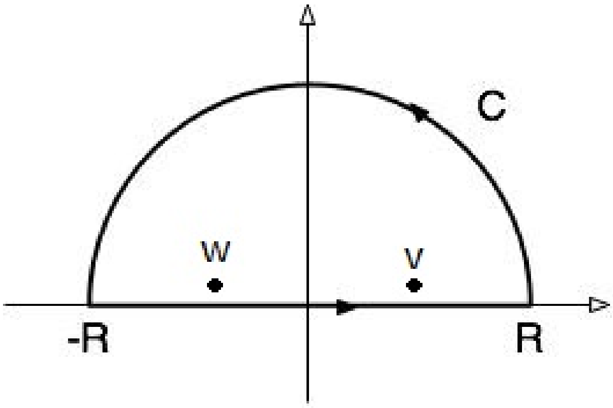

Figure 1: *The contour C used in our integration.*

If we close the contour in the lower half plance we must take $t < 0$. Jordan's lemma is proved by noting that

$$|\int_{C_R} f(z) e^{itz}| \leq \pi \cdot R \cdot \max_{C_R} |f(z)| \tag{6}$$

and using the parametrization $z = Re^{i\theta}$ (so $dz = iRe^{i\theta} d\theta$ for fixed R). Let's see how this works for a function of the form

$$f(z) = \frac{e^{itz}}{(z-w)(z-v)} \tag{7}$$

where the complex numbers w and v have a positive (!) imaginary part, so they are positioned in the upper half plane of figure 1. We parametrize the contour of figure 1 as $z = Re^{i\theta}$. We then have

$$\int_C f(z) = \int_{-R}^{+R} f(z)dz + \int_{C_R} f(z)dz = \int_{-R}^{+R} \frac{e^{itz}}{(z-w)(z-v)} dz + \int_{C_R} \frac{e^{itz}}{(z-w)(z-v)} dz\,. \tag{8}$$

The term along the half circle then becomes

$$\int_{C_R} \frac{e^{itz}}{(z-w)(z-v)} dz = \int_{C_R} \frac{e^{it(Re^{i\theta})}}{(Re^{i\theta}-w)(Re^{i\theta}-v)} iRe^{i\theta} d\theta\,. \tag{9}$$

For large R we then obtain

$$|\int_{C_R} \frac{e^{itz}}{(z-w)(z-v)} dz| \leq \pi \cdot R \cdot \max_{C_R} |f(z)| \sim \pi \cdot R \times \frac{1}{R^2} \qquad (10)$$

which indeed goes to zero in the limit $R \to \infty$. In this limit the contour integral then effectively becomes an integral along the real line

$$\lim_{R\to\infty} \int_C f(z) = \int_{-\infty}^{+\infty} f(z) dz \,, \qquad (11)$$

and we can use the powerful machinery of the residue calculus.

1.2 The forced harmonic oscillator

We first look at the free harmonic oscillator. It's displacement is called $\phi(t)$ (we're using field theory language here), and as such Newton's second law reads

$$-k\phi(t) + F_{ext} = m\ddot{\phi}(t) \,. \qquad (12)$$

For $F_{ext} = 0$ the natural frequency of the harmonic oscillator is

$$\omega_0 = \sqrt{\frac{k}{m}} \,, \qquad (13)$$

and we also define $j(t) = \frac{F_{ext}}{m}$ which can be read as the "current" or the external source. As such we can also write Newton's second law for our harmonic oscillator as

$$\ddot{\phi}(t) + \omega_0^2 \phi(t) = j(t) \,. \qquad (14)$$

We now turn this differential equation into an algebraic one by Fourier transforming:

$$\tilde{\phi}(\omega) = \int_t \phi(t) e^{-i\omega t} dt \,, \qquad \tilde{j}(\omega) = \int_t j(t) e^{-i\omega t} dt \,. \qquad (15)$$

Then

$$\tilde{\ddot{\phi}} = -\omega^2 \tilde{\phi}(\omega) \,, \qquad (16)$$

and so Newton's second law becomes

$$\tilde{j}(\omega) = [\omega_0^2 - \omega^2]\tilde{\phi}(\omega) \equiv -(\omega - \omega_+)(\omega - \omega_-)\tilde{\phi}(\omega) \,. \qquad (17)$$

where we definied $\omega_\pm \equiv \pm\omega_0$ for later convenience. The solution $\phi(t)$ for a given external source can now be written as

$$\phi(t) = \frac{1}{2\pi} \int_\omega \frac{-\tilde{j}(\omega) e^{+i\omega t}}{(\omega - \omega_+)(\omega - \omega_-)} d\omega \,. \qquad (18)$$

However, the problem is clear: we integrate along the real ω-axis, but there are poles at $\omega = \omega_\pm$. So the integrand becomes infinite at these poles. What to do? These poles are a sign that we try to do something (integrating) which doesn't make sense.Think about it: we can't expect to find a unique solution with our integral if someone throws us a $\tilde{j}(\omega)$, because we haven't imposed any boundary conditions yet. So it would be very strange indeed if we *would* be able to integrate along a contour to obtain a unique solution! We solve this issue by adding an extra friction term. This friction term will shift the poles from the real ω-axis such that we are able to perform the integral and find a solution. That means this friction term should also entail a boundary condition, and it does. After all, with friction we expect that

$$\lim_{t\to\infty} \phi(t) = 0\,. \tag{19}$$

By a continuity argument, we then expect that in the limit of vanishing ("arbitrary small") friction, we obtain the solution for the free harmonic oscillator. As such we avoid to integrate along the real axis and hitting the poles. So let's do that and add a friction term $-\mu\dot{\phi}(t)$ to our equations, where the constant μ is the friction coefficient. We also define $2\beta \equiv \frac{\mu}{m}$, such that

$$\ddot{\phi(t)} + 2\beta\dot{\phi} + \omega_0^2\phi(t) = j(t) \qquad (\beta > 0)\,. \tag{20}$$

The time derivative of the friction term now Fourier transforms as

$$\dot{\phi}(t) \to \tilde{\dot{\phi}}(\omega) = i\omega\tilde{\phi}(\omega)\,, \tag{21}$$

so Newton's second law becomes

$$\tilde{j}(\omega) = [\omega_0^2 - \omega^2 + 2\beta i\omega]\tilde{\phi}(\omega)\,. \tag{22}$$

Analogously to the frictionless case we now write

$$\tilde{j}(\omega) = -(\omega - \omega_+)(\omega - \omega_-)\tilde{\phi}(\omega)\,, \tag{23}$$

but now with $\omega_\pm$ given by the quadratic equation

$$\omega_0^2 - \omega^2 + 2\beta i\omega = 0 \ \to \omega_\pm = \beta i \pm \sqrt{\omega_0^2 - \beta^2} = \beta i \pm \omega_0\sqrt{1 - \left(\frac{\beta}{\omega_0}\right)^2}\,. \tag{24}$$

Note that the two singularities $\omega_\pm$ both lie above the real axis. We then again have the solution

$$\phi(t) = \frac{1}{2\pi}\int_\omega \frac{-\tilde{j}(\omega)e^{+i\omega t}}{(\omega - \omega_+)(\omega - \omega_-)}d\omega\,. \tag{25}$$

Now we take

$$j(t) = C\delta(t) \tag{26}$$

with C a real constant. In other words, we apply an external force at $t = 0$ (and $t = 0$ only!) on our oscillator. The Fourier transformation of this external source is simple:

$$\tilde{j}(\omega) = C\,. \tag{27}$$

So

$$\phi(t) = \frac{C}{2\pi}\int_\omega \frac{-e^{+i\omega t}}{(\omega - \omega_+)(\omega - \omega_-)}d\omega\,. \tag{28}$$

We analytically extend this integral to the complex plane and consider ω to be a complex parameter. For $t < 0$ we close the contour downwards. This means that $i\omega < 0$, and $i\omega t > 0$. Jordan's lemma ensures us that the half circle will not contribute to the integral in that case, and so our integral becomes a contour integral. However, the contour doesn't contain any poles, and as such the integral will be zero: $\phi(t) = 0$ for $t < 0$. This makes perfect sense! We only hit our oscillator at $t = 0$, so before that we don't expect any displaclement! But now we consider positive time $t > 0$. In that case we close the contour in the upper part of the complex plane, and our contour contains both poles. So we'd better look at the residues of the function

$$f(\omega) = \frac{C}{2\pi} \frac{-e^{i\omega t}}{(\omega - \omega_+)(\omega - \omega_-)} \,. \tag{29}$$

Each pole is a simple pole, as can be easily seen (the exponential function is analytic for all z). The residues can be calculated in different ways. First we note that

$$\begin{aligned} \frac{1}{(\omega - \omega_+)(\omega - \omega_-)} &= \frac{1}{(\omega_- - \omega_+)} \left[\frac{1}{(\omega - \omega_-)} - \frac{1}{(\omega - \omega_+)} \right] \\ &= \frac{1}{2\sqrt{\omega_0^2 - \beta^2}} \left[\frac{1}{(\omega - \omega_-)} - \frac{1}{(\omega - \omega_+)} \right] . \end{aligned} \tag{30}$$

So,

$$f(\omega) = \frac{Ce^{+i\omega t}}{4\pi\sqrt{\omega_0^2 - \beta^2}} \left[\frac{1}{(\omega - \omega_+)} - \frac{1}{(\omega - \omega_-)} \right] . \tag{31}$$

This gives us

$$Res[f(z)]_{\omega_\pm} = \frac{\pm Ce^{+i\omega_\pm t}}{4\pi\sqrt{\omega_0^2 - \beta^2}} \,. \tag{32}$$

Then

$$\begin{aligned} \phi(t) &= 2\pi i \Big(Res[f(z)]_{\omega_+} + Res[f(z)]_{\omega_-} \Big) \\ &= \frac{Ce^{+i\omega_+ t}}{2\sqrt{\omega_0^2 - \beta^2}} - \frac{Ce^{+i\omega_- t}}{2\sqrt{\omega_0^2 - \beta^2}} \\ &= \frac{C}{2\sqrt{\omega_0^2 - \beta^2}} \Big(e^{+i\omega_+ t} - e^{+i\omega_- t} \Big) . \end{aligned} \tag{33}$$

Remembering that

$$\omega_\pm = \beta i \pm \sqrt{\omega_0^2 - \beta^2} \equiv \beta i \pm \lambda \tag{34}$$

where we defined $\lambda \equiv \sqrt{\omega_0^2 - \beta^2}$, we finally obtain

$$\phi(t) = \theta(t) \frac{C}{\lambda} e^{-\beta t} \sin(\lambda t) \,, \tag{35}$$

with $\theta(t)$ the Heaviside step function. Note that indeed $\lim_{t\to\infty} \phi(t) = 0$ and that the amplitude of the oscillation is determined by both the source term (C) and the mass, spring constant and friction (through λ).

Now, let's go back to the free harmonic oscillator which is turned on at $t = 0$:

$$\ddot{\phi}(t) + \omega_0^2 \phi(t) = j(t), \qquad j(t) = C\delta(t) \,. \tag{36}$$

Fourier transforming this equation and using contour integration, we noted that we hit poles with our contour. So we added a friction term with parameter β, do the Fourier transformation and contour integration, and in the end we send $\beta \to 0$ with the continuity argument that this should indeed give us the solution for the free harmonic oscillator with a boundary condition (the boundary condition survives this limit). In our case the limit $\beta \to 0$ (and $\lambda \to \omega_0$) of eqn.(35) becomes

$$\phi(t) = \frac{C}{\omega_0}\theta(t)\sin(\omega_0 t)\,, \tag{37}$$

But we can also do a different calculation obtaining the same results. As an alternative we can take eqn.(36), Fourier transform it, do a contour integration, but deform the contour like in fig.2.

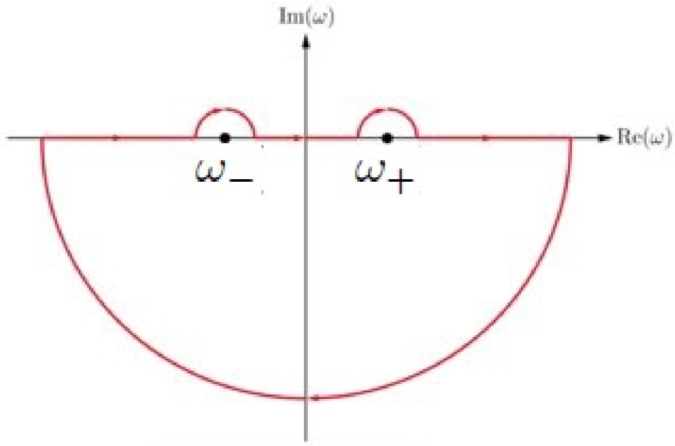

Figure 2: *The deformed contour C.*

If we make the small circles arbitrarily small, we obtain (from a continuity argument) again the undamped oscillator. The choice of the deformation determines the boundary condition we want and thus depends on the specific problem we're interested in. In this case we want that for $t < 0$ the field $\phi(t)$ remains zero, while for $t \geq 0$ it is turned on. This means that we have to close the contour again in the upper half plane, with the small semi-circles around both poles in the lower half plane. And precisely this is achieved by the $i - \epsilon$ prescription. As such this prescription builds in the causality we impose on the problem by hand, and as such we have a certain freedom to choose the contour.

2 Renormalization

In this section I'll explain renormalization with a similar algebraic problem. The equation we will analyse is the following:

$$\epsilon x^2 - 2x + 1 = 0\,. \tag{38}$$

Here ϵ is a small number. Of course, the exact solutions of x are known:

$$x = \frac{1 \pm \sqrt{1-\epsilon}}{\epsilon}\,. \tag{39}$$

But let's pretend we don't know about the quadratic formula. Instead, we will do a perturbation:

$$x = \sum_{n=0}^{\infty} x_n \epsilon^n = x_0 + x_1\epsilon + x_2\epsilon^2 + \dots \tag{40}$$

Putting this perturbation in our defining eqn.(48) and collecting the same powers of ϵ, we get

$$[-2x_0 + 1] + [x_0^2 - 2x_1]\epsilon + [2x_0x_1 - 2x_2]\epsilon^2 + \dots = 0\,. \tag{41}$$

Solving order for order, we get

$$\begin{aligned}
&\mathcal{O}(\epsilon^0) : [-2x_0 + 1] = 0 \to x_0 = \frac{1}{2}\,.\\
&\mathcal{O}(\epsilon^1) : [x_0^2 - 2x_1] = 0 \to x_1 = \frac{1}{8}\,.\\
&\mathcal{O}(\epsilon^2) : [2x_0x_1 - 2x_2] = 0 \to x_2 = \frac{1}{16}\,.\\
&\mathcal{O}(\epsilon^3) : \dots
\end{aligned} \tag{42}$$

So,

$$x = \frac{1}{2} + \frac{1}{8}\epsilon + \frac{1}{16}\epsilon^2 + \dots \tag{43}$$

Let's compare this to the series expansion of the exact solution (39):

$$x = \frac{1 \pm \sqrt{1-\epsilon}}{\epsilon} \approx \frac{1}{\epsilon} \pm \frac{1}{\epsilon}\left(1 - \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 - \frac{1}{16}\epsilon^2 + \dots\right). \tag{44}$$

So we should get the *two* solutions

$$\begin{aligned}
x^{(1)} &= \frac{2}{\epsilon} - \frac{1}{8}\epsilon - \frac{1}{16}\epsilon^2 + \dots\\
x^{(2)} &= \frac{1}{2} + \frac{1}{8}\epsilon + \frac{1}{16}\epsilon^2 + \dots
\end{aligned} \tag{45}$$

Indeed, $x^{(2)}$ is obtained by our perturbative approach. But the first solution $x^{(1)}$ which is singular in ϵ blows up in the limit $\epsilon \to 0$ and is missed in our perturbative approach. In this same limit the defining equation (48) becomes the first order equation

$$2x + 1 = 0 \quad \to \quad x = -\frac{1}{2}\,, \tag{46}$$

giving us only the leading term of the solution $x^{(2)}$, but missing every information about the second solution with the singular term. How should we then obtain both solutions in our perturbative scheme? To do that, we define a new variable y via

$$x \equiv \delta(\epsilon) y \tag{47}$$

Our defining equation then becomes

$$\epsilon\delta^2 \cdot y^2 - 2\delta \cdot y + 1 = 0\,. \tag{48}$$

We now choose our $\delta(\epsilon)$ such that we don't obtain a singular solution for y in the $\epsilon \to 0$ limit. As such, we want to avoid the singular ϵ terms, especially in the quadratic term. But otherwise, our $\delta(\epsilon)$ is arbitrary! The simplest choice is

$$\delta(\epsilon) \equiv \frac{1}{\epsilon} \quad (x = \frac{1}{\epsilon} y) \,. \tag{49}$$

Again we expand y in a series:

$$y = \sum_{n=0}^{\infty} y_n \epsilon^n = y_0 + y_1 \epsilon + y_2 \epsilon^2 + \ldots \tag{50}$$

and plug it in our defining equation with our chosen $\delta(\epsilon)$:

$$y^2 - 2y + \epsilon = 0 \,. \tag{51}$$

Note how the singular term has disappeared! Solving order for order again, we get

$$\begin{aligned} &\mathcal{O}(\epsilon^0) : [y_0(y_0 - 2)] = 0 \to y_0 = 0 \text{ or } 2 \,. \\ &\mathcal{O}(\epsilon^1) : [2y_0 y_1 - 2y_1 + 1] = 0 \to y_1 = \frac{1}{8} \text{ or } -\frac{1}{2} \,. \\ &\mathcal{O}(\epsilon^2) : [\ldots] \end{aligned} \tag{52}$$

Note that the zeroth order equation is quadratic in y_0, giving us *two* solutions for y! This is expected, because the quadratic term isn't singular anymore. Collecting same orders of ϵ, we obtain the two non-singular solutions

$$\begin{aligned} y^{(1)} &= \frac{\epsilon}{2} + \frac{\epsilon^2}{8}\epsilon + \ldots \\ y^{(2)} &= 2 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \ldots \end{aligned} \tag{53}$$

Remembering that $x = \frac{1}{\epsilon} y$ we now obtain *two* solutions, and these are indeed given by (45).

Now, as we said, our $\delta(\epsilon)$ is arbitrary, as long as it absorps the singular term in x. We can parametrise this arbitrariness by some parameter a, and write

$$\delta = \delta(a, \epsilon), \qquad y = y(a, \epsilon) \,. \tag{54}$$

In general, we can write e.g.

$$\delta(a, \epsilon) = \frac{f_{-1}(a)}{\epsilon} + \sum_i \epsilon^i f_{(i)}(a) \,, \tag{55}$$

with $f_{(i)}(a)$ analytic functions in a. But our original x doesn't depend on the parameter a! So we must have that

$$\frac{dx}{da} = \frac{d(\delta \cdot y)}{da} = \frac{d(\delta)}{da} \cdot y + \delta \cdot \frac{d(y)}{da} = 0 \tag{56}$$

at any order in ϵ. In Quantum Field Theory we would call these equations the "renormalization group equations", and a would play the role of energy scale.

3 Natural units

Let's look at some implications when we choose natural units $G = \hbar = c = 1$. We know that in SI-units

$$\begin{aligned} G &= 6,67 \times 10^{-11} \, \frac{m^3}{kg \cdot s^2} \\ c &= 299792458 \, \frac{m}{s} \\ \hbar &= 1,055 \times 10^{-34} \, \frac{kg \cdot m^2}{s} \, . \end{aligned} \tag{57}$$

So, denoting L as length, M as mass and T as time scale, we have $[c] = L/T$, $[\hbar] = ML^2/T$ and $[G] = L^3/(MT^2)$. The Planck units l_P, t_P and m_P are defined by

$$\begin{aligned} G &\equiv 1 \, \frac{l_P^3}{m_P \cdot t_P^2} \\ c &\equiv 1 \, \frac{l_P}{t_P} \\ \hbar &\equiv 1 \, \frac{m_P \cdot l_P^2}{t_P} \, . \end{aligned} \tag{58}$$

The solution of this system of three equations for three units l_P, t_P and m_P is given by

$$\begin{aligned} 1 \, l_P &= \sqrt{\frac{G\hbar}{c^3}} = 1,616 \times 10^{-35} \, m \, , \\ 1 \, t_P &= \sqrt{\frac{G\hbar}{c^5}} = 5,391 \times 10^{-44} \, s \, , \\ 1 \, m_P &= \sqrt{\frac{\hbar c}{G}} = 2,176 \times 10^{-8} \, kg \, . \end{aligned} \tag{59}$$

Note that $l_P = c \times t_P$, so c has as value one Planck length per one Planck time ($= 299792458 \, m/s$). We can invert these values such that

$$\begin{aligned} 1 \, m &= \sqrt{\frac{c^3}{G\hbar}} = 6,19 \times 10^{34} \, l_P \, , \\ 1 \, s &= \sqrt{\frac{c^5}{G\hbar}} = 1,85 \times 10^{43} \, t_P \, , \\ 1 \, kg &= \sqrt{\frac{G}{\hbar c}} = 4,60 \times 10^{7} m_P \, . \end{aligned} \tag{60}$$

Now let's calculate Earth's gravitational acceleration. In SI-units, we have $M = 5,97 \times 10^{24} \, kg$ and $R = 6,378 \times 10^6 \, m$, so

$$g = \frac{GM}{R^2} = 9,8 \, m/s^2 \, . \tag{61}$$

But what about Planck units? Well, here we put $G = 1$, so

$$g = \frac{M}{R^2} \, . \tag{62}$$

But don't be fooled by the disappearance of G; we measure mass and length now in Planck units, which contain G! So we calculate for our Earth parameters M and R that

$$M = 2,746 \times 10^{32}\, m_P\,, \quad R = 3,948 \times 10^{41}\, l_P \tag{63}$$

giving

$$g = \frac{M}{R^2} = 1,76 \times 10^{-51} \frac{m_P}{l_P^2}\,. \tag{64}$$

A tiny acceleration indeed, which is the cost of applying Planck units to Earthly problems. Note that

$$\frac{m_P}{l_P^2} = c^{7/2} \cdot \hbar^{-1/2} \cdot G^{-3/2} \tag{65}$$

and

$$\frac{l_P}{t_P^2} = c^{7/2} \cdot \hbar^{-1/2} \cdot G^{-1/2} = G \cdot \frac{m_P}{l_P^2}\,. \tag{66}$$

That extra factor of G is exactly the factor which "drops out" of the formula for g when we turn to Planck units in order to write the acceleration into the units $\frac{m_P}{l_P^2}$. The unit "Planck length per Planck time squared" would feel more "natural" for an acceleration if you compare it with SI-units than "Planck mass per Planck length squared". As a check, let's calculate $g = 9,8\, m/s^2$ explicitly in those "more natural Planck units":

$$g = 9,8 \frac{1\, m}{(1\, s)^2} = 9,8 \times \frac{6,19 \times 10^{34}\, l_P}{(1,85 \times 10^{43}\, t_P)^2} = 9,8 \times 1,81 \times 10^{-52} \frac{l_P}{t_P^2} = 1,77 \times 10^{-52} \frac{l_P}{t_P^2}\,, \tag{67}$$

which in two significant numbers indeed equals the value of g in the units $\frac{m_P}{l_P^2}$. After all, in Planck units we put $G = 1$, so

$$\frac{m_P}{l_P^2} = \frac{l_P}{t_P^2} \quad \text{(Planck units in which } G = 1)\,. \tag{68}$$

What would happen if, say, because of a change in the string length or string coupling (which, in bosonic string theory is determined by the vev of the dilaton), Newton's gravitational constant G would double its size? Under $G \to 2 \times G$ we get, in SI-units,

$$g = \frac{GM}{R^2} \to 2 \times \frac{GM}{R^2} = 2 \times 9,8\, m/s^2 = 19,6\, m/s^2\,. \tag{69}$$

However, in Planck units, this doubling of G manifests itself as

$$l_P \to \sqrt{2} l_P, \quad t_P \to \sqrt{2} t_P, \quad m_P \to \frac{1}{\sqrt{2}} m_P\,. \tag{70}$$

So looking at the Planck units of Earth's gravitational acceleration g,

$$\frac{m_P}{l_P^2} \to \frac{\frac{1}{\sqrt{2}} m_P}{(\sqrt{2} l_P)^2} = \frac{1}{2\sqrt{2}} \frac{m_P}{l_P^2}\,. \tag{71}$$

Recalculating g in the revised Planck units, we find

$$g \to 2\sqrt{2} \cdot g = 2\sqrt{2} \times 1,76 \times 10^{-51} \frac{m_P}{l_P^2} = 4,98 \times 10^{-51} \frac{m_P}{l_P^2}\,. \tag{72}$$

So whereas in SI-units the value of g doubles if we double G, in Planck units the value of g changes by a factor of $2\sqrt{2}$. That's the "cost" of choosing units containing G.