The probability for a transition from an initial state α to a final state β is

$$S_{\alpha\beta} = (\Psi^{+}_{\alpha}, \Psi^{-}_{\beta}) = (\Phi_{\alpha}, S\Phi_{\beta}), \tag{1}$$

where Ψ_α are the states of the full Hamiltonia H and Φ_α are the states of the free Hamiltonia H_0. According to the Dyson Series the perturbative expansion of the S operator appearing in the above expression is

$$S = 1 + \sum_{n=1}^{\infty} \left(\frac{-i}{n!}\right)^{n} \int_{-\infty}^{+\infty} dt_1 ... dt_n T\{V(t_1)...V(t_n)\}, \tag{2}$$

where

$$V(t) = e^{iH_0 t} V e^{-iH_0 t}, \tag{3}$$

and $V = H - H_0$.

So far all the above hold for a general QFT. Now consider a Lagrangian of the form

$$L = \frac{1}{2}(\partial\Phi_B)^2 + \lambda_B \Phi_B^4. \tag{4}$$

The subscript "B" denotes that all these quantities are bare leading to wrong poles in the scattering amplitude. In order to shift the poles to the right position we rescale the field and the parameters and we define the renormalized quantities as

$$\Phi_B = Z^{1/2}\Phi_R, \quad \lambda_B = \frac{Z_\lambda}{Z^2}\lambda_R. \tag{5}$$

Under this rescaling the Lagrangian is written as

$$L = \frac{1}{2} Z (\partial\Phi_R)^2 + Z_\lambda \lambda_R \Phi_R^4. \tag{6}$$

We quantize the theory by imposing equal-time commutation relations

$$[\Pi(x,t), \Phi(y,t)] = i\delta^3(x-y), \tag{7}$$

where Π is the canonical momentum to the field Φ defined by

$$\Pi(x,t) = \frac{\delta L}{\delta \dot{\Phi}(x,t)}. \tag{8}$$

Using the bare Lagrangian we have that

$$\Pi_B(x,t) = \dot{\Phi}_B(x,t), \tag{9}$$

and therefore

$$[\dot{\Phi}_B(x,t), \Phi_B(y,t)] = i\delta^3(x-y). \tag{10}$$

The same procedure for the renormalized Lagrangian leads to

$$\Pi_R(x,t) = Z\dot{\Phi}_R(x,t), \quad [\dot{\Phi}_R(x,t), \Phi_R(y,t)] = \frac{i}{Z}\delta^3(x-y). \tag{11}$$

Now we proceed to identify the free part of the Hamiltonian and the interaction appearing in the Dyson series. The renormalized Lagrangian leads to the following Hamiltonian

$$H = \Pi_R\dot{\Phi}_R - L_R = \frac{1}{2Z}\Pi_R^2 + \frac{1}{2}Z(\nabla\Phi_R)^2 - Z_\lambda\lambda_R\Phi_R^4. \tag{12}$$

Hence

$$H_0 = \frac{1}{2}\Pi_R^2 + \frac{1}{2}(\nabla\Phi_R)^2, \tag{13}$$

$$\begin{aligned} V &= \frac{1}{2}(\frac{1}{Z} - 1)\Pi_R^2 + \frac{1}{2}(Z-1)(\nabla\Phi_R)^2 - Z_\lambda\lambda_R\Phi_R^4 & (14)\\ &= \frac{1}{2}\frac{1-Z}{Z}\Pi_R^2 + \frac{1}{2}(Z-1)(\nabla\Phi_R)^2 - Z_\lambda\lambda_R\Phi_R^4 & (15)\end{aligned}$$

At this time we make the transition to the Interaction picture fields defined by

$$\begin{aligned} &\Pi_I(x,0) = \Pi_R(x,0), \quad \Pi_I(x,t) = e^{iH_0t}\Pi_I(x,0)e^{-iH_0t},\\ &\Phi_I(x,0) = \Phi_R(x,0), \quad \Phi_I(x,t) = e^{iH_0t}\Phi_I(x,0)e^{-iH_0t}. \end{aligned} \tag{16}$$

So the fields in the interaction picture are free fields that at time zero are equal to the renormalized interacting fields. Since the Hamiltonian H does not depend on time we can set $t = 0$ and replace all the fields in H by the corresponding fields in the interaction picture. Namely

$$\begin{aligned} H_0 &= \frac{1}{2}(\Pi_I(x,0))^2 + \frac{1}{2}\left(\nabla\Phi_I(x,0)\right)^2,\\ V &= \frac{1}{2}\frac{1-Z}{Z}(\Pi_I(x,0))^2 + \frac{1}{2}(Z-1)(\nabla\Phi_I(x,0))^2 - Z_\lambda\lambda_R(\Phi_I(x,0))^4 \end{aligned} \tag{17}$$

The commutation relations obeyed by the fields in the interaction picture are

$$[\Pi_R(x,0), \Phi_R(y,0)] = [\Pi_I(x,0), \Phi_I(y,0)] = i\delta^3(x-y), \tag{18}$$

and performing the similarity transformation (16) we get

$$[\Pi_I(x,t), \Phi_I(y,t)] = i\delta^3(x-y). \tag{19}$$

Since the fields in the interaction picture evolve with the free Hamiltonian and they obey the commutation relations (19) they have the usual free field expansion in terms of raising and lowering operators. The final step now in order to get ready to compute the S-matrix is to write the interaction V in terms of the fields in the interaction picture. We can easily see that

$$V(t) = e^{iH_0t}Ve^{-iH_0t} = \frac{1}{2}\frac{1-Z}{Z}(\Pi_I(x,t))^2 + \frac{1}{2}(Z-1)(\nabla\Phi_I(x,t))^2 - Z_\lambda\lambda_R(\Phi_I(x,t))^4. \quad (20)$$

In order to write the interaction in terms of the field and its derivatives only we first note that

$$H_0 = e^{iH_0t}H_0e^{-iH_0t} = \frac{1}{2}(\Pi_I(x,t))^2 + \frac{1}{2}\left(\nabla\Phi_I(x,t)\right)^2, \quad (21)$$

and from Heisenberg equation of motion we get

$$\dot{\Phi}_I(x,t) = i[H_0, \Phi_I(x,t)] = \Pi_I(x,t). \quad (22)$$

Thus, finally we have that

$$V(t) = \frac{1}{2}\frac{1-Z}{Z}(\dot{\Phi}_I(x,t))^2 + \frac{1}{2}(Z-1)(\nabla\Phi_I(x,t))^2 - Z_\lambda\lambda_R(\Phi_I(x,t))^4. \quad (23)$$

So it seems that the interaction is not even Lorentz Invariant... A Professor told me that in the Canonical Formalism that happens and that there is a theorem called Mathieu Theorem that states that at the end the symmetry is restored but he couldn't provide reference. Any idea?