

Schwarzschild metric

4-dimensionas space-time

3-dimensional space-time

Space coordinates

$$r, \theta, \phi$$

$$r, \phi$$

Assumptions

1). Spherical symmetry

metric is independent of θ, ϕ :

$$\frac{\partial g_{\mu\nu}}{\partial \theta} = 0, \quad \frac{\partial g_{\mu\nu}}{\partial \phi} = 0; \quad \mu, \nu = 0..3$$

1). Cylindrical symmetry:

metric is independent of ϕ :

$$\frac{\partial g_{\mu\nu}}{\partial \phi} = 0; \quad \mu, \nu = 0..2$$

2). Static solution: metric is independent of time: $\frac{\partial g_{\mu\nu}}{\partial t} = 0$

3). Vacuum outside of the body producing the field: $T_{\mu\nu} = 0 \Leftrightarrow R_{\mu\nu} = 0$

General form of metric

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (1)$$

$$ds^2 = e^{\nu(r)} dt^2 - e^{\lambda(r)} dr^2 - r^2 d\phi^2 \quad (1^*)$$

$e^{\nu(r)}$ and $e^{\lambda(r)}$ are unknown functions of r . [for proof see [General form of metric in symmetrical space](#), and the literature cited there]

Ricci tensor

$$R_{00} = e^{\nu-\lambda} \left(\frac{1}{2} \nu'' + \frac{1}{4} \nu'^2 - \frac{1}{4} \nu' \lambda' + \frac{\nu'}{r} \right) = 0$$

$$R_{11} = -\frac{1}{2} \nu'' - \frac{1}{4} \nu'^2 + \frac{1}{4} \nu' \lambda' + \frac{\lambda'}{r} = 0$$

$$R_{22} = -e^{-\lambda} \left(1 + \frac{1}{2} r (\nu' - \lambda') \right) + 1 = 0$$

$$R_{00} = e^{\nu-\lambda} \left(\frac{1}{2} \nu'' + \frac{1}{4} \nu'^2 - \frac{1}{4} \nu' \lambda' + \frac{1}{2} \frac{\nu'}{r} \right) = 0$$

$$R_{11} = -\frac{1}{2} \nu'' - \frac{1}{4} \nu'^2 + \frac{1}{4} \nu' \lambda' + \frac{1}{2} \frac{\lambda'}{r} = 0$$

$$R_{22} = \frac{1}{2} r e^{-\lambda} (\lambda' - \nu') = 0$$

where prime denotes differentiation with respect to r : $\nu' \equiv \frac{d\nu}{dr}$. These equations are transformed into

$$\lambda' = -\nu' \quad (2)$$

$$e^{\nu} (1 + r \nu') = 1 \quad (3)$$

$$\lambda' = -\nu' \quad (2^*)$$

$$\lambda' = \nu' \quad (3^*)$$

Solving this system of equations and remembering that at $r \rightarrow \infty$: $e^{\nu} = e^{\lambda} = 1$ (for metric is Euclidean there), we obtain

$$e^{\nu} + r(e^{\nu})' = 1 \Rightarrow \frac{de^{\nu}}{1 - e^{\nu}} = \frac{dr}{r} \Rightarrow 1 - e^{\nu} = r_g r^{-1} \Rightarrow e^{\nu} = 1 - \frac{r_g}{r}$$

$$\lambda' = \nu' = 0 \Rightarrow \lambda = C_1, \nu = C_2$$

$$\lambda = -\nu \Rightarrow e^{\lambda} = \frac{1}{1 - \frac{r_g}{r}} \quad (4)$$

$$e^{\lambda} = e^{\nu} = 1 \quad (4^*)$$

So, the metric is given by

$$ds^2 = \left(1 - \frac{r_g}{r} \right) dt^2 - \left(1 - \frac{r_g}{r} \right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (5)$$

$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 \quad (5^*)$$

General form of metric in symmetrical space

4-dimensional space-time

3-dimensional space-time

1). metric is diagonal

Let us consider the transformation $t \rightarrow -t$. By tensor definition, the metric tensor is transformed as

$$g'_{0\nu} = \frac{\partial x^\alpha}{\partial x'^0} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = -g_{0\nu}, \quad \nu \neq 0$$

But because of the static nature assumption $g'_{0\nu} = g_{0\nu}$ (metric is independent of t). Therefore $g_{0\nu} = 0$, ($\nu \neq 0$).
Considering analogous transformations for θ, ϕ and using the symmetry assumption, we obtain $g_{2\nu} = 0$, ($\nu \neq 2$); $g_{3\nu} = 0$, ($\nu \neq 3$);

Thus

$$g_{\mu\nu} = 0, \quad (\mu \neq \nu) \quad (6)$$

metric assumes the form

$$ds^2 = g_{00}dt^2 - g_{11}dr^2 - g_{22}d\theta^2 - g_{33}d\phi^2 \quad (7) \quad ds^2 = g_{00}dt^2 - g_{11}dr^2 - g_{22}d\phi^2 \quad (7^*)$$

2). General form of metric

As $g_{\mu\nu}$ depends only on r , applying the symmetry assumption yields:

$$ds^2 = W(r)dt^2 - U(r)dr^2 - V(r)(r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2) \quad (8) \quad ds^2 = W(r)dt^2 - U(r)dr^2 - V(r)r^2d\phi^2 \quad (8^*)$$

U, W and V are arbitrary functions of r . Because we can choose any coordinate system (as long as it's symmetric), we may introduce new coordinate r_1 so that

$$r_1^2 = r^2V(r)$$

Having coordinates thus defined (and dropping 1 for convenience), the metric becomes

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (9) \quad ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2d\phi^2 \quad (9^*)$$

where $e^{\nu(r)}$ and $e^{\lambda(r)}$ - are as yet unknown functions of r .

References

- [1] Misner, Thorne, Wheeler: "Gravitation" (1973), §23.2, p.594
- [2] [Eddington: "The mathematical theory of relativity" \(1923\), §38, p.83](#) (online)
- [3] Landau, Lifshitz: "Classical theory of fields" (1975), §100, p.320