TABLE 6.1 Summary of Results for Series and Parallel Resonators

Quantity	Series Resonator	Parallel Resonator
Input impedance/admittance	$Z_{\text{in}} = R + j\omega L - j\dfrac{1}{\omega C}$	$Y_{\text{in}} = \dfrac{1}{R} + j\omega C - j\dfrac{1}{\omega L}$
	$\simeq R + j\dfrac{2RQ_0\Delta\omega}{\omega_0}$	$\simeq \dfrac{1}{R} + j\dfrac{2Q_0\Delta\omega}{R\omega_0}$
Power loss	$P_{\text{loss}} = \dfrac{1}{2}\|I\|^2 R$	$P_{\text{loss}} = \dfrac{1}{2}\dfrac{\|V\|^2}{R}$
Stored magnetic energy	$W_m = \dfrac{1}{4}\|I\|^2 L$	$W_m = \dfrac{1}{4}\|V\|^2 \dfrac{1}{\omega^2 L}$
Stored electric energy	$W_e = \dfrac{1}{4}\|I\|^2 \dfrac{1}{\omega^2 C}$	$W_e = \dfrac{1}{4}\|V\|^2 C$
Resonant frequency	$\omega_0 = \dfrac{1}{\sqrt{LC}}$	$\omega_0 = \dfrac{1}{\sqrt{LC}}$
Unloaded Q	$Q_0 = \dfrac{\omega_0 L}{R} = \dfrac{1}{\omega_0 RC}$	$Q_0 = \omega_0 RC = \dfrac{R}{\omega_0 L}$
External Q	$Q_e = \dfrac{\omega_0 L}{R_L}$	$Q_e = \dfrac{R_L}{\omega_0 L}$

external load resistor, R_L. If the resonator is a series RLC circuit, the load resistor R_L adds in series with R, so the effective resistance in (6.8) is $R + R_L$. If the resonator is a parallel RLC circuit, the load resistor R_L combines in parallel with R, so the effective resistance in (6.18) is $RR_L/(R + R_L)$. If we define an *external* Q, Q_e, as

$$Q_e = \begin{cases} \dfrac{\omega_0 L}{R_L} & \text{for series circuits} \\ \dfrac{R_L}{\omega_0 L} & \text{for parallel circuits,} \end{cases} \tag{6.22}$$

then the loaded Q can be expressed as

$$\frac{1}{Q_L} = \frac{1}{Q_e} + \frac{1}{Q_0}. \tag{6.23}$$

Table 6.1 summarizes the above results for series and parallel resonant circuits.

6.2 TRANSMISSION LINE RESONATORS

As we have seen, ideal lumped circuit elements are often unattainable at microwave frequencies, so distributed elements are frequently used. In this section we will study the use of transmission line sections with various lengths and terminations (usually open- or short-circuited) to form resonators. Because we are interested in the Q of these resonators, we must consider transmission lines with losses.

Short-Circuited $\lambda/2$ Line

A length of lossy transmission line, short circuited at one end, is shown in Figure 6.4. The line has a characteristic impedance, Z_0, propagation constant, β, and attenuation

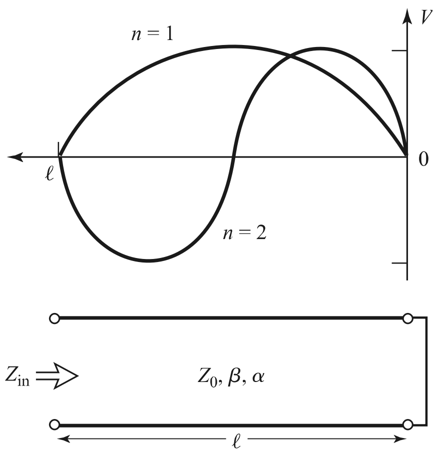

FIGURE 6.4 A short-circuited length of lossy transmission line, and the voltage distributions for $n = 1$ ($\ell = \lambda/2$) and $n = 2$ ($\ell = \lambda$) resonators.

constant, α. At the resonant frequency $\omega = \omega_0$, the length of the line is $\ell = \lambda/2$. From (2.91), the input impedance is

$$Z_{in} = Z_0 \tanh(\alpha + j\beta)\ell.$$

Using an identity for the hyperbolic tangent gives

$$Z_{in} = Z_0 \frac{\tanh \alpha\ell + j \tan \beta\ell}{1 + j \tan \beta\ell \tanh \alpha\ell}. \tag{6.24}$$

Observe that $Z_{in} = jZ_0 \tan \beta\ell$ if $\alpha = 0$ (a lossless line).

In practice it is usually desirable to use a low-loss transmission line, so we assume that $\alpha\ell \ll 1$, and then $\tanh \alpha\ell \simeq \alpha\ell$. Again let $\omega = \omega_0 + \Delta\omega$, where $\Delta\omega$ is small. Then, assuming a TEM line, we have

$$\beta\ell = \frac{\omega\ell}{v_p} = \frac{\omega_0\ell}{v_p} + \frac{\Delta\omega\ell}{v_p},$$

where v_p is the phase velocity of the transmission line. Because $\ell = \lambda/2 = \pi v_p/\omega_0$ for $\omega = \omega_0$, we have

$$\beta\ell = \pi + \frac{\Delta\omega\pi}{\omega_0},$$

and then

$$\tan \beta\ell = \tan\left(\pi + \frac{\Delta\omega\pi}{\omega_0}\right) = \tan \frac{\Delta\omega\pi}{\omega_0} \simeq \frac{\Delta\omega\pi}{\omega_0}.$$

Using these results in (6.24) gives

$$Z_{in} \simeq Z_0 \frac{\alpha\ell + j(\Delta\omega\pi/\omega_0)}{1 + j(\Delta\omega\pi/\omega_0)\alpha\ell} \simeq Z_0 \left(\alpha\ell + j\frac{\Delta\omega\pi}{\omega_0}\right), \tag{6.25}$$

since $\Delta\omega\alpha\ell/\omega_0 \ll 1$.

Equation (6.25) is of the form

$$Z_{in} = R + 2jL\Delta\omega,$$

which is the input impedance of a series *RLC* resonant circuit, as given by (6.9). We can identify the resistance of the equivalent circuit as

$$R = Z_0 \alpha \ell, \tag{6.26a}$$

and the inductance of the equivalent circuit as

$$L = \frac{Z_0 \pi}{2\omega_0}. \tag{6.26b}$$

The capacitance of the equivalent circuit can be found from (6.6) as

$$C = \frac{1}{\omega_0^2 L}. \tag{6.26c}$$

The resonator of Figure 6.4 thus resonates for $\Delta\omega = 0$ ($\ell = \lambda/2$), and its input impedance at resonance is $Z_{\text{in}} = R = Z_0 \alpha \ell$. Resonance also occurs for $\ell = n\lambda/2$, $n = 1, 2, 3, \ldots$. The voltage distributions for the $n = 1$ and $n = 2$ resonant modes are shown in Figure 6.4. The unloaded Q of this resonator can be found from (6.8) and (6.26) as

$$Q_0 = \frac{\omega_0 L}{R} = \frac{\pi}{2\alpha\ell} = \frac{\beta}{2\alpha}, \tag{6.27}$$

since $\beta\ell = \pi$ at the first resonance. This result shows that the Q decreases as the attenuation of the line increases, as expected.

EXAMPLE 6.1 *Q* OF HALF-WAVE COAXIAL LINE RESONATORS

A $\lambda/2$ resonator is made from a piece of copper coaxial line having an inner conductor radius of 1 mm and an outer conductor radius of 4 mm. If the resonant frequency is 5 GHz, compare the unloaded Q of an air-filled coaxial line resonator to that of a Teflon-filled coaxial line resonator.

Solution

We first compute the attenuation of the coaxial line, using the results of Examples 2.6 or 2.7. From Appendix F, the conductivity of copper is $\sigma = 5.813 \times 10^7$ S/m. The surface resistivity at 5 GHz is

$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = 1.84 \times 10^{-2}\ \Omega.$$

The attenuation due to conductor loss for the air-filled line is

$$\begin{aligned} \alpha_c &= \frac{R_s}{2\eta \ln b/a}\left(\frac{1}{a} + \frac{1}{b}\right) \\ &= \frac{1.84 \times 10^{-2}}{2(377) \ln (0.004/0.001)}\left(\frac{1}{0.001} + \frac{1}{0.004}\right) = 0.022 \text{ Np/m.} \end{aligned}$$

For Teflon, $\epsilon_r = 2.08$ and $\tan\delta = 0.0004$, so the attenuation due to conductor loss for the Teflon-filled line is

$$\alpha_c = \frac{1.84 \times 10^{-2}\sqrt{2.08}}{2(377)\ \ln\ (0.004/0.001)}\left(\frac{1}{0.001} + \frac{1}{0.004}\right) = 0.032\ \text{Np/m}.$$

The dielectric loss of the air-filled line is zero, but the dielectric loss of the Teflon-filled line is

$$\begin{aligned}\alpha_d &= k_0 \frac{\sqrt{\epsilon_r}}{2} \tan\delta \\ &= \frac{(104.7)\sqrt{2.08}(0.0004)}{2} = 0.030\ \text{Np/m}.\end{aligned}$$

Finally, from (6.27), the unloaded Qs can be computed as

$$Q_{\text{air}} = \frac{\beta}{2\alpha} = \frac{104.7}{2(0.022)} = 2380,$$

$$Q_{\text{Teflon}} = \frac{\beta}{2\alpha} = \frac{104.7\sqrt{2.08}}{2(0.032 + 0.030)} = 1218.$$

Thus it is seen that the Q of the air-filled line is almost twice that of the Teflon-filled line. The Q can be further increased by using silver-plated conductors. ■

Short-Circuited λ/4 Line

A parallel type of resonance (antiresonance) can be achieved using a short-circuited transmission line of length $\lambda/4$. The input impedance of a shorted line of length ℓ is

$$\begin{aligned}Z_{\text{in}} &= Z_0 \tanh(\alpha + j\beta)\ell \\ &= Z_0 \frac{\tanh\alpha\ell + j\tan\beta\ell}{1 + j\tan\beta\ell\tanh\alpha\ell} \\ &= Z_0 \frac{1 - j\tanh\alpha\ell\cot\beta\ell}{\tanh\alpha\ell - j\cot\beta\ell},\end{aligned} \tag{6.28}$$

where the last result was obtained by multiplying both numerator and denominator by $-j\cot\beta\ell$. Now assume that $\ell = \lambda/4$ at $\omega = \omega_0$, and let $\omega = \omega_0 + \Delta\omega$. Then, for a TEM line,

$$\beta\ell = \frac{\omega_0\ell}{v_p} + \frac{\Delta\omega\ell}{v_p} = \frac{\pi}{2} + \frac{\pi\Delta\omega}{2\omega_0},$$

and so

$$\cot\beta\ell = \cot\left(\frac{\pi}{2} + \frac{\pi\Delta\omega}{2\omega_0}\right) = -\tan\frac{\pi\Delta\omega}{2\omega_0} \simeq \frac{-\pi\Delta\omega}{2\omega_0}.$$

Also, as before, $\tanh\alpha\ell \simeq \alpha\ell$ for small loss. Using these results in (6.28) gives

$$Z_{\text{in}} = Z_0 \frac{1 + j\alpha\ell\pi\Delta\omega/2\omega_0}{\alpha\ell + j\pi\Delta\omega/2\omega_0} \simeq \frac{Z_0}{\alpha\ell + j\pi\Delta\omega/2\omega_0}, \tag{6.29}$$

since $\alpha\ell\pi\Delta\omega/2\omega_0 \ll 1$. This result is of the same form as the impedance of a parallel RLC circuit, as given in (6.19):

$$Z_{\text{in}} = \frac{1}{(1/R) + 2j\Delta\omega C}.$$

We can identify the resistance of the equivalent circuit as

$$R = \frac{Z_0}{\alpha \ell} \tag{6.30a}$$

and the capacitance of the equivalent circuit as

$$C = \frac{\pi}{4\omega_0 Z_0}. \tag{6.30b}$$

The inductance of the equivalent circuit can be found as

$$L = \frac{1}{\omega_0^2 C}. \tag{6.30c}$$

The resonator of Figure 6.4 therefore has a parallel-type resonance for $\ell = \lambda/4$, with an input impedance at resonance of $Z_{in} = R = Z_0/\alpha\ell$. From (6.18) and (6.30) the unloaded Q of this resonator is

$$Q_0 = \omega_0 RC = \frac{\pi}{4\alpha\ell} = \frac{\beta}{2\alpha}, \tag{6.31}$$

since $\ell = \pi/2\beta$ at resonance.

Open-Circuited λ/2 Line

A practical resonator that is often used in microstrip circuits consists of an open-circuited length of transmission line, as shown in Figure 6.5. This resonator will behave as a parallel resonant circuit when the length is λ/2, or multiples of λ/2.

The input impedance of an open-circuited lossy transmission line of length ℓ is

$$Z_{in} = Z_0 \coth(\alpha + j\beta)\ell = Z_0 \frac{1 + j \tan\beta\ell \tanh\alpha\ell}{\tanh\alpha\ell + j\tan\beta\ell}. \tag{6.32}$$

As before, assume that $\ell = \lambda/2$ at $\omega = \omega_0$, and let $\omega = \omega_0 + \Delta\omega$. Then,

$$\beta\ell = \pi + \frac{\pi\Delta\omega}{\omega_0},$$

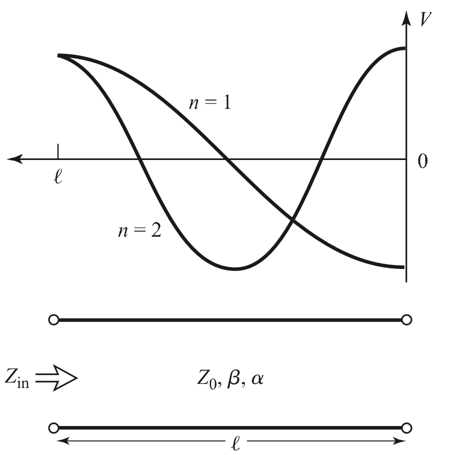

FIGURE 6.5 An open-circuited length of lossy transmission line, and the voltage distributions for $n = 1$ ($\ell = \lambda/2$) and $n = 2$ ($\ell = \lambda$) resonators.

and so

$$\tan \beta\ell = \tan \frac{\Delta\omega\pi}{\omega} \simeq \frac{\Delta\omega\pi}{\omega_0},$$

and $\tanh \alpha\ell \simeq \alpha\ell$. Using these results in (6.32) gives

$$Z_{\text{in}} = \frac{Z_0}{\alpha\ell + j(\Delta\omega\pi/\omega_0)}. \tag{6.33}$$

Comparison with the input impedance of a parallel resonant circuit, as given by (6.19), suggests that the resistance of the equivalent *RLC* circuit is

$$R = \frac{Z_0}{\alpha\ell}, \tag{6.34a}$$

and the capacitance of the equivalent circuit is

$$C = \frac{\pi}{2\omega_0 Z_0}. \tag{6.34b}$$

The inductance of the equivalent circuit is

$$L = \frac{1}{\omega_0^2 C}. \tag{6.34c}$$

From (6.18) and (6.34) the unloaded Q is

$$Q_0 = \omega_0 RC = \frac{\pi}{2\alpha\ell} = \frac{\beta}{2\alpha}, \tag{6.35}$$

since $\ell = \pi/\beta$ at resonance.

EXAMPLE 6.2 A HALF-WAVE MICROSTRIP RESONATOR

Consider a microstrip resonator constructed from a $\lambda/2$ length of 50 Ω open-circuited microstrip line. The substrate is Teflon ($\epsilon_r = 2.08$, $\tan\delta = 0.0004$), with a thickness of 0.159 cm, and the conductors are copper. Compute the required length of the line for resonance at 5 GHz, and the unloaded Q of the resonator. Ignore fringing fields at the end of the line.

Solution

From (3.197), the width of a 50 Ω microstrip line on this substrate is found to be $W = 0.508$ cm, and the effective permittivity is $\epsilon_e = 1.80$. The resonant length can then be calculated as

$$\ell = \frac{\lambda}{2} = \frac{v_p}{2f} = \frac{c}{2f\sqrt{\epsilon_e}} = \frac{3 \times 10^8}{2(5 \times 10^9)\sqrt{1.80}} = 2.24 \text{ cm}.$$

The propagation constant is

$$\beta = \frac{2\pi f}{v_p} = \frac{2\pi f\sqrt{\epsilon_e}}{c} = \frac{2\pi(5 \times 10^9)\sqrt{1.80}}{3 \times 10^8} = 151.0 \text{ rad/m}.$$

From (3.199), the attenuation due to conductor loss is

$$\alpha_c = \frac{R_s}{Z_0 W} = \frac{1.84 \times 10^{-2}}{50(0.00508)} = 0.0724 \text{ Np/m},$$

where we used R_s from Example 6.1. From (3.198), the attenuation due to dielectric loss is

$$\alpha_d = \frac{k_0 \epsilon_r (\epsilon_e - 1) \tan \delta}{2\sqrt{\epsilon_e}(\epsilon_r - 1)} = \frac{(104.7)(2.08)(0.80)(0.0004)}{2\sqrt{1.80}(1.08)} = 0.024 \text{ Np/m}.$$

Then from (6.35) the unloaded Q is

$$Q_0 = \frac{\beta}{2\alpha} = \frac{151.0}{2(0.0724 + 0.024)} = 783.$$ ■

6.3 RECTANGULAR WAVEGUIDE CAVITY RESONATORS

Microwave resonators can also be constructed from closed sections of waveguide. Because radiation loss from an open-ended waveguide can be significant, waveguide resonators are usually short circuited at both ends, thus forming a closed box, or *cavity*. Electric and magnetic energy is stored within the cavity enclosure, and power is dissipated in the metallic walls of the cavity as well as in the dielectric material that may fill the cavity. Coupling to a cavity resonator may be by a small aperture, or a small probe or loop. We will see that there are many possible resonant modes for a cavity resonator, corresponding to field variations along the three dimensions of the structure.

We will first derive the resonant frequencies for a general TE or TM resonant mode of a rectangular cavity, and then derive an expression for the unloaded Q of the $TE_{10\ell}$ mode. A complete treatment of the unloaded Q for arbitrary TE and TM modes can be made using the same procedure, but is not included here because of its length and complexity.

Resonant Frequencies

The geometry of a rectangular cavity is shown in Figure 6.6. It consists of a length, d, of rectangular waveguide shorted at both ends ($z = 0, d$). We will find the resonant

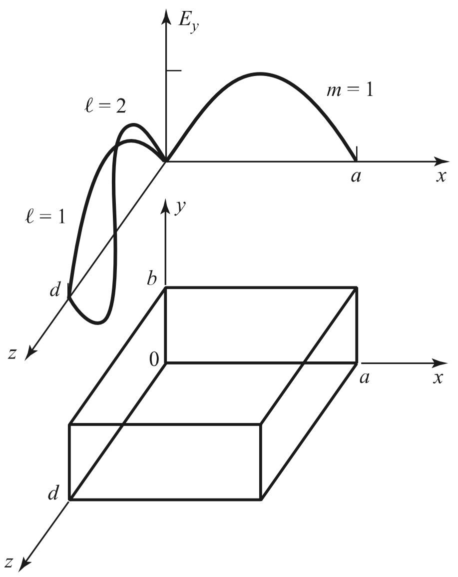

FIGURE 6.6 A rectangular cavity resonator, and the electric field variations for the TE_{101} and TE_{102} resonant modes.

frequencies of this cavity under the assumption that the cavity is lossless, then determine the unloaded Q using the perturbation method outlined in Section 2.7. Although we could begin with the Helmholtz wave equation and the method of separation of variables to solve for the electric and magnetic fields that satisfy the boundary conditions of the cavity, it is easier to start with the fields of the TE or TM waveguide modes since these already satisfy the necessary boundary conditions on the side walls ($x = 0, a$ and $y = 0, b$) of the cavity. Then it is only necessary to enforce the boundary conditions that $E_x = E_y = 0$ on the end walls at $z = 0, d$.

From Table 3.2 the transverse electric fields (E_x, E_y) of the TE_{mn} or TM_{mn} rectangular waveguide mode can be written as

$$\bar{E}_t(x, y, z) = \bar{e}(x, y) \left(A^+ e^{-j\beta_{mn}z} + A^- e^{j\beta_{mn}z}\right), \tag{6.36}$$

where $\bar{e}(x, y)$ is the transverse variation of the mode, and A^+, A^- are arbitrary amplitudes of the forward and backward traveling waves. The propagation constant of the m, nth TE or TM mode is

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}, \tag{6.37}$$

where $k = \omega\sqrt{\mu\epsilon}$, and μ and ϵ are the permeability and permittivity of the material filling the cavity.

Applying the condition that $\bar{E}_t = 0$ at $z = 0$ to (6.36) implies that $A^+ = -A^-$ (as we should expect for reflection from a perfectly conducting wall). Then the condition that $\bar{E}_t = 0$ at $z = d$ leads to the equation

$$\bar{E}_t(x, y, d) = -\bar{e}(x, y)A^+ 2j \sin\beta_{mn}d = 0.$$

The only nontrivial ($A^+ \neq 0$) solution occurs for

$$\beta_{mn}d = \ell\pi, \quad \ell = 1, 2, 3, \ldots, \tag{6.38}$$

which implies that the cavity must be an integer multiple of a half-guide wavelength long at the resonant frequency. No nontrivial solutions are possible for other lengths, or for frequencies other than the resonant frequencies.

A resonance wave number for the rectangular cavity can be defined as

$$k_{mn\ell} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}. \tag{6.39}$$

Then we can refer to the $TE_{mn\ell}$ or $TM_{mn\ell}$ resonant mode of the cavity, where the indices m, n, ℓ indicate the number of variations in the standing wave pattern in the x, y, z directions, respectively. The resonant frequency of the $TE_{mn\ell}$ or $TM_{mn\ell}$ mode is given by

$$f_{mn\ell} = \frac{ck_{mn\ell}}{2\pi\sqrt{\mu_r\epsilon_r}} = \frac{c}{2\pi\sqrt{\mu_r\epsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}. \tag{6.40}$$

If $b < a < d$, the dominant resonant mode (lowest resonant frequency) will be the TE_{101} mode, corresponding to the TE_{10} dominant waveguide mode in a shorted guide of length $\lambda_g/2$, and is similar to the short-circuited $\lambda/2$ transmission line resonator. The dominant TM resonant mode is the TM_{110} mode.

Unloaded Q of the $TE_{10\ell}$ Mode

From Table 3.2, (6.36), and the fact that $A^- = -A^+$, the total fields for the $TE_{10\ell}$ resonant mode can be written as

$$E_y = A^+ \sin\frac{\pi x}{a}\left(e^{-j\beta z} - e^{j\beta z}\right), \tag{6.41a}$$

$$H_x = \frac{-A^+}{Z_{TE}}\sin\frac{\pi x}{a}\left(e^{-j\beta z} + e^{j\beta z}\right), \tag{6.41b}$$

$$H_z = \frac{j\pi A^+}{k\eta a}\cos\frac{\pi x}{a}\left(e^{-j\beta z} - e^{j\beta z}\right). \tag{6.41c}$$

Letting $E_0 = -2jA^+$ and using (6.38) allows these expressions to be simplified to

$$E_y = E_0 \sin\frac{\pi x}{a}\sin\frac{\ell\pi z}{d}, \tag{6.42a}$$

$$H_x = \frac{-jE_0}{Z_{TE}}\sin\frac{\pi x}{a}\cos\frac{\ell\pi z}{d}, \tag{6.42b}$$

$$H_z = \frac{j\pi E_0}{k\eta a}\cos\frac{\pi x}{a}\sin\frac{\ell\pi z}{d}, \tag{6.42c}$$

which clearly show that the fields form standing waves inside the cavity. We can now compute the unloaded Q of this mode by finding the stored electric and magnetic energies, and the power lost in the conducting walls and the dielectric filling.

The stored electric energy is, from (1.84),

$$W_e = \frac{\epsilon}{4}\int_V E_y E_y^* dv = \frac{\epsilon abd}{16}E_0^2, \tag{6.43a}$$

while the stored magnetic energy is, from (1.86),

$$W_m = \frac{\mu}{4}\int_V (H_x H_x^* + H_z H_z^*)dv$$

$$= \frac{\mu abd}{16}E_0^2\left(\frac{1}{Z_{TE}^2} + \frac{\pi^2}{k^2\eta^2 a^2}\right). \tag{6.43b}$$

Because $Z_{TE} = k\eta/\beta$, with $\beta = \beta_{10} = \sqrt{k^2 - (\pi/a)^2}$, the quantity in parentheses in (6.43b) can be reduced to

$$\left(\frac{1}{Z_{TE}^2} + \frac{\pi^2}{k^2\eta^2 a^2}\right) = \frac{\beta^2 + (\pi/a)^2}{k^2\eta^2} = \frac{1}{\eta^2} = \frac{\epsilon}{\mu},$$

showing that $W_e = W_m$ at resonance. The condition of equal stored electric and magnetic energies at resonance also applied to the *RLC* resonant circuits of Section 6.1.

For small losses we can find the power dissipated in the cavity walls using the perturbation method of Section 2.7. Thus, the power lost in the conducting walls is given by (1.131) as

$$P_c = \frac{R_s}{2}\int_{\text{walls}} |H_t|^2 ds, \tag{6.44}$$

where $R_s = \sqrt{\omega\mu_0/2\sigma}$ is the surface resistivity of the metallic walls, and H_t is the tangential magnetic field at the surface of the walls. Using (6.42b), (6.42c) in (6.44) gives

$$P_c = \frac{R_s}{2}\left\{2\int_{y=0}^{b}\int_{x=0}^{a}|H_x(z=0)|^2 dxdy + 2\int_{z=0}^{d}\int_{y=0}^{b}|H_z(x=0)|^2 dydz \right.$$
$$\left. +2\int_{z=0}^{d}\int_{x=0}^{a}\left[|H_x(y=0)|^2 + |H_z(y=0)|^2\right] dxdz\right\}$$
$$= \frac{R_s E_0^2 \lambda^2}{8\eta^2}\left(\frac{\ell^2 ab}{d^2} + \frac{bd}{a^2} + \frac{\ell^2 a}{2d} + \frac{d}{2a}\right), \tag{6.45}$$

where use has been made of the symmetry of the cavity in doubling the contributions from the walls at $x = 0$, $y = 0$, and $z = 0$ to account for the contributions from the walls at $x = a$, $y = b$, and $z = d$, respectively. The relations $k = 2\pi/\lambda$ and $Z_{TE} = k\eta/\beta = 2d\eta/\ell\lambda$ were also used in simplifying (6.45). Then, from (6.7), the unloaded Q of the cavity with lossy conducting walls but lossless dielectric can be found as

$$Q_c = \frac{2\omega_0 W_e}{P_c}$$
$$= \frac{k^3 abd\eta}{4\pi^2 R_s}\frac{1}{[(\ell^2 ab/d^2) + (bd/a^2) + (\ell^2 a/2d) + (d/2a)]}$$
$$= \frac{(kad)^3 b\eta}{2\pi^2 R_s}\frac{1}{(2\ell^2 a^3 b + 2bd^3 + \ell^2 a^3 d + ad^3)}. \tag{6.46}$$

Next we compute the power lost in the dielectric material that may fill the cavity. As discussed in Chapter 1, a lossy dielectric has an effective conductivity $\sigma = \omega\epsilon'' = \omega\epsilon_r\epsilon_0 \tan\delta$, where $\epsilon = \epsilon' - j\epsilon'' = \epsilon_r\epsilon_0(1 - j\tan\delta)$, and $\tan\delta$ is the loss tangent of the material. The power dissipated in the dielectric is, from (1.92),

$$P_d = \frac{1}{2}\int_V \bar{J}\cdot\bar{E}^* dv = \frac{\omega\epsilon''}{2}\int_V |\bar{E}|^2 dv = \frac{abd\omega\epsilon''|E_0|^2}{8}, \tag{6.47}$$

where $\bar{E}$ is given by (6.42a). Then from (6.7) the unloaded Q of the cavity with a lossy dielectric filling, but with perfectly conducting walls, is

$$Q_d = \frac{2\omega W_e}{P_d} = \frac{\epsilon'}{\epsilon''} = \frac{1}{\tan\delta}. \tag{6.48}$$

The simplicity of this result is due to the fact that the integral in (6.43a) for W_e cancels with the identical integral in (6.47) for P_d. This result therefore applies to Q_d for an arbitrary resonant cavity mode. When both wall losses and dielectric losses are present, the total power loss is $P_c + P_d$, so (6.7) gives the total unloaded Q as

$$Q_0 = \left(\frac{1}{Q_c} + \frac{1}{Q_d}\right)^{-1}. \tag{6.49}$$

EXAMPLE 6.3 DESIGN OF A RECTANGULAR CAVITY RESONATOR

A rectangular waveguide cavity is made from a piece of copper WR-187 H-band waveguide, with $a = 4.755$ cm and $b = 2.215$ cm. The cavity is filled with polyethylene ($\epsilon_r = 2.25$, $\tan\delta = 0.0004$). If resonance is to occur at $f = 5$ GHz, find the required length, d, and the resulting unloaded Q for the $\ell = 1$ and $\ell = 2$ resonant modes.

Solution
The wave number k is

$$k = \frac{2\pi f \sqrt{\epsilon_r}}{c} = 157.08 \text{ m}^{-1}.$$

From (6.40) the required length for resonance can be found as ($m = 1, n = 0$)

$$d = \frac{\ell\pi}{\sqrt{k^2 - (\pi/a)^2}},$$

$$\text{for } \ell = 1, \quad d = \frac{\pi}{\sqrt{(157.08)^2 - (\pi/0.04755)^2}} = 2.20 \text{ cm},$$

$$\text{for } \ell = 2, \quad d = 2(2.20) = 4.40 \text{ cm}.$$

From Example 6.1, the surface resistivity of copper at 5 GHz is $R_s = 1.84 \times 10^{-2}\ \Omega$. The intrinsic impedance is

$$\eta = \frac{377}{\sqrt{\epsilon_r}} = 251.3\ \Omega.$$

Then from (6.46) the Q due to conductor loss only is

$$\text{for } \ell = 1, \quad Q_c = 8{,}403,$$
$$\text{for } \ell = 2, \quad Q_c = 11{,}898.$$

From (6.48) the Q due to dielectric loss only is, for both $\ell = 1$ and $\ell = 2$,

$$Q_d = \frac{1}{\tan\delta} = \frac{1}{0.0004} = 2500.$$

Then total unloaded Qs are, from (6.49)

$$\text{for } \ell = 1, \quad Q_0 = \left(\frac{1}{8403} + \frac{1}{2500}\right)^{-1} = 1927,$$
$$\text{for } \ell = 2, \quad Q_0 = \left(\frac{1}{11{,}898} + \frac{1}{2500}\right)^{-1} = 2065.$$

Note that the dielectric loss has the dominant effect on the Q; higher Q could be obtained using an air-filled cavity. These results can be compared to those of Examples 6.1 and 6.2, which used similar types of materials at the same frequency. ■

6.4 CIRCULAR WAVEGUIDE CAVITY RESONATORS

A cylindrical cavity resonator can be constructed from a section of circular waveguide shorted at both ends, similar to rectangular cavities. Because the dominant circular waveguide mode is the TE_{11} mode, the dominant cylindrical cavity mode is the TE_{111} mode. We will derive the resonant frequencies for the $TE_{nm\ell}$ and $TM_{nm\ell}$ circular cavity modes, and an expression for the unloaded Q of the $TE_{nm\ell}$ mode.

Circular cavities are often used for microwave frequency meters. The cavity is constructed with a movable top wall to allow mechanical tuning of the resonant frequency, and the cavity is loosely coupled to a waveguide through a small aperture. In operation, power will be absorbed by the cavity as it is tuned to the operating frequency of the system; this absorption can be monitored with a power meter elsewhere in the system. The

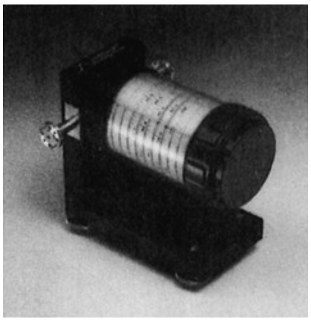

FIGURE 6.7 Photograph of a W-band waveguide frequency meter. The knob rotates to change the length of the circular cavity resonator; the scale gives a readout of the frequency.
Photograph courtesy of Millitech Inc., Northampton, Mass.

mechanical tuning dial is usually directly calibrated in frequency, as in the model shown in Figure 6.7. Because frequency resolution is determined by the Q of the resonator, the TE_{011} mode is often used for frequency meters because its Q is much higher than the Q of the dominant circular cavity mode. This is also the reason for a loose coupling to the cavity.

Resonant Frequencies

The geometry of a cylindrical cavity is shown in Figure 6.8. As in the case of the rectangular cavity, the solution is simplified by beginning with the circular waveguide modes, which already satisfy the necessary boundary conditions on the wall of the circular waveguide. From Table 3.5, the transverse electric fields (E_ρ, E_ϕ) of the TE_{nm} or TM_{nm} circular waveguide mode can be written as

$$\bar{E}_t(\rho, \phi, z) = \bar{e}(\rho, \phi)\left(A^+ e^{-j\beta_{nm}z} + A^- e^{j\beta_{nm}z}\right), \tag{6.50}$$

where $\bar{e}(\rho, \phi)$ represents the transverse variation of the mode, and A^+ and A^- are arbitrary amplitudes of the forward and backward traveling waves. The propagation constant of the TE_{nm} mode is, from (3.126),

$$\beta_{nm} = \sqrt{k^2 - \left(\frac{p'_{nm}}{a}\right)^2}, \tag{6.51a}$$

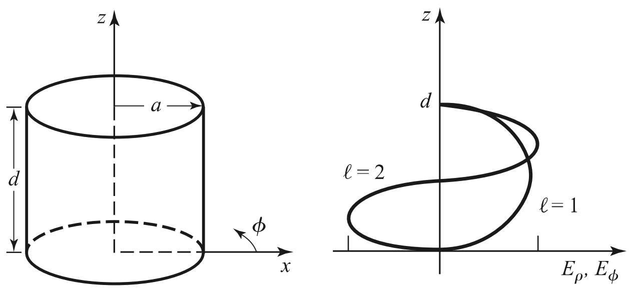

FIGURE 6.8 A cylindrical resonant cavity, and the electric field distribution for resonant modes with $\ell = 1$ or $\ell = 2$.

while the propagation constant of the TM_{nm} mode is, from (3.139),

$$\beta_{nm} = \sqrt{k^2 - \left(\frac{p_{nm}}{a}\right)^2}, \tag{6.51b}$$

where $k = \omega\sqrt{\mu\epsilon}$.

In order to have $\bar{E}_t = 0$ at $z = 0, d$, we must choose $A^+ = -A^-$, and $A^+ \sin \beta_{nm} d = 0$,

or

$$\beta_{nm} d = \ell\pi, \quad \text{for } \ell = 0, 1, 2, 3, \ldots, \tag{6.52}$$

which implies that the waveguide must be an integer number of half-guide wavelengths long. Thus, the resonant frequency of the $TE_{nm\ell}$ mode is

$$f_{nm\ell} = \frac{c}{2\pi\sqrt{\mu_r \epsilon_r}} \sqrt{\left(\frac{p'_{nm}}{a}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}, \tag{6.53a}$$

and the resonant frequency of the $TM_{nm\ell}$ mode is

$$f_{nm\ell} = \frac{c}{2\pi\sqrt{\mu_r \epsilon_r}} \sqrt{\left(\frac{p_{nm}}{a}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}. \tag{6.53b}$$

Thus the dominant TE mode is the TE_{111} mode, while the dominant TM mode is the TM_{010} mode. Figure 6.9 shows a *mode chart* for the lower order resonant modes of a cylindrical cavity. Such a chart is useful for the design of circular cavity resonators, as it shows what modes can be excited at a given frequency for a given cavity size.

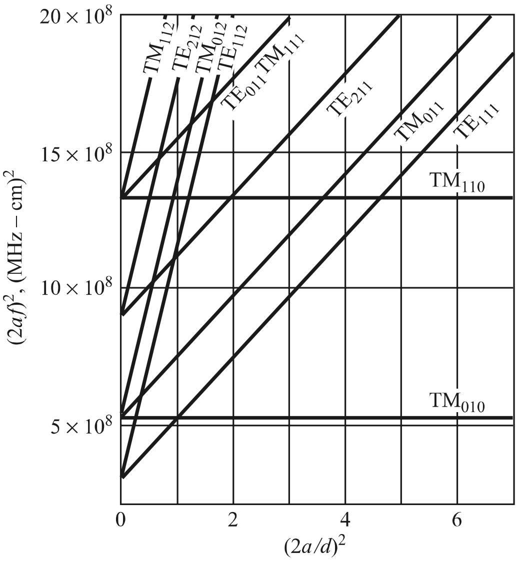

FIGURE 6.9 Resonant mode chart for a cylindrical cavity.

Adapted from data from R. E. Collin, *Foundations for Microwave Engineering,* 2nd edition, Wiley–IEEE Press, Hoboken, N.J., 2001. Used with permission.