# The spherical pendulum

We want to solve the spherical pendulum and express its solution in terms of Jacobi elliptic functions. The equation of motion is given by:

$$L=\frac{m}{2}l^2(\dot{\theta}^2+\sin^2(\theta)\dot{\phi}^2)+mgl\cos(\theta)$$

from this one obtains via Euler-Lagrange equations:

$$\frac{d}{dt}P_\theta=\frac{d}{dt}ml^2\dot{\theta}=-mgl\sin(\theta)+ml^2\sin(\theta)\cos(\theta)\dot{\phi}^2$$

$$\frac{d}{dt}P_\phi=\frac{d}{dt}ml^2\sin(\theta)^2\dot{\phi}=0$$

From the second equation follows $P_\phi=const$ and therefore:

$$\dot{\phi}=\frac{P_\phi}{ml^2\sin^2(\theta)} \tag{1}$$

Plugging this into the θ equation leads to:

$$\frac{d}{dt}P_\theta=-mgl\sin(\theta)+\frac{P_\phi^2}{ml^2\sin^3(\theta)}\cos(\theta)$$

Now multiplying both sides by $\dot{\theta}$ and using gives:

$$\frac{d}{dt}\frac{1}{2}ml^2\dot{\theta}^2=\frac{d}{dt}(mgl\cos(\theta)-\frac{P_\phi^2}{2ml^2\sin^2(\theta)})$$

Integrating both sides yields the constant energy E of the system. Setting $\omega:=\sqrt{\frac{g}{l}}$ , solving for $\dot{\theta}$ and separation of variables yields the integral:

$$t=\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{\frac{2E}{ml^2}+2\omega^2\cos(\theta)-\frac{P_\phi^2}{m^2l^4\sin^2(\theta)}}} \tag{2}$$

where $t_0=0$ without loss of generality. We now want to express this integral as an elliptic integral of the first kind so that we can afterwards invert it to express $\theta(t)$ in terms of Jacobi elliptic functions. To do so requires to transform the integral into the form:

$$\int_{\beta_3}^{x}\frac{du}{\sqrt{P(u)}}$$

$$P(u)=(u-\beta_1)(u-\beta_2)(u-\beta_3)$$

$$\beta_1>\beta_2>\beta_3 \tag{3}$$

to use the formula (17.4.62) in Abramowitz&Stegun. For simplicity reasons, we will assume that the movement of the pendulum is initiated only by a displacement but not a velocity. We can then write:

$$E=-m\omega^2l^2\cos(\theta_0)+\frac{P_\phi^2}{2ml^2\sin^2(\theta_0)}$$

where $\theta_0$ is the initial displacement. Plugging this into (2) yields:

$$t=\frac{1}{\sqrt{2}\omega}\cdot\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{\cos(\theta)-\cos(\theta_0)-\frac{P_\phi^2}{2m^2l^4\omega^2}\cdot(\frac{1}{\sin^2(\theta)}-\frac{1}{\sin^2(\theta_0)})}}$$

Now we express the sines and cosines by using the half angle formulas $\cos(\theta)=1-2\cdot\sin^2(\frac{\theta}{2})$ , $\sin(\theta)=2\cdot\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$ and make the substitution $u=(\frac{\sin(\frac{\theta}{2})}{\sin(\frac{\theta_0}{2})})^2$ . This yields after a bit of calculation:

$$t=\frac{1}{2\omega}\cdot\int_{1}^{x}\frac{du}{\sqrt{(1-u)u(1-k_0^2u)-\alpha^2(1-k_0^2)(1-k_0^2-u\cdot(1-k_0^2u))}}$$

with

$$x:=\frac{1}{k_0^2}\cdot\sin^2(\frac{\theta}{2}) \tag{4}$$

$$k_0:=\sin(\frac{\theta_0}{2})$$

$$\alpha^2:=\frac{P_\phi^2}{16m^2l^4\omega^2\sin^4(\frac{\theta_0}{2})\cos^4(\frac{\theta_0}{2})}=(\frac{\dot{\phi}_0}{\omega})^2$$

In the last equality eq. (1) for the initial values $\theta_0,\dot{\phi}_0$ has been used. Now we have a polynomial of 3rd degree in the square root in the denominator. By pulling out the factor $k_0^2$ standing by u^3 it can be brought into standard form:

$$t=\frac{1}{2k_0\omega}\cdot\int_{1}^{x}\frac{du}{\sqrt{P(u)}} \tag{5}$$

with:

$$P(u)=\sum_{k=0}^{3}a_ku^k$$

$$a_3=1 \qquad a_2=-\alpha^2(1-k_0^2)-\frac{1+k_0^2}{k_0^2} \qquad a_1=\frac{\alpha^2(1-k_0^2)+1}{k_0^2} \qquad a_0=-\alpha^2\frac{(1-k_0^2)^2}{k_0^2}$$

It can be simply checked now that:

$$\sum_{k=0}^{3}a_k=0$$

holds. This can be used to write $a_2=-a_0-a_1-a_3$ and consequently the polynomial as:

$$-a_0\cdot(1+u)(1-u)-a_1\cdot u(1-u)+u^2\cdot(1-u)=0$$

From this we can read off the first zero as $u_1=1$ and after dividing by $(1-u)$ and applying pq-formula the two other ones as:

$$u_{2,3}=\frac{a_0+a_1}{2}\pm\sqrt{\left(\frac{a_0+a_1}{2}\right)^2+a_0}$$

After plugging in the values of the coefficients $a_k$ this reads as:

$$u_{2,3}=\frac{1}{2k_0^2}\cdot\left(1+\alpha^2(1-k_0^2)k_0^2\pm\sqrt{(1+\alpha^2(1-k_0^2)k_0^2)^2-4\alpha^2\cdot(1-k_0^2)^2\cdot k_0^2}\right) \tag{6}$$

With the help of this we can write P in the zero form:

$$P(u)=(u-u_1)(u-u_2)(u-u_3)$$

To finally apply the formula (17.4.62) in Abramowitz&Stegun we have to keep the right order of the zeros according to (3). First, one checks through a simple calculation that the radicand in (6) is always positive and therefore all solutions are real. Since we have $u_1=1$ , $u_2>1$ and $u_2>u_3$ in either case, we just have to distinguish the cases: $u_3<1$ , $u_3>1$ and $u_3=1$ . One can show from (6) that

$$u_3\sim 1\Leftrightarrow\left(\frac{\dot{\phi}_0}{\omega}\right)^2\sim\frac{1}{1-2k_0^2}=\frac{1}{\cos(\theta_0)}$$

meaning that the relation between the $u_3$ and 1 is the same as between the terms on the right side. Now:

$$\cos(\theta_E)=\left(\frac{\omega}{\dot{\phi}_0}\right)^2=\frac{g}{l\dot{\phi}_0^2}$$

is just the angle at which there is an equilibrium between gravity and centrifugal force on the mass. Consequently we can rewrite above relation as:

$$u_3\sim 1\Leftrightarrow\theta_E\sim\theta_0$$

since the cosine is a strictly monotonously falling function in the relevant interval $[0,\pi]$ . Therefore in the case $u_3=1$ we have $\theta_0=\theta_E$ and expect to have just a constant rotation in ϕ phi direction. The two different other cases each correspond to a displacement above the equilibrium for $u_3<1\Leftrightarrow\theta_E<\theta_0$ and below the equilibrium for $u_3>1\Leftrightarrow\theta_E>\theta_0$ .
Now, in the first case we have $u_2>u_1>u_3$ and can apply formula 17.4.62 with $\beta_1=u_2$ , $\beta_2=u_1$ , $\beta_3=u_3$ . This yields:

$$\int_1^x\frac{du}{\sqrt{P(u)}}=\int_{u_3}^x\frac{du}{\sqrt{P(u)}}-\int_{u_3}^1\frac{du}{\sqrt{P(u)}}=\frac{1}{\lambda}(F(\varphi,k)-F(\varphi_0,k))$$

with:

$$\sin^2(\varphi)=\frac{x-u_3}{1-u_3} \tag{7}$$

$$\lambda=\frac{1}{2}\sqrt{u_2-u_3}=\frac{1}{2k_0}\sqrt[4]{(1+\alpha^2(1-k_0^2)k_0^2)^2-4\alpha^2\cdot(1-k_0^2)^2\cdot k_0^2}$$

$$k=\sqrt{\frac{1-u_3}{u_2-u_3}}$$

The first equation implies that $\sin^2(\varphi_0)=1$ and therefore $\varphi_0=\frac{\pi}{2}$ . Now $F(\frac{\pi}{2},k)=E(k)$ is by definition the complete elliptic integral of the first kind. We can now insert the found expressions into (5) and obtain:

$$2\lambda k_0 \omega t + E(k) = F(\varphi, k)$$

which finally expresses t as an elliptic integral of first order. To invert this and find the formula for $\theta$ we can use the definition of the Jacobi elliptic function and write:

$$\sin^2(\varphi) = sn^2(\Omega t + E(k))$$

$$\Omega := 2\lambda k_0 \omega = \sqrt[4]{(1+\alpha^2(1-k_0^2)k_0^2)^2 - 4\alpha^2 \cdot (1-k_0^2)^2 \cdot k_0^2} \cdot \omega$$

Plugging this into eq. (7), solve for x and use $cn^2 = 1 - sn^2$ we have:

$$x = sn^2(\Omega t + E(k)) + u_3 \cdot cn^2(\Omega t + E(k))$$

and finally by substituting back x according to (4):

$$\sin(\frac{\theta}{2}) = k_0 \sqrt{sn^2(\Omega t + E(k)) + u_3 \cdot cn^2(\Omega t + E(k))} \qquad (8)$$

which is the final expression for $\theta(t)$ . Since the period for the elliptic functions sn(z,k), cn(z,k) is $4E(k)$ , the period duration of above motion is given by:

$$T = \frac{4E(k)}{\Omega} = \frac{4E(k)}{\sqrt[4]{(1+\alpha^2(1-k_0^2)k_0^2)^2 - 4\alpha^2 \cdot (1-k_0^2)^2 \cdot k_0^2} \cdot \omega}$$

The second case works completely analogous just with exchanging $u_3 \Leftrightarrow u_1$ since we now have $u_2 > u_3 > u_1$ . The last case is $u_3 = u_1 = 1$ . In this case the integral is elementary an yields:

$$\int_1^x \frac{du}{(x-1)\cdot\sqrt{x-u_2}} = \frac{2}{\sqrt{u_2-1}} \cdot \arctan(\frac{\sqrt{x-u_2}}{\sqrt{u_2-1}})\Bigg|_1^x$$

Now since $u_2 > 1$ holds, the lower limit is imaginary. The only way to prevent the whole integral to become imaginary (time must not be imaginary) is to require $x = 1$ which yields zero for the integral. Substituting back with the help of eq. (4) yields $\theta = \theta_0$ staying constant as expected.
From eq. (1) one can now also obtain the angular velocity in $\phi$ direction by inserting eq. (8) and using the half angle formula:

$$\dot{\phi} = \dot{\phi}_0 (1-k_0^2) \frac{1}{sn^2 + u_3 \cdot cn^2} \frac{1}{1 - k_0^2 sn^2 - k_0^2 u_3 \cdot cn^2}$$

The arguments of sn and cn are the same as above. In principle, from this equation one could also obtain $\phi(t)$ by integration:

$$\phi(t) = \dot{\phi}_0 (1-k_0^2) \int_0^t \frac{1}{sn^2 + u_3 \cdot cn^2} \frac{1}{1 - k_0^2 sn^2 - k_0^2 u_3 \cdot cn^2} dt'$$

for which i dont now if any closed form solutions in terms of Jacobi elliptic functions or other special functions exist.
In the limit of vanishing angular momentum $\alpha \to 0$ , in which always the first case $u_3 < 1$ holds, we have:

$$\lim_{\alpha \to 0} u_{2,3} = \frac{1 \pm 1}{2k_0^2} \qquad \lim_{\alpha \to 0} k(\alpha) = k_0 \qquad \lim_{\alpha \to 0} \lambda(\alpha) = \frac{1}{2k_0}$$

and therefore:

$$\lim_{\alpha \to 0} \sin(\frac{\theta}{2}) = k_0 \cdot sn(\omega t + E(k_0))$$

which is indeed just the solution for the mathematical pendulum. In the case of small initial displacements $k_0 \approx \frac{\theta_0}{2}$ we have $\sin(\frac{\theta}{2}) \approx \frac{\theta}{2}$ as well as $sn(\omega t) \approx \sin(\omega t)$ and $E(k_0) \approx E(0) = \frac{\pi}{2}$ . Then the above equation reduces further to:

$$\theta(t) \approx \theta_0 \sin(\omega t + \frac{\pi}{2}) = \theta_0 \cos(\omega t)$$

which is the well known solution to the mathematical pendulum when small-angle approximation is applied.