Solution of 1D problem

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{1}$$

IC $$T(x,0) = T_0 \tag{2}$$

BC $$T(L,t) = T_0 \tag{3}$$

BC $$\left.\frac{\partial T}{\partial x}\right|_{x=0} = -\frac{A}{\kappa}(cos(wt) + 1) \tag{4}$$

By defining $T(x,t) \ - T_0 = \theta(x,t)$, we get

$$\frac{\partial \theta}{\partial t} = \alpha \frac{\partial^2 \theta}{\partial x^2} \tag{5}$$

IC $$\theta(x,0) = 0 \tag{6}$$

BC $$\theta(L,t) = 0 \tag{7}$$

BC $$\left.\frac{\partial \theta}{\partial \mathrm{x}}\right|_{\mathrm{x}=0} = -\frac{A}{\kappa}(cos(wt) + 1) \tag{8}$$

We assume a form of solution by focusing exclusively on the long-time solution when the system has reached oscillatory steady state. So, the solution will take the following form.

$$\theta(x,t) = a(x)\cos(\omega t) + b(x)\sin(\omega t) + \frac{A}{k}(L - x) \tag{9}$$

Taking temporal and spatial derivative of equation (9), we get

$$\frac{\partial \theta}{\partial t} = -a(x)\omega\sin(\omega t) + b(x)\omega\cos(\omega t) \tag{10}$$

$$\frac{\partial \theta}{\partial x} = \frac{da(x)}{dx}\cos(\omega t) + \frac{db(x)}{dx}\sin(\omega t) + \frac{A}{k} \tag{11}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 a(x)}{\partial x^2}\cos(\omega t) + \frac{\partial^2 b(x)}{\partial x^2}\sin(\omega t) \tag{12}$$

Applying equation (12) and (10) into equation (5), we get

$$-a(x)\omega\sin(\omega t) + b(x)\omega\cos(\omega t) = \alpha\frac{\partial^2 a(x)}{\partial x^2}\cos(\omega t) + \alpha\frac{\partial^2 b(x)}{\partial x^2}\sin(\omega t) \tag{13}$$

At this point, we want to obtain the value of a(x) and b(x). To do that, we can separate the coefficients of $cos(\omega t)$ and $sin(\omega t)$.

$$\alpha\frac{\partial^2 a(x)}{\partial x^2} = b(x)\omega \tag{14}$$

$$\alpha\frac{\partial^2 b(x)}{\partial x^2} = -a(x)\omega \tag{15}$$

We can imply b(x) value from equation (14) to (15), so that equation (15) becomes solely dependent on a(x).

$$\frac{\partial^4 a(x)}{\partial x^4} + \left(\frac{\omega}{\alpha}\right)^2 a(x) = 0 \tag{16}$$

This ordinary differential equation can be solved by taking complementary and particular solution. But since this is also a homogeneous equation, particular solution will be zero. By looking at the roots of characteristic polynomial, we can easily find the complementary solution of this equation.

$$a(x) = C_1 e^{-\sqrt{\frac{\omega}{\alpha}}x} cos\left(\sqrt{\frac{\omega}{\alpha}}x\right) + C_2 e^{-\sqrt{\frac{\omega}{\alpha}}x} sin\left(\sqrt{\frac{\omega}{\alpha}}x\right) \tag{17}$$

We can eliminate b(x) ?? By considering b(x)=0;

From equation (9) and ((15), we get

$$\theta(x,t) = \left[C_1 e^{-\sqrt{\frac{\omega}{\alpha}}x} cos\left(\sqrt{\frac{\omega}{\alpha}}x\right) + C_2 e^{-\sqrt{\frac{\omega}{\alpha}}x} sin\left(\sqrt{\frac{\omega}{\alpha}}x\right)\right]\cos(\omega t) + \frac{A}{k}(L-x) \tag{18}$$

Applying second boundary condition,

$$\frac{\partial\theta(x,t)}{\partial \mathrm{x}} = \left[-\sqrt{\frac{\omega}{\alpha}}C_1 e^{-\sqrt{\frac{\omega}{\alpha}}x}\left[sin\left(\sqrt{\frac{\omega}{\alpha}}x\right) + cos\left(\sqrt{\frac{\omega}{\alpha}}x\right)\right] - \sqrt{\frac{\omega}{\alpha}}C_2 e^{-\sqrt{\frac{\omega}{\alpha}}x}\left[sin\left(\sqrt{\frac{\omega}{\alpha}}x\right) - cos\left(\sqrt{\frac{\omega}{\alpha}}x\right)\right]\right]\cos(\omega t) + \frac{A}{k} \tag{19}$$

At x=0

$$-\frac{A}{\kappa}(cos(wt)+1) = \left[-\sqrt{\frac{\omega}{\alpha}}C_1 - \sqrt{\frac{\omega}{\alpha}}C_2\right]\cos(\omega t) + \frac{A}{k} \tag{20}$$

Taking the coefficient of cos(ωt), we get

$$(C_1 - C_2)\sqrt{\frac{\omega}{\alpha}} = -\frac{A}{\kappa} \tag{21}$$

Or,

$$(C_1 - C_2) = -\frac{A}{\kappa}\sqrt{\frac{\omega}{\alpha}} \tag{22}$$

By applying equation (22) in equation (18), and applying initial condition, we get

$$\theta(x,0) = \left[\left(C_2 + \frac{A}{\kappa}\sqrt{\frac{\omega}{\alpha}}\right)e^{-\sqrt{\frac{\omega}{\alpha}}x} cos\left(\sqrt{\frac{\omega}{\alpha}}x\right) + C_2 e^{-\sqrt{\frac{\omega}{\alpha}}x} sin\left(\sqrt{\frac{\omega}{\alpha}}x\right)\right] + \frac{A}{k}(L-x) = 0 \tag{23}$$

Again, separating the coefficient of $cos\left(\sqrt{\frac{\omega}{\alpha}}x\right)$ and $sin\left(\sqrt{\frac{\omega}{\alpha}}x\right)$, we get

$$\left(C_2 + \frac{A}{\kappa}\sqrt{\frac{\omega}{\alpha}}\right)e^{-\sqrt{\frac{\omega}{\alpha}}x} = 0 \tag{24}$$

$$C_2 e^{-\sqrt{\frac{\omega}{\alpha}}x} = 0 \tag{25}$$

From equation (25), we get

$$C_2 = 0, \text{ since } e^{-\sqrt{\frac{\omega}{\alpha}}x} \neq 0 \tag{26}$$

Equation (18) turns out to be

$$\theta(x,t) = \frac{A}{\kappa}\sqrt{\frac{\omega}{\alpha}} e^{-\sqrt{\frac{\omega}{\alpha}}x} cos\left(\sqrt{\frac{\omega}{\alpha}}x\right)\cos(\omega t) + \frac{A}{k}(L-x) \tag{27}$$

By applying trigonometric rule

$$\cos(\mathrm{A})\cos(\mathrm{B}) = \frac{cos(A-B) + cos(A+B)}{2} \tag{28}$$

Equation (23) become

$$\theta(x,t) = \frac{1}{2}\frac{A}{\kappa}\sqrt{\frac{\omega}{\alpha}} e^{-\sqrt{\frac{\omega}{\alpha}}x} cos\left(\omega t - \sqrt{\frac{\omega}{\alpha}}x\right)\cos\left(\omega t + \sqrt{\frac{\omega}{\alpha}}x\right) + \frac{A}{k}(L-x) \tag{29}$$

Somehow, $\cos\left(\omega t + \sqrt{\frac{\omega}{\alpha}}x\right)$ *become zero*??

The final solution

$$\theta(x,t) = \frac{1}{2}\frac{A}{\kappa}\sqrt{\frac{\omega}{\alpha}}e^{-\sqrt{\frac{\omega}{\alpha}}x}cos\left(\omega t - \sqrt{\frac{\omega}{\alpha}}x\right) + \frac{A}{k}(L - x) \qquad (30)$$