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Full Length Research Paper

Schwarzschild-like solution for ellipsoidal celestial objects

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Schwarzschild solution is the simplest solution of Einstein's field equations, but it has not been able to be describe any non-spherical in shape as in the real are existing. Many objects like stars and/or galaxies are in the form of ellipsoidal form and consequently, the gravitational lines around the objects are different in comparison with spherical form. In this paper a line element has been constructed with the intention, not only to describe a spherical form but also to explain an ellipsoidal system in more accurate and complete form. In fact, Schwarzschild line element and its solution is only a part of the whole work, which I have done. Applying this metric for more consideration an arbitrary object is the next step. The solution for planetary orbit of an ellipsoid planet by using Einstein's field equations has also been done. We attempt to solve these equations as the exterior solution for an ellipsoidal planet.

Key words: General relativity, Schwarzschild-like solution, elliptical stars, planetary orbits.

INTRODUCTION

Gravitational field equations described by general theory of relativity (Einstein, 1916). These equations are able to explain the properties of gravitational field around celestial objects. The first person who applied these equations was the German astrophysicist Karl Schwarzschild (Schwarzschild, 1916). He solved these equations for the first time and described the gravitational fields only around spherically symmetric and non-rotating objects in the static form such as stars and planets. In fact, the exact solution to the Einstein field equations is the Schwarzschild metric. This solution is corresponding to the external gravitational field of a stationary and uncharged object. Schwarzschild in his solution ignored the effects of the star's interior. However, Schwarzschild solution is the simplest solution of Einstein's field equations, but it is not able to describe non-spherical objects such as elliptical objects like stars and /or galaxies. The simple structure of elliptical galaxies is reflected in their place in Hubble's Classification. They are characterized by a single number, the ellipticity $\varepsilon = 10(1 - b/a)$, where b and a are the projected angular extent of the short and long axis of the galaxy on the sky (Roger, 2006).

In fact, most of the celestial objects like stars and planets are not exactly spherical but fairly ellipsoidal in shape. Therefore, the Schwarzschild solution is unsuitable for elliptical objects in shape. Certainly, to obtain the gravitational field around these types of objects, we need some more modification in our metric and line element too. The purpose of the present paper is to construct a framework for considering ellipsoidal shapes in general theory of relativity, which covers situations studied for all ellipsoidal objects.

SCHWARZSCHILD-LIKE SOLUTION

As we know huge bodies like galaxies and/or stars-cluster, which are very far away from us, are rotating with a period of the order of a billion years or more. Thus it is a good approximation if we consider these types of celestial objects which are in the static form. Therefore, assumed rotation of these objects may not play an important role for general purpose. Some of these objects are in the form elliptic. In the Euclidean geometry, the concept of an ellipsoid object, completely is clear but in the curved spaces, it has some different meaning. Since, the geometry of General Theory of Relativity (GTR) is based on Riemannian geometry, therefore, the curved space and its analysis is necessary. In this case, the perfect-fluid bodies, having an ellipsoidal shape and it is esential to determine the curvature of space and time in the presence of an ellipsoid (Zsigrai, 2008).

We, therefore, try to find out a solution to the Einstein's field equations for static and ellipsoidal shaped heavenly bodies where we are not much concerned with its rotation. In the absence of any mass point, the space time would be flat. Consider ordinary Minkowski space-time, described by the coordinates (x, y, z), where the static line element is defined as:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

By performing the following coordinate transformations, (Landau and Lifshitz, 1987)

$$x \to (r^2+a^2)^{1/2}\sin\theta\cos\phi\,,\ \ y \to (r^2+a^2)^{1/2}\sin\theta\sin\phi\,,\ \ z \to r\cos\theta\,,\ \ t \to t \tag{2}$$

on the metric given in (Equation 1), the metric in the new coordinate is, (Nikouravan, 2001)

$$ds^2 = c^2dt^2 - \left[\frac{r^2 + a^2\cos^2\theta}{r^2+a^2}\right]dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 - (r^2+a^2)\sin^2\theta\, d\phi^2 \tag{3}$$

In this coordinate frame, a is a constant in the, $x\,y$ surface. The aforeseen line element (3) is valid only for a vacuum space and time. This line element in the presence of a mass point takes the following form:

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}\left[\frac{r^2 + a^2\cos^2\theta}{r^2+a^2}\right]dr^2 - (r^2 + a^2\cos^2\theta)d\theta^2 - (r^2+a^2)\sin^2\theta\, d\phi^2 \tag{4}$$

Here e^{ν} and e^{λ} are as coefficient the parameters λ and ν, are function of r and θ only and c is the velocity of light and supposed as unit $(c = 1)$. The covariant, g_{ij} and contravariant, g^{ij} components of the metric tensors (Equation 4) are subsequently shown respectively,

$$g_{11} = -e^{\lambda}\frac{(r^2+a^2\cos^2\theta)}{(r^2+a^2)}\ ,\ \ g_{22} = -(r^2+a^2\cos^2\theta)\ ,\ \ g_{33} = -(r^2+a^2)\sin^2\theta\ ,\ \ g_{44} = e^{\nu}$$
$$g^{11} = -e^{-\lambda}\frac{(r^2+a^2)}{(r^2+a^2\cos^2\theta)}\ ,\ g^{22} = -\frac{1}{(r^2+a^2\cos^2\theta)}\ ,\ \ g^{33} = -\frac{1}{(r^2+a^2)\sin^2\theta}\ ,\ \ g^{44} = e^{-\nu} \tag{5}$$

The indexes i and j are in different suffixes $(i, j = 1,2,3,4)$. The non-vanishing first kinds of Christofell's symbols are;

$$\Gamma_{1/11} = -e^{\lambda}\left[\frac{\lambda'(r^2+a^2\cos^2\theta)}{2(r^2+a^2)} + \frac{r}{(r^2+a^2)} - \frac{r(r^2+a^2\cos^2\theta)}{(r^2+a^2)^2}\right],\ \ \Gamma_{3/23} = \Gamma_{3/32} = -\frac{1}{2}(r^2+a^2)\sin 2\theta,$$
$$\Gamma_{1/12} = \Gamma_{1/21} = -e^{\lambda}\left[\frac{\dot\lambda(r^2+a^2\cos^2\theta)}{2(r^2+a^2)} - \frac{a^2\sin 2\theta}{2(r^2+a^2)}\right]\ ,\ \ \ \Gamma_{2/11} = e^{\lambda}\left[\frac{\dot\lambda(r^2+a^2\cos^2\theta)}{2(r^2+a^2)} - \frac{a^2\sin 2\theta}{2(r^2+a^2)}\right],$$
$$\Gamma_{2/12} = \Gamma_{2/21} = -r\ ,\ \ \Gamma_{2/22} = \frac{1}{2}a^2\sin 2\theta\ ,\ \Gamma_{2/33} = \frac{1}{2}(r^2+a^2)\sin 2\theta\ ,\ \ \Gamma_{2/44} = -e^{\nu}(\frac{\dot\upsilon}{2})\ ,\ \Gamma_{1/44} = -e^{\upsilon}(\frac{\upsilon'}{2}),$$
$$\Gamma_{3/13} = \Gamma_{3/13} = -r\sin^2\theta\ ,\ \Gamma_{1/22} = r\ \ ,\ \ \Gamma_{1/33} = r\sin^2\theta\ ,\ \Gamma_{4/14} = \Gamma_{4/14} = e^{\nu}(\frac{\upsilon'}{2})\ ,\ \Gamma_{4/24} = \Gamma_{4/42} = e^{\upsilon}(\frac{\dot\upsilon}{2}) \tag{6}$$

And also the second kinds of Christofell's symbols are as,

$$\Gamma^1_{11} = \frac{\lambda'}{2} + \frac{r}{(r^2+a^2\cos^2\theta)} - \frac{r}{(r^2+a^2)}\ ,\qquad \Gamma^1_{12} = \Gamma^1_{21} = \frac{\dot\lambda}{2} - \frac{a^2\sin 2\theta}{2(r^2+a^2\cos^2\theta)}$$
$$\Gamma^1_{22} = \frac{-r(r^2+a^2)}{e^{\lambda}(r^2+a^2\cos^2\theta)}\ ,\ \ \Gamma^1_{33} = \frac{-r(r^2+a^2)\sin^2\theta}{e^{\lambda}(r^2+a^2\cos^2\theta)}\ ,\ \ \Gamma^1_{44} = \frac{\upsilon' e^{\upsilon}(r^2+a^2)}{2e^{\lambda}(r^2+a^2\cos^2\theta)}\,,$$
$$\Gamma^2_{44} = \frac{\dot\upsilon e^{\upsilon}}{2(r^2+a^2\cos^2\theta)}\ ,\ \ \Gamma^2_{33} = \frac{-(r^2+a^2)\sin 2\theta}{2(r^2+a^2\cos^2\theta)}\ ,\ \ \Gamma^2_{12} = \Gamma^2_{21} = \frac{r}{(r^2+a^2\cos^2\theta)}\,,$$
$$\Gamma^2_{11} = \frac{-\dot\lambda e^{\lambda}}{2(r^2+a^2)} + \frac{e^{\lambda}a^2\sin 2\theta}{2(r^2+a^2)(r^2+a^2\cos^2\theta)}\ ,\qquad \Gamma^3_{13} = \Gamma^3_{31} = \frac{r}{(r^2+a^2)}\ ,$$
$$\Gamma^3_{23} = \Gamma^3_{32} = \cot g\theta\ ,\qquad \Gamma^4_{14} = \Gamma^4_{41} = \frac{\upsilon'}{2}\,,\qquad \Gamma^4_{24} = \Gamma^4_{42} = \frac{\dot\upsilon}{2} \tag{7}$$

Here $\lambda' = \partial\lambda/\partial r$, $\dot\lambda = \partial\lambda/\partial\theta$, $\upsilon' = \partial\upsilon/\partial r$ and $\dot\upsilon = \partial\upsilon/\partial\theta$ have their usual meaning. Consequently different values of Ricci tensors are:

$$R_{11} = \frac{\upsilon''}{2} + \frac{\upsilon'^2}{4} - \frac{\lambda'\upsilon'}{4} - \frac{\lambda' r}{2(r^2+a^2\cos^2\theta)} - \frac{\lambda' r}{2(r^2+a^2)} - \frac{\upsilon' r}{2(r^2+a^2\cos^2\theta)} + \frac{\upsilon' r}{2(r^2+a^2)} +$$
$$\frac{1}{(r^2+a^2\cos^2\theta)} - \frac{2r^2}{(r^2+a^2\cos^2\theta)^2} + \frac{1}{(r^2+a^2)} + e^{\lambda}\left[\frac{\ddot\lambda}{2(r^2+a^2)} + \frac{\dot\lambda^2}{4(r^2+a^2)} + \right.$$
$$\frac{\dot\lambda\dot\upsilon}{4(r^2+a^2)} - \frac{\dot\lambda a^2\sin 2\theta}{4(r^2+a^2)(r^2+a^2\cos^2\theta)} + \frac{\dot\lambda\cot g\theta}{2(r^2+a^2)} - \frac{\dot\upsilon a^2\sin 2\theta}{4(r^2+a^2)(r^2+a^2\cos^2\theta)} +$$
$$\left.\frac{a^2\sin^2\theta}{(r^2+a^2)(r^2+a^2\cos^2\theta)} - \frac{2a^2\cos^2\theta}{(r^2+a^2)(r^2+a^2\cos^2\theta)} - \frac{a^4\sin^2 2\theta}{2(r^2+a^2)(r^2+a^2\cos^2\theta)^2}\right] \tag{8}$$

$$R_{22} = e^{-\lambda}\left[\frac{\upsilon' r(r^2+a^2)}{2(r^2+a^2\cos^2\theta)} - \frac{\lambda' r(r^2+a^2)}{2(r^2+a^2\cos^2\theta)} + \frac{(3r^2+a^2)}{(r^2+a^2\cos^2\theta)} - \frac{2r^2(r^2+a^2)}{(r^2+a^2\cos^2\theta)^2}\right] +$$
$$\frac{\ddot\lambda}{2} + \frac{\ddot\upsilon}{2} + \frac{\dot\lambda^2}{4} + \frac{\dot\upsilon^2}{4} - 1 + \frac{\dot\upsilon a^2\sin 2\theta}{4(r^2+a^2\cos^2\theta)} - \frac{\dot\lambda a^2\sin 2\theta}{4(r^2+a^2\cos^2\theta)} + \frac{a^2\sin^2\theta}{(r^2+a^2\cos^2\theta)} \tag{9}$$

$$R_{33} = e^{-\lambda}\left[\frac{\upsilon' r(r^2+a^2)}{2(r^2+a^2\cos^2\theta)} - \frac{\lambda' r(r^2+a^2)}{2(r^2+a^2\cos^2\theta)} + \frac{(r^2+a^2)}{(r^2+a^2\cos^2\theta)}\right]\sin^2\theta +$$
$$+ \frac{\dot\upsilon(r^2+a^2)\sin 2\theta}{4(r^2+a^2\cos^2\theta)} + \frac{\dot\lambda(r^2+a^2)\sin 2\theta}{4(r^2+a^2\cos^2\theta)} - \frac{(r^2+a^2)\sin^2\theta}{(r^2+a^2\cos^2\theta)} \tag{10}$$

$$R_{44} = -e^{\upsilon-\lambda}\left[\frac{\upsilon''(r^2+a^2)}{2(r^2+a^2\cos^2\theta)} + \frac{\upsilon'^2(r^2+a^2)}{4(r^2+a^2\cos^2\theta)} + \frac{r\upsilon'}{(r^2+a^2\cos^2\theta)} - \right.$$
$$e^{\upsilon}\left[\frac{\ddot\upsilon}{2(r^2+a^2\cos^2\theta)} + \frac{\dot\upsilon^2}{4(r^2+a^2\cos^2\theta)} + \frac{\dot\upsilon\cot g\theta}{2(r^2+a^2\cos^2\theta)} + \frac{\dot\lambda\dot\upsilon}{4(r^2+a^2\cos^2\theta)}\right] \tag{11}$$

Nikouravan (2009) equipped a computer software programming by *Mathematica* to calculate components of Ricci tensors. All components of Ricci tensors R_{ij}'s, Equations (8), (9), (10) and (11), in the empty space and for θ approximately constant, are identically to zero ($R_{ij} = 0$) and simplifies to the following form,

$$R_{11} = \frac{\upsilon''}{2} + \frac{\upsilon'^2}{4} - \frac{\lambda'\upsilon'}{4} - \frac{\lambda'}{r} \tag{12}$$

$$R_{22} = e^{-\lambda}\left(\frac{\upsilon' r}{2} - \frac{\lambda' r}{2} + 1\right) - 1 \tag{13}$$

$$R_{33} = \left[e^{-\lambda}\left(\frac{\upsilon' r}{2} - \frac{\lambda' r}{2} + 1\right) - 1\right]\sin^2\theta \tag{14}$$

$$R_{44} = -\frac{1}{2}e^{\upsilon-\lambda}\left(\upsilon'' + \frac{\upsilon'^2}{2} - \frac{\lambda'\upsilon'}{2} + \frac{2\upsilon'}{r}\right) \tag{15}$$

The solution of these Equations (12), (13), (14) and (15), for λ and υ we get, $r^{-1}(\partial\upsilon/\partial r) + (\partial\lambda/r\partial r) = 0$. After integration we have $\lambda + \upsilon = A$. The value of A is a constant of integration which may be set equal to zero. For large r , the values $\lambda = 0$ and $\upsilon = 0$, then $\upsilon = -\lambda$. By substituting in the previous equations we get, $e^{\nu}(1 + r\partial\upsilon'/\partial r) = 1$. After integrating we have $re^{\upsilon} = r + B$. Here the value of B being constant of integration that is, $e^{\upsilon} = e^{-\lambda} = 1 - 2m/r$, where we have put $B = -2m$.

This is done in order to facilitate the physical interpretation of m as the mass of the gravitating particle. Finally, the suggested line element due to a static and ellipsoidal isolated gravitating mass point get in the following form (Nikouravan, 2011).

$$ds^2 = (1-\frac{2m}{r})dt^2 - \frac{1}{(1-\frac{2m}{r})}\left[\frac{r^2+a^2\cos^2\theta}{r^2+a^2}\right]dr^2 - (r^2+a^2\cos^2\theta)d\theta - (r^2+a^2)\sin^2\theta d\phi^2 \quad (16)$$

Here the meaning of m is as Schwarzschild equation. Indeed, if we set $m = GM/c^2$ then we see that, for large value r, $g \approx GM/r^2$ so that M is the mass of central body. In relativistic units $(c = G = 1)$ we simply have $m = M$ (Wolfgang, 2006) and dimensionally are correct.

In Equation (16), if we put $a = 0$, then it become Schwarzschild line element. In fact Schwarzschild metric is only a spherically symmetric solution of Einstein's equation for the vacuum space (Chandrasekhar, 1983).

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{(1-\frac{2m}{r})}dr^2 - r^2 d\theta - r^2\sin^2\theta d\phi^2 \quad (17)$$

Equation (16) is more general and complete as compared with Equation (17). The line element (16) is not only valid for ellipsoidal form of any object but also it can explain the spherical form of object too.

Planetary orbits

Here we consider a solution for planetary orbits by using Einstein's field equations (Einstein, 1916). In terms of curved coordinate system x^i, we start with the line element (16) and attempt to solve these equations as exterior solution for a planet going around an ellipsoidal star. Indeed, we need to have equations that determine the connection field surrounding a heavy elliptical object such that it describes the gravitational field correctly (Hooft, 2009). Consequently it is assumed that the ellipsoidal star remains at the center and the planet is rotating around the star. Therefore, the geodesic differential equations of the ellipsoid and their space-time trajectories are given by:

$$\frac{dx^{\alpha}}{ds^2} + \Gamma^{\alpha}_{\mu\upsilon}\frac{dx^{\mu}}{ds}\frac{dx^{\upsilon}}{ds} = 0 \quad (18)$$

In the line element (16), m and r, are the mass and radius of ellipsoidal star, respectively. By using (Equation 7) for the non-vanishing Christofell's symbols of second kind, and (Equation 18), we have four differential equation of motion as below.

$$\frac{d^2r}{ds^2} + \left[\Gamma^1_{11}\left(\frac{dr}{ds}\right)^2 + \Gamma^1_{22}\left(\frac{d\theta}{ds}\right)^2 + \Gamma^1_{33}\left(\frac{d\phi}{ds}\right)^2 + \Gamma^1_{44}\left(\frac{dt}{ds}\right)^2 + 2\Gamma^1_{12}\left(\frac{dr}{ds}\frac{d\theta}{ds}\right)\right] = 0 \quad (19)$$

$$\frac{d^2\theta}{ds^2} + e^{\lambda}\left[\Gamma^2_{11}\left(\frac{dr}{ds}\right)^2 + \Gamma^2_{22}\left(\frac{d\theta}{ds}\right)^2 + \Gamma^2_{33}\left(\frac{d\phi}{ds}\right)^2 + \Gamma^2_{44}\left(\frac{dt}{ds}\right)^2 + 2\Gamma^2_{12}\left(\frac{dr}{ds}\frac{d\theta}{ds}\right)\right] = 0 \quad (20)$$

$$\frac{d^2\phi}{ds^2} + \frac{2r}{(r^2+a^2)}\left(\frac{dr}{ds}\frac{d\phi}{ds}\right) + 2\cot g\theta\left(\frac{d\theta}{ds}\frac{d\phi}{ds}\right) = 0 \quad (21)$$

$$\frac{d^2t}{ds^2} + \upsilon'\left(\frac{dr}{ds}\frac{dt}{ds}\right) + \dot{\upsilon}\left(\frac{d\theta}{ds}\frac{dt}{ds}\right) = 0 \quad (22)$$

The aforeseen relations are the equations of motion of a secondary going around an ellipsoidal star. If the planet moves initially in a plane $\theta = \pi/2$, then $d\theta/ds = 0$, and the previous equations are as:

$$\frac{d^2r}{ds^2} + \frac{\lambda'}{2}\left(\frac{dr}{ds}\right)^2 - re^{-\lambda}\sin^2\theta\left(\frac{d\phi}{ds}\right)^2 + e^{\upsilon-\lambda}\frac{\upsilon'}{2}\left(\frac{dt}{ds}\right)^2 = 0 \quad (23)$$

$$-e^{\lambda}(\frac{\dot{\lambda}}{2r^2})\left(\frac{dr}{ds}\right)^2 + e^{\upsilon}(\frac{\dot{\upsilon}}{2r^2})\left(\frac{dt}{ds}\right) = 0 \quad (24)$$

$$\frac{d^2\phi}{ds^2} + \frac{2r}{(r^2+a^2)}\left(\frac{dr}{ds}\frac{d\phi}{ds}\right) = 0 \quad (25)$$

$$\frac{d^2t}{ds^2} + \upsilon'\left(\frac{dr}{ds}\frac{dt}{ds}\right) = 0 \quad (26)$$

The solution of these equations yields

$$\begin{cases} \frac{1}{(r^2+a^2)}\frac{d}{ds}\left((r^2+a^2)\frac{d\phi}{ds}\right) = 0 \\ \frac{1}{e^{\upsilon}}\frac{d}{ds}(e^{\upsilon}\frac{dt}{ds}) = 0 \end{cases} \quad i.e. \begin{cases} \frac{d}{ds}\left((r^2+a^2)\frac{d\phi}{ds}\right) = 0 \\ i.e. \frac{d}{ds}(e^{\upsilon}\frac{dt}{ds}) = 0 \end{cases}$$

By integrating the previous equations we have:

$$(r^2+a^2)\frac{d\phi}{ds} = h \quad , \quad e^{\upsilon}\frac{dt}{ds} = k \,, \quad (27)$$

Where h and k are constants of integration. The constant h is a measure of the angular momentum of the motion.

Further, instead of working with Equation (25) due to its troublesome integration, we use our line element (4). By using $d\theta/ds = 0$, $\lambda = -\upsilon$, h and k as constant of integration, we get,

$$\left(\frac{r^2}{r^2+a^2}\right)\left(\frac{dr}{ds}\right)^2 + e^{\upsilon}\frac{h^2}{(r^2+a^2)} - k^2 + e^{\upsilon} = 0 \quad (28)$$

Using $(dr/ds) = (dr/d\phi)(d\phi/ds) = (h/r^2)(dr/d\phi)$ and $e^{\nu} = 1 - 2m/r$ the equation (28) becomes

$$\left(\frac{r^2h^2}{(r^2+a^2)^3}\right)\left(\frac{dr}{d\phi}\right)^2 + \left(1 - \frac{2m}{r}\right)\left[1 + \frac{h^2}{(r^2+a^2)}\right] - k^2 = 0 \quad (29)$$

Now, if we substitute $u = 1/r$ in the previous equation and after rearranging, we get

$$\left(\frac{du}{d\phi}\right)^2 + \left(1+a^2u^2\right)^2 u^2 = \left(\frac{k^2-1}{h^2}\right)\left(1+a^2u^2\right)^3 + \left(\frac{2mu}{h^2}\right)\left(1+a^2u^2\right)^3 + 2mu^3\left(1+a^2u^2\right)^2 \quad (30)$$

Differentiating the previous equation with respect to φ we can easily get

$$\frac{d^2u}{d\phi^2} + u\left(1+a^2u^2\right)\left(1+3a^2u^2\right) = \left(\frac{m}{h^2}\right)\left(1+a^2u^2\right)^3 + mu^2\left(1+a^2u^2\right)\left(3+7a^2u^2\right) + 3a^2u\left(\frac{k^2-1}{h^2}\right)\left(1+a^2u^2\right)^2 + \left(\frac{6ma^2u^2}{h^2}\right)\left(1+a^2u^2\right)^2 \quad (31)$$

Equation (31) represents the relativistic differential equation of the path of a planet going around an ellipsoidal star. Here r and φ are the special coordinates, and ds is an element of the proper time as measured by a clock moving with the planet. In approximation small terms multiplied with u^3 and greater powers of u in Equation (31) are still small and hence can be approximated to zero and get (Nikouravan, 2001),

$$\left(\frac{d^2u}{d\phi^2}\right) + u\left(1 - \frac{3a^2(K^2-1)}{h^2}\right) = \frac{m}{h^2} + 3mu^2\left(1 + \frac{2a^2}{h^2}\right) \quad (32)$$

Therefore, the relativistic differential equation of the orbit of the planet, Equation (32), can be compared with the corresponding Schwarzschild's line element for a spherical planet which is given subsequently.

$$\frac{d^2u}{d\phi^2} + u = \left(\frac{m}{h^2}\right) + 3mu^2 \quad (33)$$

and Newtonian equation (Ramsey,1961),

$$\left(\frac{d^2u}{d\phi^2}\right) + u = \frac{m}{h^2} \quad (34)$$

RESULTS AND DISCUSSION

By comparing line elements (4) and (16) we get the values of e^{λ} and e^{v} are in terms of mass of the object or mass of the gravitating particle, like Schwarzschild. The relation between these two factors, is $e^{v} = e^{-\lambda} = 1 - 2m/r$. By applying these values, we get the line element in the form of (Equation 17) as follows:

$$ds^2 = (1-\frac{2m}{r})dt^2 - \frac{1}{(1-\frac{2m}{r})}\left[\frac{r^2+a^2\cos^2\theta}{r^2+a^2}\right]dr^2 - (r^2+a^2\cos^2\theta)d\theta - (r^2+a^2)\sin^2\theta d\phi^2$$

The line element (16) is more complete than Schwarzschild line element (17). The line element (17) can be obtained by substituting $a = 0$ in (Equation 16). But the main difference between (Equations 16 and 17) is the value of a. For an ellipsoid, $a \neq 0$ and for a spherical object $a = 0$. The Equation (17) is valid only for spherical and is not possible to find out any non-spherical line element. The lines of gravitational field around the spherical and elliptical objects certainly are in different form. Therefore, the motion of secondary around the first object, in case of elliptical and/or spherical, certainly should be different. Hence the equations of motion, for elliptical in shape, is calculated using general theory of relativity and the result is (Equation 32). The Equation (32) simply shows the effect of elliptical objects in the space and also is the differential equation of motion. By substituting, $a = 0$ in (Equation 32), we get (Equation 33), which describes the same conditions for the Schwarzschild line element. It means, Schwarzschild line element (17) and it's relating calculated equations of motion, for planetary orbit (Equation 33), all are results of (Equations 16 and 32), respectively. Certainly the Newtonian differential equation of motion (Equation 34) also is the next result of Equations (16) and (32). Figure 1 shows the results of the calculations and differential equations of motion for flat, spherical and elliptical objects separately.

CONCLUSION

The aim of this work was to obtain a new form of line element which should be able to describe gravitational field around an elliptical object with solution of general theory of relativity. This solution not only describes the gravitational field around elliptical objects in shape but also it can explain the field around spherical objects too. One of the applications of this line element is planetary orbit of an object around the elliptical object. Differential equations of motion for planetary orbits in elliptical and spherical are different. For elliptical objects we found a new term as mentioned in the equation (32). The differential equation of motion of elliptical objects for ($a = 0$) is same as differential equation of motion for spherical objects. Consequently, elliptical line element (16) and differential equation of motion (32) are more general and accurate than spherical form.

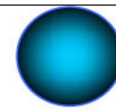
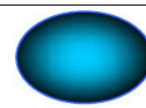
	Geometry of space	Differential equation of motion	
1	Newtonian equation	$\left(\frac{d^2u}{d\phi^2}\right)+u=\frac{m}{h^2}$	
2	Spherical Schwarzschild	$\left(\frac{d^2u}{d\phi^2}\right)+u=\frac{m}{h^2}+3mu^2$	
3	Elliptical	$\left(\frac{d^2u}{d\phi^2}\right)+u\left(1-\frac{3a^2(k^2-1)}{h^2}\right)=\frac{m}{h^2}+3mu^2\left(1+\frac{2a^2}{h^2}\right)$	

Figure 1. Shows the results of the calculations and differential equations of motion for flat, spherical and elliptical objects separately.

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