Introduce a function $g \in L^p(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ and a number

$$\alpha \in (0,3), \quad p > \frac{3}{3-\alpha}.$$

THEOREM 1.

$$\lim_{|x|\to\infty} \int_{\mathbb{R}^3} \frac{g(y)dy}{|x-y|^\alpha} = 0.$$

PROOF.

Equip the space $X = L^p(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ with a norm

$$\|f\| = \|f\|_{L^p(\mathbb{R}^3)} + \|f\|_{L^1(\mathbb{R})}.$$

So that X is a Banach space.

Consider a sequence $\{x_j\} \subset \mathbb{R}^3$ such that: $|x_j| \to \infty$.

And let:

$$u_j : X \to \mathbb{R}; \quad u_j(f) = \int_{\mathbb{R}^3} \frac{f(y)dy}{|x_j - y|^\alpha} = \int_{\mathbb{R}^3} \frac{f(\xi + x_j)d\xi}{|\xi|^\alpha}$$

be a sequence of linear functions. This sequence is uniformly bounded. Indeed, let $B_n \subset \mathbb{R}^3$ be an open ball of radius n centered at the the origin, $n \in \mathbb{N}$;

$$|u_j(f)| \le \int_{B_1} \frac{|f(\xi + x_j)|d\xi}{|\xi|^\alpha} + \int_{\mathbb{R}^3\setminus B_1} \frac{|f(\xi + x_j)|d\xi}{|\xi|^\alpha} \le \Big(\int_{B_1} \frac{d\xi}{|\xi|^{q\alpha}}\Big)^{1/q} \|f\|_{L^p(\mathbb{R}^3)} + \|f\|_{L^1(\mathbb{R}^3)},$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let $X_0 \subset X$ denote a set of continuous in $\mathbb{R}^3$ functions that have compact support. The set X_0 is dense in X.

If $h \in X_0$ then $u_j(h) \to 0$. Indeed let $\operatorname{supp} h \subset B_n$ then

$$|u_j(h)| \le \int_{B_n} \frac{|h(\xi + x_j)|d\xi}{|\xi|^\alpha} + \int_{\mathbb{R}^3\setminus B_n} \frac{|h(\xi + x_j)|d\xi}{|\xi|^\alpha}.$$

The first summand in the right side of this inequality vanishes as $|x_j|$ is big enough;

$$\int_{\mathbb{R}^3\setminus B_n} \frac{|h(\xi + x_j)|d\xi}{|\xi|^\alpha} \le \frac{1}{n^\alpha}\|h\|_{L^1(\mathbb{R}^3)} \to 0 \quad \text{as} \quad n \to \infty.$$

By the Uniform Boundedness Principle the theorem is proved.