Grasshopper on a Straw

The statement:

A small grasshopper of mass m sits on a tip of a straw. The straw is a thin homogeneous rod of mass M and of length $2l$. The straw lies on a smooth horizontal floor. What smallest initial speed must the grasshopper have to jump to other tip of the straw?

By $\boldsymbol{v}_G$ denote an initial velocity of the grasshopper.

The answer:

If $M \geq 2m$ then

$$\min|\boldsymbol{v}_G|^2 = \frac{2lgM}{M+m};$$

if $M < 2m$ then

$$\min|\boldsymbol{v}_G|^2 = \frac{lgM}{M+m}\sqrt{2(1+\cos\varphi_*)},$$

where $\varphi_* \in (0,\pi)$ is a root of equation

$$\Big(\frac{M}{m}+1\Big)\varphi = 3\sin\varphi. \tag{0.1}$$

The solution.

The grasshopper jumps at moment $t=0$.

Let $Oxyz$ stand for a fixed coordinate frame such that initially ($t=0$) the straw AB is situated along the axis Ox and

$$A = O, \quad B = (2l, 0, 0).$$

The axis Oz is directed vertically such that $\boldsymbol{g} = -g\boldsymbol{e}_z$.

By S denote the center of mass of the straw, initially it follows that $S = (l, 0, 0)$.

Initially the grasshopper G sits at point A and has velocity

$$\boldsymbol{v}_G = \boldsymbol{u} + w\boldsymbol{e}_z, \quad \boldsymbol{u} = u_x\boldsymbol{e}_x + u_y\boldsymbol{e}_y,$$

so that

$$\boldsymbol{OG} = (wt - gt^2/2)\boldsymbol{e}_z + u_xt\boldsymbol{e}_x + u_yt\boldsymbol{e}_y.$$

When $t=0$ we have

$$M\boldsymbol{v}_S + m\boldsymbol{u} = 0, \quad J\boldsymbol{\omega} = -\boldsymbol{SA}\times(m\boldsymbol{u}), \quad \boldsymbol{SA} = -l\boldsymbol{e}_x,$$

where $J = Ml^2/3$ is the moment of inertia of the straw about the point S; $\boldsymbol{\omega} = \omega\boldsymbol{e}_z$ is a angular velocity of the straw. Thus we obtain

$$\boldsymbol{v}_S = -\frac{m}{M}u_x\boldsymbol{e}_x - \frac{m}{M}u_y\boldsymbol{e}_y, \quad \omega = \frac{ml}{J}u_y.$$

The angle of straw's turn is

$$\varphi = \omega t.$$

It follows that

$$u_y t = \frac{Ml}{3m}\varphi. \tag{0.2}$$

Further we have

$$\boldsymbol{OS} = l\boldsymbol{e}_x + \int_0^t \boldsymbol{v}_S d\xi = \Big(l - \frac{m}{M}u_x t\Big)\boldsymbol{e}_x - \frac{m}{M}u_y t\boldsymbol{e}_y,$$

and

$$\boldsymbol{OB} = \boldsymbol{OS} + l(\cos\varphi\boldsymbol{e}_x + \sin\varphi\boldsymbol{e}_y).$$

Equation $\boldsymbol{OB} = \boldsymbol{OG}$ implies

$$w = gt/2, \tag{0.3}$$

$$u_x t = l - \frac{m}{M}u_x t + l\cos\varphi, \tag{0.4}$$

$$u_y t = -\frac{m}{M}u_y t + l\sin\varphi. \tag{0.5}$$

Substituting formula (0.2) to equation (0.5) we obtain equation (0.1).

Consider nontrivial case when the root $\varphi_* \in (0,\pi)$ exists. From formula (0.4) we get

$$u_x = \frac{l(1+\cos\varphi_*)}{(1+m/M)t}.$$

Using this formula together with formulas (0.3) and (0.2) let us construct a function

$$f(t) = |\boldsymbol{v}_G|^2 = u_x^2 + u_y^2 + w^2.$$

We must minimize this function on the interval $t > 0$. Let us do that in standard way:

$$\dot f(t) = 0 \Longrightarrow t = t_* = \frac{\sqrt{2(M+m)gMl\sqrt{2(1+\cos\varphi_*)}}}{(M+m)g};$$

and

$$f(t_*) = \frac{lgM}{M+m}\sqrt{2(1+\cos\varphi_*)}.$$