1 Lognormal stock process with normal volatility process

The next model we consider is a lognormal stock process with a normal volatility process, given by

$$\begin{cases} \mathrm{d}S = \mu S\ \mathrm{d}t + \sigma S\ \mathrm{d}W, & \mathrm{d}W \sim N(0, \mathrm{d}t), \quad (1.1) \\ \mathrm{d}\sigma = a\ \mathrm{d}t + b\ \mathrm{d}\widetilde{W}, & \mathrm{d}\widetilde{W} \sim N(0, \mathrm{d}t). \quad (1.2) \end{cases}$$

Here a and b are constants. The stochastic parts $\mathrm{d}W$ and $\mathrm{d}\widetilde{W}$ have correlation ρ, such that

$$\mathrm{E}\left[\mathrm{d}W\mathrm{d}\widetilde{W}\right] = \rho\ \mathrm{d}t.$$

Applying Itô's formula (**??**) again, we get

$$\begin{aligned} \mathrm{d}V &= \frac{\partial V}{\partial t}\mathrm{d}t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\mathrm{d}S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\mathrm{d}\sigma^2 + \frac{\partial^2 V}{\partial S \partial \sigma}\mathrm{d}S\mathrm{d}\sigma + \frac{\partial V}{\partial S}\mathrm{d}S + \frac{\partial V}{\partial \sigma}\mathrm{d}\sigma. \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial \sigma^2} + b\sigma S\rho\frac{\partial^2 V}{\partial S \partial \sigma} + \mu S\frac{\partial V}{\partial S} + a\frac{\partial V}{\partial \sigma}\right)\mathrm{d}t \\ &+ \sigma S\frac{\partial V}{\partial S}\mathrm{d}W + b\frac{\partial V}{\partial \sigma}\mathrm{d}\widetilde{W}. \end{aligned} \tag{1.3}$$

The next step is to construct a portfolio

$$\Pi = V - \Delta S - \Delta_1 V_1, \tag{1.4}$$

which contains the option $V(S, \sigma, t)$, a quantity $-\Delta$ of the stock S, and a quantity $-\Delta_1$ of another asset whose value V_1 depends on the volatility σ.

The change $\mathrm{d}\Pi$ in this portfolio in a time $\mathrm{d}t$ is given by

$$\mathrm{d}\Pi = \mathrm{d}V - \Delta\mathrm{d}S - \Delta_1\mathrm{d}V_1.$$

Also we know that $\mathrm{d}\Pi = r\Pi\mathrm{d}t = r\left(V - \Delta S - \Delta_1 V_1\right)\mathrm{d}t$, such that we have

$$r\left(V - \Delta S - \Delta_1 V_1\right)\mathrm{d}t = \mathrm{d}V - \Delta\mathrm{d}S - \Delta_1\mathrm{d}V_1.$$

To make the portfolio instantaneously risk-free, we must eliminate all terms with $\mathrm{d}W_t$ and $\mathrm{d}\widetilde{W}_t$. So we choose

$$\begin{aligned} \Delta &= \frac{\partial V}{\partial S} - \Delta_1\frac{\partial V_1}{\partial S}, \quad \text{and} \\ \Delta_1 &= \frac{\partial V}{\partial \sigma}\Big/\frac{\partial V_1}{\partial \sigma} = 0. \end{aligned}$$

Collecting all V-terms on the left-hand side and all V_1-terms on the right-hand side, we get

$$\begin{aligned} &\frac{\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial \sigma^2} + b\sigma S\rho\frac{\partial^2 V}{\partial S\partial \sigma_t} + rS\frac{\partial V}{\partial S} - rV}{\frac{\partial V}{\partial \sigma}} \\ = &\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V_1}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 V_1}{\partial \sigma^2} + b\sigma S\rho\frac{\partial^2 V_1}{\partial S\partial \sigma} + rS\frac{\partial V_1}{\partial S} - rV_1}{\frac{\partial V_1}{\partial \sigma}}. \end{aligned} \tag{1.5}$$

The left-hand side of equation (1.5) is a function of V only and the right-hand side is a function of V_1 only. The only way that this can be is for both sides to be equal to some function f of the independent variables S, σ and t. We deduce that

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\frac{1}{2}b^2\frac{\partial^2 V}{\partial \sigma^2}+b\sigma S\rho\frac{\partial^2 V}{\partial S\partial\sigma}+rS\frac{\partial V}{\partial S}-rV=-\Big(a-\lambda b\sigma\Big)\frac{\partial V}{\partial \sigma}, \tag{1.6}$$

where, without loss of generality, we have written the arbitrary function f of S, σ and t as $\Big(a-\lambda b\sigma\Big)$, where a and b are the constant drift and volatility from the SDE (1.2) for instantaneous variance.

So the price $V(S,\sigma,t)$ satisfies the following differential equation:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\frac{1}{2}b^2\frac{\partial^2 V}{\partial \sigma^2}+b\sigma S\rho\frac{\partial^2 V}{\partial S\partial\sigma}+rS\frac{\partial V}{\partial S}+\Big(a-\lambda b\sigma\Big)\frac{\partial V}{\partial \sigma}-rV=0. \tag{1.7}$$

1.1 Scalings

Suppose that σ and b are small volatilities, and write

$$\Sigma=\frac{\sigma}{\varepsilon^\eta}\Leftrightarrow\sigma=\varepsilon^\eta\Sigma \quad\text{and}\quad B=\frac{b}{\varepsilon^\nu}\Leftrightarrow b=\varepsilon^\nu B. \tag{1.8}$$

Then the partial differential equation (1.7) becomes

$$\begin{aligned}\varepsilon^\eta\frac{\partial V}{\partial t}+\frac{1}{2}\varepsilon^{3\eta}\Sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\frac{1}{2}\varepsilon^{2\nu-\eta}B^2\frac{\partial^2 V}{\partial \Sigma^2}+\varepsilon^{\eta+\nu}\Sigma BS\rho\frac{\partial^2 V}{\partial S\partial\Sigma}+\varepsilon^\eta rS\frac{\partial V}{\partial S}&\\+\Big(a-\varepsilon^{\eta+\nu}\lambda B\Sigma\Big)\frac{\partial V}{\partial\Sigma}-\varepsilon^\eta rV=0.&\end{aligned} \tag{1.9}$$

Maximum balance gives

$$0=2\nu-\eta\quad\Leftrightarrow\quad 2\nu=\eta\quad\Leftrightarrow\quad\nu=1,\ \eta=2, \tag{1.10}$$

such that equation (1.9) becomes

$$\begin{aligned}\varepsilon^2\frac{\partial V}{\partial t}+\frac{1}{2}\varepsilon^{3}\Sigma^2S^2\frac{\partial^2 V}{\partial S^2}+\frac{1}{2}B^2\frac{\partial^2 V}{\partial \Sigma^2}+\varepsilon^{3}\Sigma BS\rho\frac{\partial^2 V}{\partial S\partial\Sigma}+\varepsilon^2 rS\frac{\partial V}{\partial S}&\\+\Big(a-\varepsilon^{3}\lambda B\Sigma\Big)\frac{\partial V}{\partial\Sigma}-\varepsilon^2 rV=0.&\end{aligned} \tag{1.11}$$

1.2 Outer expansion

Expand $V_\varepsilon=V_0+\varepsilon V_1+\ldots$, such that the $\mathcal{O}(1)$ equation becomes

$$\frac{1}{2}B^2\frac{\partial^2 V_0}{\partial\Sigma^2}+a\frac{\partial V_0}{\partial\Sigma}=0. \tag{1.12}$$

After integrating twice, we have

$$V_0(S,\Sigma,t)=-c_1(S,t)\frac{B^2}{2a}e^{-\frac{2a}{B^2}\Sigma}+c_2(S,t), \tag{1.13}$$

with terminal condition $V_0(S,\Sigma,T)=P(S)=\max(S-K,0)$.

This solution should also satisfy the following limits:[1]

$$\lim_{\Sigma\to 0} V_0(S,\Sigma,t) = \max(S-K,0)\cdot e^{-r(T-t)}, \tag{1.14}$$

and

$$\lim_{\Sigma\to \infty} V_0(S,\Sigma,t) = S, \tag{1.15}$$

which gives $c_2(S,t)$ and

$$c_1(S,t) = \frac{2a}{B^2}\left(c_2(S,t) - \max(S-K,0)\cdot e^{-r(T-t)}\right) = \frac{2a}{B^2}\left(S - \max(S-K,0)\cdot e^{-r(T-t)}\right).$$

The solution of the $\mathcal{O}(1)$ equation (1.12) thus becomes

$$V_0(S,\Sigma,t) = \left(\max(S-K,0)\cdot e^{-r(T-t)} - S\right)e^{-\frac{2a}{B^2}\Sigma} + S. \tag{1.16}$$

The $\mathcal{O}(\varepsilon)$ equation is given by

$$\frac{1}{2}B^2\frac{\partial^2 V_1}{\partial \Sigma^2} + a\frac{\partial V_1}{\partial \Sigma} = 0. \tag{1.17}$$

After integrating twice, again we have

$$V_1(S,\Sigma,t) = -c_3(S,t)\frac{B^2}{2a}e^{-\frac{2a}{B^2}\Sigma} + c_4(S,t). \tag{1.18}$$

Now the terminal condition is given by $V_1(S,\Sigma,T)=0$.

Also we know that

$$\lim_{\Sigma\to 0} V_1 = 0, \tag{1.19}$$

and

$$\lim_{\Sigma\to \infty} V_1 = 0, \tag{1.20}$$

such that $c_4(S,t)=0$ and $c_3(S,t)=0$, and thus the solution of the $\mathcal{O}(\varepsilon)$ equation (1.17) is given by

$$V_1(S,\Sigma,t) = 0. \tag{1.21}$$

The $\mathcal{O}(\varepsilon^2)$ equation is given by

$$\frac{\partial V_0}{\partial t} + \frac{1}{2}B^2\frac{\partial^2 V_2}{\partial \Sigma^2} + rS\frac{\partial V_0}{\partial S} + a\frac{\partial V_2}{\partial \Sigma} - rV_0 = 0. \tag{1.22}$$

Now that we know the $\mathcal{O}(1)$ solution V_0, we can substitute it, such that the $\mathcal{O}(\varepsilon^2)$ equation becomes[2]

$$\frac{1}{2}B^2\frac{\partial^2 V_2}{\partial \Sigma^2} + a\frac{\partial V_2}{\partial \Sigma} = \left(-2\max(S-K,0)\ e^{-r(T-t)} + S\ H(S-K)\ e^{-r(T-t)} - S\right)re^{-\frac{2a}{B^2}\Sigma} + 1.$$

!SOLUTION: MAYBE NOT NECESSARY FOR OUTER EXPANSION?!

[1]This can be explained by considering te case $\Sigma = 0$, which also causes $\mathrm{d}\Sigma = 0$. The SDE now becomes $\mathrm{d}S = \mu S\mathrm{d}t$, which is deterministic, with terminal condition $V(S,T) = \max(S-K,0)$. And when we take a look at the case $\Sigma\to\infty$ in the exact Black-Scholes solution, we get $V\to S$.

[2]Here $H(\cdot)$ is the **Heaviside function**.

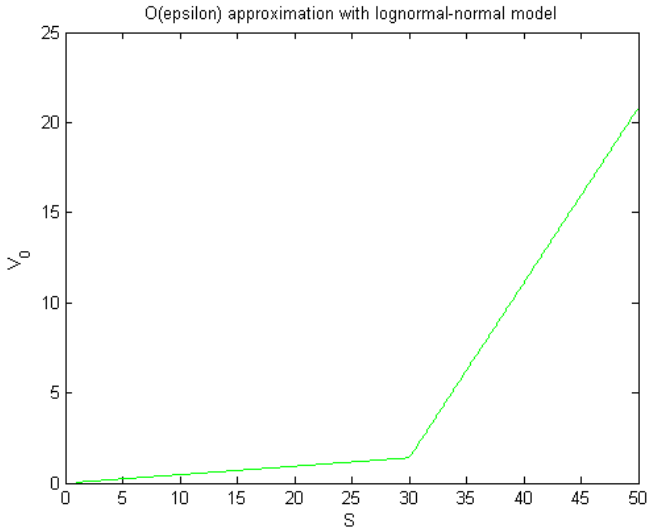

Figure 1: The $\mathcal{O}(\varepsilon)$ approximation $V_0 + \varepsilon V_1$.

1.3 Inner expansion

In order to create an interior boundary layer near $S = K$, we introduce a local variable

$$x = \frac{S-K}{\varepsilon^\alpha K}, \tag{1.23}$$

such that we can replace S by $K\left(1 + \varepsilon^\alpha x\right)$.

The partial differential equation (1.11) now becomes

$$\varepsilon^2 \frac{\partial V}{\partial t} + \frac{1}{2}\varepsilon^{3-2\alpha}\Sigma^2 \frac{\partial^2 V}{\partial x^2} + \varepsilon^{3-\alpha} x^2 \Sigma^2 \frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\varepsilon^3 x^2 \Sigma^2 \frac{\partial^2 V}{\partial x^2} + \frac{1}{2}B^2 \frac{\partial^2 V}{\partial \Sigma^2} + \varepsilon^{3-\alpha}\Sigma B\rho \frac{\partial^2 V}{\partial x \partial \Sigma}$$
$$+\varepsilon^3 \Sigma B \rho x \frac{\partial^2 V}{\partial x \partial \Sigma} + \varepsilon^{2-\alpha} r \frac{\partial V}{\partial x} + \varepsilon^2 r x \frac{\partial V}{\partial x} + \left(a - \varepsilon^3 \lambda B \Sigma\right) \frac{\partial V}{\partial \Sigma} - \varepsilon^2 r V = 0.$$

Maximum balance gives $\alpha = \frac{3}{2}$, such that we have

$$\varepsilon^2 \frac{\partial V}{\partial t} + \frac{1}{2}\Sigma^2 \frac{\partial^2 V}{\partial x^2} + \varepsilon^{\frac{3}{2}} x^2 \Sigma^2 \frac{\partial^2 V}{\partial x^2} + \frac{1}{2}\varepsilon^3 x^2 \Sigma^2 \frac{\partial^2 V}{\partial x^2} + \frac{1}{2}B^2 \frac{\partial^2 V}{\partial \Sigma^2} + \varepsilon^{\frac{3}{2}}\Sigma B\rho \frac{\partial^2 V}{\partial x \partial \Sigma}$$
$$+\varepsilon^3 \Sigma B \rho x \frac{\partial^2 V}{\partial x \partial \Sigma} + \varepsilon^{\frac{1}{2}} r \frac{\partial V}{\partial x} + \varepsilon^2 r x \frac{\partial V}{\partial x} + \left(a - \varepsilon^3 \lambda B \Sigma\right) \frac{\partial V}{\partial \Sigma} - \varepsilon^2 r V = 0.$$

If we expand $V_\varepsilon = V_0 + \sqrt{\varepsilon} V_1 + \varepsilon V_2 + \ldots$, the $\mathcal{O}(1)$ equation becomes

$$\frac{1}{2}\Sigma^2 \frac{\partial^2 V_0}{\partial x^2} + \frac{1}{2}B^2 \frac{\partial^2 V_0}{\partial \Sigma^2} + a\frac{\partial V_0}{\partial \Sigma} = 0, \tag{1.24}$$

subject to the boundary conditions

$$\lim_{\Sigma\to 0} V_0(x,\Sigma,t) = \varepsilon^{\frac{3}{2}}\max(x,0)e^{-r(T-t)}, \tag{1.25}$$

$$\lim_{\Sigma\to\infty} V_0(x,\Sigma,t) = K(1+\varepsilon^{\frac{3}{2}}x), \tag{1.26}$$

and terminal condition

$$V_0(x,\Sigma,T) = \varepsilon^{\frac{3}{2}}K\max(x,0). \tag{1.27}$$

For notational reasons, we rewrite equation (1.24)

$$\frac{1}{2}\sigma^2\frac{\partial^2 V_0}{\partial x^2} + \frac{1}{2}B^2\frac{\partial^2 V_0}{\partial\sigma^2} + a\frac{\partial V_0}{\partial\sigma} = 0. \tag{1.28}$$

To solve this, we use separation of variables, writing

$$V_0 = X(x)Y(\sigma). \tag{1.29}$$

After substitution into equation (1.28), we obtain

$$\frac{1}{2}\sigma^2 X''(x)Y(\sigma) + \frac{1}{2}B^2X(x)Y''(\sigma) + aX(x)Y'(\sigma) = 0, \tag{1.30}$$

$$\Rightarrow \frac{X''(x)}{X(x)} = -\frac{B^2}{\sigma^2}\frac{Y''(\sigma)}{Y(\sigma)} - \frac{2a}{\sigma^2}\frac{Y'(\sigma)}{Y(\sigma)} = k. \tag{1.31}$$

So we have to solve the following two ODEs:

$$\frac{X''(x)}{X(x)} = k, \tag{1.32}$$

$$-\frac{B^2}{\sigma^2}\frac{Y''(\sigma)}{Y(\sigma)} - \frac{2a}{\sigma^2}\frac{Y'(\sigma)}{Y(\sigma)} = k. \tag{1.33}$$

The solution of the first ODE (1.32) for $X(x)$ is given by

$$X(x) = z_1\sinh(\sqrt{k}x) + z_2\cosh(\sqrt{k}x), \quad \text{with} \quad z_1, z_2 \in \mathbb{R}. \tag{1.34}$$

The solution of the ODE for $Y(\sigma)$ can be found by constructing a power series in σ

$$Y(\sigma) = \sum_{n=0}^{\infty} c_n\sigma^n, \tag{1.35}$$

First we rewrite the ODE (1.33) as

$$B^2Y''(\sigma) + 2aY'(\sigma) + k\sigma^2Y(\sigma) = 0, \tag{1.36}$$

which after substitution of the series (1.35) becomes

$$B^2\sum_{n=2}^{\infty} n(n-1)c_n\sigma^{n-2} + 2a\sum_{n=0}^{\infty} nc_n\sigma^{n-1} + k\sigma^2\sum_{n=0}^{\infty} c_n\sigma^n = 0. \tag{1.37}$$

Now, changing coefficients so all powers are the same, we have

$$\sum_{n=0}^{\infty}\left((n+2)B^2c_{n+2} + 2ac_n\right)(n+1)\sigma^n + \sum_{n=2}^{\infty} kc_{n-2}\sigma^n = 0. \tag{1.38}$$

For $n = 0$ and $n = 1$ we obtain

$$\begin{array}{lrcl} n = 0: & 2B^2c_2 + 2ac_0 & = & 0 \quad \Rightarrow c_2 = -\frac{a}{B^2}c_0, \\ n = 1: & (6B^2c_3 + 4ac_1)\sigma & = & 0 \quad \Rightarrow c_3 = -\frac{2a}{3B^2}c_1, \end{array}$$

and for $n > 1$ we have the following recurrency relation:

$$(n+1)(n+2)B^2c_{n+2} + 2ac_n(n+1) + kc_{n-2} = 0 \Rightarrow c_{n+2} = -\frac{2a}{(n+2)B^2} - \frac{kc_{n-2}}{(n+1)(n+2)B^2}.$$

!BCs? GENERAL SOLUTION?!

The $\mathcal{O}(\sqrt{\varepsilon})$ equation is given by

$$\frac{1}{2}\Sigma^2\frac{\partial^2 V_1}{\partial x^2} + \frac{1}{2}B^2\frac{\partial^2 V_1}{\partial \Sigma^2} + r\frac{\partial V_0}{\partial x} + a\frac{\partial V_0}{\partial \Sigma} = 0. \tag{1.39}$$

!SOLUTION?!

1.4 Matching

References

A List of symbols

Greek symbols

Γ:	Second order derivative of the option price, one of the Greeks.
γ:	Parameter in the CEV model.
Δ:	First order derivative of the option price, one of the Greeks.
δ:	Dividend.
ε:	Stretching parameter.
μ_t:	Drift.
σ_t:	Volatility.

Latin symbols

B_t:	Bond price.
K:	Strike price.
$N(\cdot)$:	Standard normal cumulative distribution function, $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$.
$N(\mu, \sigma)$:	Normal distribution with mean μ and variance σ^2.
$n(\cdot)$:	Standard normal probability density function, $n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$.
$P(S)$:	Payoff.
r_t:	Risk-free interest rate.
S_t:	Stock price.
T:	Expiry time.
t:	Time.
V:	General option price.
V^{call}:	Price of a call option.
V^{put}:	Price of a put option.
W_t:	Wiener Process.