Stability Theorem

Consider a smooth system of ordinary differential equations

$$\dot{x} = v(x), \quad x \in D,$$

in a domain $D \subset \mathbb{R}^m$.

Suppose that this system admits $n+1$ first integrals:

$$f_1, \ldots, f_n, f \colon D \to \mathbb{R}.$$

We introduce the joint level sets

$$M_c = \{x \in D \mid f_i(x) = c_i, \quad i = 1, \ldots, n\}, \quad c = (c_1, \ldots, c_n).$$

Theorem 1. *Suppose that for some $\tilde{c}$, the restriction of the function f to the set $M_{\tilde{c}}$ attains an isolated local minimum at a point $\tilde{x} \in M_{\tilde{c}}$, that is, there exists $\varepsilon > 0$ such that*

$$(0 < |x - \tilde{x}| < \varepsilon) \wedge (x \in M_{\tilde{c}}) \Longrightarrow f(x) > f(\tilde{x}). \tag{1}$$

Then the point $\tilde{x}$ is an equilibrium point ($v(\tilde{x}) = 0$), and this equilibrium is Lyapunov stable.

The condition of a local minimum can be replaced by that of a local maximum by passing from the function f to $-f$.

Proof of Theorem 1

Suppose that $\tilde{x}$ is not an equilibrium point; then a solution $x(t)$ with $x(0) = \tilde{x}$ emanates from this point and, for all admissible $t > 0$, lies in the set $M_{\tilde{c}}$:

$$x(t) \in M_{\tilde{c}}, \quad x(t) \neq \tilde{x}, \quad t > 0.$$

Then for small $t > 0$ we have $f(x(t)) > f(x(0))$, which is impossible since f is a first integral.

The function

$$W(x) = (f(x) - f(\tilde{x}))^2 + \sum_{k=1}^{n} (f_k(x) - f_k(\tilde{x}))^2$$

is a Lyapunov function for this equilibrium point. Indeed, this function remains constant along the solutions of the system (as a combination of first integrals), with $W(\tilde{x}) = 0$ and $W(x) \geq 0$ on D.

Choose a punctured neighborhood U of the point $\tilde{x}$ in the domain D such that the inequality $f(x) > f(\tilde{x})$ holds for all $x \in U \cap M_{\tilde{c}}$. Such a neighborhood exists by the hypothesis of the theorem.

In this punctured neighborhood, the following inequality holds:

$$W(x) > W(\tilde{x}).$$

Indeed, suppose the contrary: there exists a point $x \in U$ such that $W(x) = W(\tilde{x}) = 0$. Consequently, $f_k(x) = f_k(\tilde{x})$ for all $k = 1, \ldots, n$ (which implies $x \in M_{\tilde{c}}$) and simultaneously $f(x) = f(\tilde{x})$, contradicting the choice of the neighborhood U.

The theorem is proved.