



Figure 11.10 A general volume ΔV in orthogonal curvilinear coordinates u_1, u_2, u_3 . PT gives the vector $h_1 \Delta u_1 \hat{\mathbf{e}}_1$, PS gives $h_2 \Delta u_2 \hat{\mathbf{e}}_2$ and PQ gives $h_3 \Delta u_3 \hat{\mathbf{e}}_3$.

► By considering the infinitesimal planar surface element $PQRS$ in figure 11.10, show that (11.17) leads to the usual expression for $\nabla \times \mathbf{a}$ in orthogonal curvilinear coordinates.

The planar surface $PQRS$ is defined by the orthogonal vectors $h_2 \Delta u_2 \hat{\mathbf{e}}_2$ and $h_3 \Delta u_3 \hat{\mathbf{e}}_3$ at the point P . If we traverse the loop in the direction $PSRQ$ then, by the right-hand convention, the unit normal to the plane is $\hat{\mathbf{e}}_1$. Writing $\mathbf{a} = a_1 \hat{\mathbf{e}}_1 + a_2 \hat{\mathbf{e}}_2 + a_3 \hat{\mathbf{e}}_3$, the line integral around the loop in this direction is given by

$$\begin{aligned} \oint_{PSRQ} \mathbf{a} \cdot d\mathbf{r} &= a_2 h_2 \Delta u_2 + \left[a_3 h_3 + \frac{\partial}{\partial u_2} (a_3 h_3) \Delta u_3 \right] \Delta u_3 \\ &\quad - \left[a_2 h_2 + \frac{\partial}{\partial u_3} (a_2 h_2) \Delta u_3 \right] \Delta u_2 - a_3 h_3 \Delta u_3 \\ &= \left[\frac{\partial}{\partial u_2} (a_3 h_3) - \frac{\partial}{\partial u_3} (a_2 h_2) \right] \Delta u_2 \Delta u_3. \end{aligned}$$

Therefore from (11.17) the component of $\nabla \times \mathbf{a}$ in the direction $\hat{\mathbf{e}}_1$ at P is given by

$$\begin{aligned} (\nabla \times \mathbf{a})_1 &= \lim_{\Delta u_2, \Delta u_3 \rightarrow 0} \left[\frac{1}{h_2 h_3 \Delta u_2 \Delta u_3} \oint_{PSRQ} \mathbf{a} \cdot d\mathbf{r} \right] \\ &= \frac{1}{h_2 h_3} \left[\frac{\partial}{\partial u_2} (h_3 a_3) - \frac{\partial}{\partial u_3} (h_2 a_2) \right]. \end{aligned}$$

The other two components are found by cyclically permuting the subscripts 1, 2, 3. ◀

Finally, we note that we can also write the ∇^2 operator as a surface integral by setting $\mathbf{a} = \nabla \phi$ in (11.15), to obtain

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \lim_{V \rightarrow 0} \left(\frac{1}{V} \oint_S \nabla \phi \cdot d\mathbf{S} \right).$$