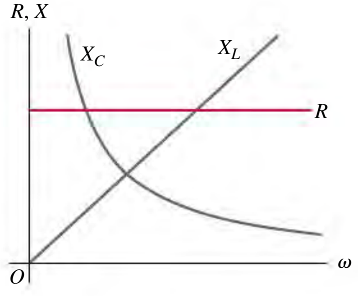
(The 80-Ω reactance of the capacitor is 40% of the resistor's 200-Ω resistance, so V_C is 40% of V_R.) The instantaneous capacitor voltage is given by Eq. (31.16):

$$v_C = V_C \cos(\omega t - 90°) = (0.48\text{ V}) \cos[(2500\text{ rad/s})t - \pi/2\text{ rad}]$$

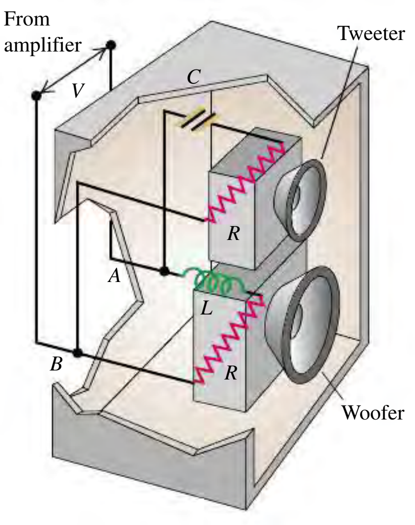
EVALUATE: Although the same *current* passes through both the capacitor and the resistor, the *voltages* across them are different in both amplitude and phase. Note that in the expression for v_C we converted the 90° to $\pi/2$ rad so that all the angular quantities have the same units. In ac circuit analysis, phase angles are often given in degrees, so be careful to convert to radians when necessary.

31.11 Graphs of R, X_L, and X_C as functions of angular frequency ω.

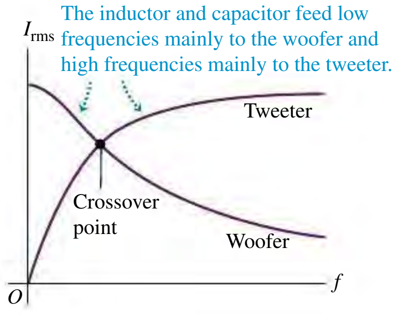


31.12 (a) The two speakers in this loudspeaker system are connected in parallel to the amplifier. (b) Graphs of current amplitude in the tweeter and woofer as functions of frequency for a given amplifier voltage amplitude.

(a) A crossover network in a loudspeaker system



(b) Graphs of rms current as functions of frequency for a given amplifier voltage



Comparing ac Circuit Elements

Table 31.1 summarizes the relationships of voltage and current amplitudes for the three circuit elements we have discussed. Note again that *instantaneous* voltage and current are proportional in a resistor, where there is zero phase difference between v_R and i (see Fig. 31.7b). The instantaneous voltage and current are *not* proportional in an inductor or capacitor, because there is a 90° phase difference in both cases (see Figs. 31.8b and 31.9b).

TABLE 31.1 Circuit Elements with Alternating Current

Circuit Element	Amplitude Relationship	Circuit Quantity	Phase of v
Resistor	$V_R = IR$	R	In phase with i
Inductor	$V_L = IX_L$	$X_L = \omega L$	Leads i by 90°
Capacitor	$V_C = IX_C$	$X_C = 1/\omega C$	Lags i by 90°

Figure 31.11 shows how the resistance of a resistor and the reactances of an inductor and a capacitor vary with angular frequency ω. Resistance R is independent of frequency, while the reactances X_L and X_C are not. If $\omega = 0$, corresponding to a dc circuit, there is *no* current through a capacitor because $X_C \to \infty$, and there is no inductive effect because $X_L = 0$. In the limit $\omega \to \infty$, X_L also approaches infinity, and the current through an inductor becomes vanishingly small; recall that the self-induced emf opposes rapid changes in current. In this same limit, X_C and the voltage across a capacitor both approach zero; the current changes direction so rapidly that no charge can build up on either plate.

Figure 31.12 shows an application of the above discussion to a loudspeaker system. Low-frequency sounds are produced by the *woofer,* which is a speaker with large diameter; the *tweeter,* a speaker with smaller diameter, produces high-frequency sounds. In order to route signals of different frequency to the appropriate speaker, the woofer and tweeter are connected in parallel across the amplifier output. The capacitor in the tweeter branch blocks the low-frequency components of sound but passes the higher frequencies; the inductor in the woofer branch does the opposite.

TEST YOUR UNDERSTANDING OF SECTION 31.2 An oscillating voltage of fixed amplitude is applied across a circuit element. If the frequency of this voltage is increased, will the amplitude of the current through the element (i) increase, (ii) decrease, or (iii) remain the same if it is (a) a resistor, (b) an inductor, or (c) a capacitor?

31.3 THE *L-R-C* SERIES CIRCUIT

Many ac circuits used in practical electronic systems involve resistance, inductive reactance, and capacitive reactance. **Figure 31.13a** shows a simple example: a series circuit containing a resistor, an inductor, a capacitor, and an ac source. (In Section 30.6 we studied an *L-R-C* series circuit *without* a source.)

To analyze this circuit, we'll use a phasor diagram that includes the voltage and current phasors for each of the components. Because of Kirchhoff's loop rule, the instantaneous *total* voltage v_{ad} across all three components is equal to the source voltage at that instant. We will show that the phasor representing this total voltage is the *vector sum* of the phasors for the individual voltages.

Figures 31.13b and 31.13c show complete phasor diagrams for the circuit of Fig. 31.13a. We assume that the source supplies a current i given by $i = I\cos\omega t$. Because the circuit elements are connected in series, the current at any instant is the same at every point in the circuit. Thus a *single phasor I*, with length proportional to the current amplitude, represents the current in *all* circuit elements.

As in Section 31.2, we use the symbols v_R, v_L, and v_C for the instantaneous voltages across R, L, and C, and the symbols V_R, V_L, and V_C for the maximum voltages. We denote the instantaneous and maximum *source* voltages by v and V. Then, in Fig. 31.13a, $v = v_{ad}$, $v_R = v_{ab}$, $v_L = v_{bc}$, and $v_C = v_{cd}$.

The potential difference between the terminals of a resistor is *in phase* with the current in the resistor. Its maximum value V_R is given by Eq. (31.7):

$$V_R = IR$$

The phasor V_R in Fig. 31.13b, in phase with the current phasor I, represents the voltage across the resistor. Its projection onto the horizontal axis at any instant gives the instantaneous potential difference v_R.

The voltage across an inductor *leads* the current by 90°. Its voltage amplitude is given by Eq. (31.13):

$$V_L = IX_L$$

The phasor V_L in Fig. 31.13b represents the voltage across the inductor, and its projection onto the horizontal axis at any instant equals v_L.

The voltage across a capacitor *lags* the current by 90°. Its voltage amplitude is given by Eq. (31.19):

$$V_C = IX_C$$

The phasor V_C in Fig. 31.13b represents the voltage across the capacitor, and its projection onto the horizontal axis at any instant equals v_C.

The instantaneous potential difference v between terminals a and d is equal at every instant to the (algebraic) sum of the potential differences v_R, v_L, and v_C. That is, it equals the sum of the *projections* of the phasors V_R, V_L, and V_C. But the sum of the projections of these phasors is equal to the *projection* of their *vector sum*. So the vector sum V must be the phasor that represents the source voltage v and the instantaneous total voltage v_{ad} across the series of elements.

To form this vector sum, we first subtract the phasor V_C from the phasor V_L. (These two phasors always lie along the same line, with opposite directions.) This gives the phasor $V_L - V_C$. This is always at right angles to the phasor V_R, so from the Pythagorean theorem the magnitude of the phasor V is

$$V = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{(IR)^2 + (IX_L - IX_C)^2} \quad \text{or}$$

$$V = I\sqrt{R^2 + (X_L - X_C)^2} \qquad (31.20)$$

We define the **impedance** Z of an ac circuit as the ratio of the voltage amplitude across the circuit to the current amplitude in the circuit. From Eq. (31.20) the impedance of the *L-R-C* series circuit is

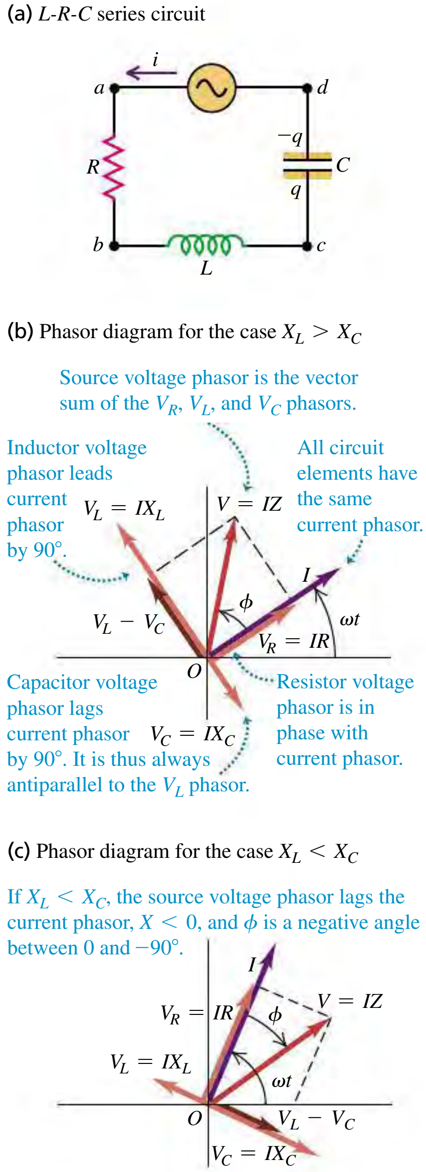
$$Z = \sqrt{R^2 + (X_L - X_C)^2} \qquad (31.21)$$

so we can rewrite Eq. (31.20) as

$$V = IZ \qquad (31.22)$$

Amplitude of voltage across an ac circuit → V; **Current amplitude** → I; **Impedance of circuit** → Z

31.13 An *L-R-C* series circuit with an ac source.



MP

PhET: Circuit Construction Kit (AC+DC)
PhET: Faraday's Electromagnetic Lab

31.14 This gas-filled glass sphere has an alternating voltage between its surface and the electrode at its center. The glowing streamers show the resulting alternating current that passes through the gas. When you touch the outside of the sphere, your fingertips and the inner surface of the sphere act as the plates of a capacitor, and the sphere and your body together form an *L-R-C* series circuit. The current (which is low enough to be harmless) is drawn to your fingers because the path through your body has a low impedance.

While Eq. (31.21) is valid only for an *L-R-C* series circuit, we can use Eq. (31.22) to define the impedance of *any* network of resistors, inductors, and capacitors as the ratio of the amplitude of the voltage across the network to the current amplitude. The SI unit of impedance is the ohm.

The Meaning of Impedance and Phase Angle

Equation (31.22) has a form similar to $V = IR$, with impedance Z in an ac circuit playing the role of resistance R in a dc circuit. Just as direct current tends to follow the path of least resistance, so alternating current tends to follow the path of lowest impedance (**Fig. 31.14**). Note, however, that impedance is actually a function of R, L, and C, as well as of the angular frequency ω. We can see this by substituting Eq. (31.12) for X_L and Eq. (31.18) for X_C into Eq. (31.21), giving the following complete expression for Z for a series circuit:

Impedance of an *L-R-C* series circuit (Resistance: R; Inductance: L; Capacitance: C; Angular frequency: ω)

$$Z = \sqrt{R^2 + [\omega L - (1/\omega C)]^2} \qquad (31.23)$$

Hence for a given amplitude V of the source voltage applied to the circuit, the amplitude $I = V/Z$ of the resulting current will be different at different frequencies. We'll explore this frequency dependence in detail in Section 31.5.

In Fig. 31.13b, the angle ϕ between the voltage and current phasors is the phase angle of the source voltage v with respect to the current i; that is, it is the angle by which the source voltage leads the current. From the diagram,

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{I(X_L - X_C)}{IR} = \frac{X_L - X_C}{R}$$

Phase angle of voltage with respect to current in an *L-R-C* series circuit (Inductance: L; Capacitance: C; Resistance: R; Angular frequency: ω)

$$\tan\phi = \frac{\omega L - 1/\omega C}{R} \qquad (31.24)$$

If the current is $i = I\cos\omega t$, then the source voltage v is

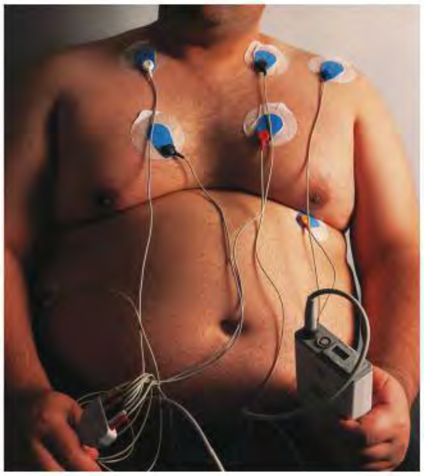
$$v = V\cos(\omega t + \phi) \qquad (31.25)$$

Figure 31.13b shows the behavior of an *L-R-C* series circuit in which $X_L > X_C$. Figure 31.13c shows the behavior when $X_L < X_C$; the voltage phasor V lies on the opposite side of the current phasor I and the voltage *lags* the current. In this case, $X_L - X_C$ is *negative,* $\tan\phi$ is negative, and ϕ is a negative angle between 0° and −90°. Since X_L and X_C depend on frequency, the phase angle ϕ depends on frequency as well. We'll examine the consequences of this in Section 31.5.

All of the expressions that we've developed for an *L-R-C* series circuit are still valid if one of the circuit elements is missing. If the resistor is missing, we set $R = 0$; if the inductor is missing, we set $L = 0$. But if the capacitor is missing, we set $C = \infty$, corresponding to the absence of any potential difference ($v_C = q/C = 0$) or any capacitive reactance ($X_C = 1/\omega C = 0$).

In this entire discussion we have described magnitudes of voltages and currents in terms of their *maximum* values, the voltage and current *amplitudes*. But we remarked at the end of Section 31.1 that these quantities are usually described in terms of rms values, not amplitudes. For any sinusoidally varying quantity, the rms value is always $1/\sqrt{2}$ times the amplitude. All the relationships between voltage and current that we have derived in this and the preceding sections are

BIO Application Measuring Body Fat by Bioelectric Impedance Analysis The electrodes attached to this overweight patient's chest are applying a small ac voltage of frequency 50 kHz. The attached instrumentation measures the amplitude and phase angle of the resulting current through the patient's body. These depend on the relative amounts of water and fat along the path followed by the current, and so provide a sensitive measure of body composition.



still valid if we use rms quantities throughout instead of amplitudes. For example, if we divide Eq. (31.22) by $\sqrt{2}$, we get

$$\frac{V}{\sqrt{2}} = \frac{I}{\sqrt{2}}Z$$

which we can rewrite as

$$V_{\text{rms}} = I_{\text{rms}}Z \qquad (31.26)$$

We can translate Eqs. (31.7), (31.13), and (31.19) in exactly the same way.

We have considered only ac circuits in which an inductor, a resistor, and a capacitor are in series. You can do a similar analysis for an *L-R-C parallel* circuit; see Problem 31.54.

PROBLEM-SOLVING STRATEGY 31.1 ALTERNATING-CURRENT CIRCUITS

IDENTIFY *the relevant concepts:* In analyzing ac circuits, we can apply all of the concepts used to analyze direct-current circuits, particularly those in Problem-Solving Strategies 26.1 and 26.2. But now we must distinguish between the amplitudes of alternating currents and voltages and their instantaneous values, and among resistance (for resistors), reactance (for inductors or capacitors), and impedance (for composite circuits).

SET UP *the problem* using the following steps:

1. Draw a diagram of the circuit and label all known and unknown quantities.
2. Identify the target variables.

EXECUTE *the solution* as follows:

1. Use the relationships derived in Sections 31.2 and 31.3 to solve for the target variables, using the following hints.
2. It's almost always easiest to work with angular frequency $\omega = 2\pi f$ rather than ordinary frequency f.
3. Keep in mind the following phase relationships: For a resistor, voltage and current are *in phase,* so the corresponding phasors always point in the same direction. For an inductor, the voltage *leads* the current by 90° (i.e., $\phi = +90° = \pi/2$ radians), so the voltage phasor points 90° counterclockwise from the current phasor. For a capacitor, the voltage *lags* the current by 90° (i.e., $\phi = -90° = -\pi/2$ radians), so the voltage phasor points 90° clockwise from the current phasor.
4. Kirchhoff's rules hold *at each instant*. For example, in a series circuit, the instantaneous current is the same in all circuit elements; in a parallel circuit, the instantaneous potential difference is the same across all circuit elements.
5. Inductive reactance, capacitive reactance, and impedance are analogous to resistance; each represents the ratio of voltage amplitude V to current amplitude I in a circuit element or combination of elements. However, phase relationships are crucial. In applying Kirchhoff's loop rule, you must combine the effects of resistance and reactance by *vector* addition of the corresponding voltage phasors, as in Figs. 31.13b and 31.13c. When several circuit elements are in series, for example, you can't *add* all the numerical values of resistance and reactance to get the impedance; that would ignore the phase relationships.

EVALUATE *your answer:* When working with an *L-R-C series* circuit, you can check your results by comparing the values of the inductive and capacitive reactances X_L and X_C. If $X_L > X_C$, then the voltage amplitude across the inductor is greater than that across the capacitor and the phase angle ϕ is positive (between 0° and 90°). If $X_L < X_C$, then the voltage amplitude across the inductor is less than that across the capacitor and the phase angle ϕ is negative (between 0° and −90°).

EXAMPLE 31.4 AN *L-R-C* SERIES CIRCUIT I

In the series circuit of Fig. 31.13a, suppose $R = 300\ \Omega$, $L = 60$ mH, $C = 0.50\ \mu$F, $V = 50$ V, and $\omega = 10{,}000$ rad/s. Find the reactances X_L and X_C, the impedance Z, the current amplitude I, the phase angle ϕ, and the voltage amplitude across each circuit element.

SOLUTION

IDENTIFY and SET UP: This problem uses the ideas developed in Section 31.2 and this section about the behavior of circuit elements in an ac circuit. We use Eqs. (31.12) and (31.18) to determine X_L and X_C, and Eq. (31.23) to find Z. We then use Eq. (31.22) to find the current amplitude and Eq. (31.24) to find the phase angle. The relationships in Table 31.1 then yield the voltage amplitudes.

EXECUTE: The inductive and capacitive reactances are

$$X_L = \omega L = (10{,}000\ \text{rad/s})(60\ \text{mH}) = 600\ \Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(10{,}000\ \text{rad/s})(0.50 \times 10^{-6}\ \text{F})} = 200\ \Omega$$

The impedance Z of the circuit is then

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(300\ \Omega)^2 + (600\ \Omega - 200\ \Omega)^2}$$
$$= 500\ \Omega$$

Continued