Uncertainties Manual

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Uncertainties

When a quantity, such as the length of a piece of string, is measured, there is some uncertainty in the result. One intuitively knows one just cannot measure such a thing to an accuracy of a millionth of an inch. How well CAN it be measured?

1. Statistical, or Random, Uncertainties

One very good way to assess the uncertainty is to repeat the measurement, many times if possible. Here is a table of repeated measurements of the length:

Measurement No.	Length (mm)
1	384
2	381
3	383
4	383
5	385
6	382
7	384
8	383
9	384
10	382

2. Best estimate for the Length

The **best estimate for the length turns out to be the average:** $\overline{L} = \sum_{i=1,n} \frac{L_i}{n}$ **[U1]**

For our string, we get $\overline{L} = \sum_{i=1,n} \frac{L_i}{n} = \frac{3831}{10} = 383.1$ mm.

3. Best estimate for the uncertainty

Looking at the table, one can see at a glance that each measurement differs from many of the others by something like 1 or 2 mm. That is a very good habit to get into: make a quick eyeball estimate of the uncertainty just by looking at how the data are scattered.

A quantitative estimate can be made by calculating the "root-mean-square" (rms) deviation of measurements from the mean. First calculate the mean (as above), then calculate how far each measurement is away from the mean, square that difference, take the average (mean), and finally the square root of that mean. Here is a table with extra columns showing the two additional steps:

Measurement No.	Length (mm)	Difference from mean (mm)	Square of difference
1	384	0.9	0.81
2	381	-2.1	4.41
3	383	-0.1	0.01
4	383	-0.1	0.01
5	385	1.9	3.61
6	382	-1.1	1.21
7	384	0.9	0.81
8	383	-0.1	0.01
9	384	0.9	0.81
10	382	-1.1	1.21

The mean of the squared differences (last column) is 1.3 mm^2, and the square root of that is 1.14 mm, or approximately 1.1 mm. That is the "rms error" or "rms uncertainty" and is a good estimate for the uncertainty in any single measurement. In other words, subsequent measurements, if taken, would "often" be within 1.1 mm of the mean we estimated above (there is no point in using more than 1 or 2 significant figures – uncertainties are only estimates and not precisely known). Mathematically, the rms uncertainty (also called the 1-standard-deviation uncertainty, or "1 σ" for short) is:

$$\sigma_{L_i}^2 = \sum_i \frac{(L_i - \bar{L})^2}{n-1} \qquad \textbf{[U2]}$$

Our measurements give

$$\sigma_{L_i}^2 = \sum_i \frac{(L_i - \bar{L})^2}{n-1} = 1.43 \text{ mm}^2$$

$$\sigma_{L_i} = 1.2 \text{ mm}$$

Note in the formal expression we take the average by dividing by n-1 (9) instead of n (10). The reason is, we already made use of the data once in order to calculate the mean itself. That uses up one of the so-called "degrees of freedom", and we take account of that by reducing the effective number of measurements by 1 when it comes time to calculate the uncertainty. Confusing? Consider what would happen if we only had one measurement – there is no way to estimate the uncertainty at all, and the formula with n-1 in place of n gets it right. When n is as large as 10, it doesn't matter very much whether you use n or n-1.

And what did we mean by "often" when discussing how often new measurements would be less than 1 σ away from the mean? It turns out to be 2/3 of the time (actually 68.3%). That gives another good way to estimate the error, and easier. Find the 2/3 of the measurements that are closest to the mean: they are measurements No. 3, 4, 8, 1, 6, 7, 9, 10. That is 8 out of 10 measurements, a number reasonably close to 2/3 of the total. They range from 382 to 384 mm. This range is then approximately equal to the mean plus-or-minus σ, a total span of 2σ.

$$\pm\sigma = \textbf{range that includes 2/3 of the measurements} \qquad \textbf{[U3]}$$

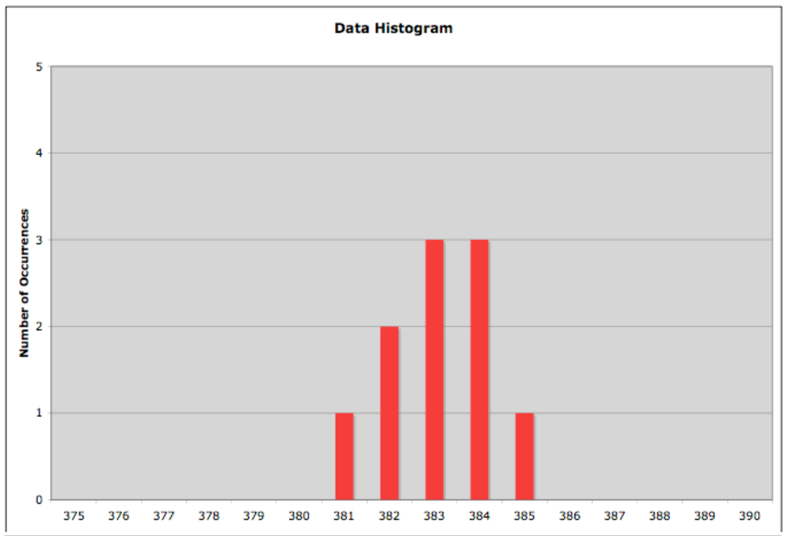
So we would conclude that

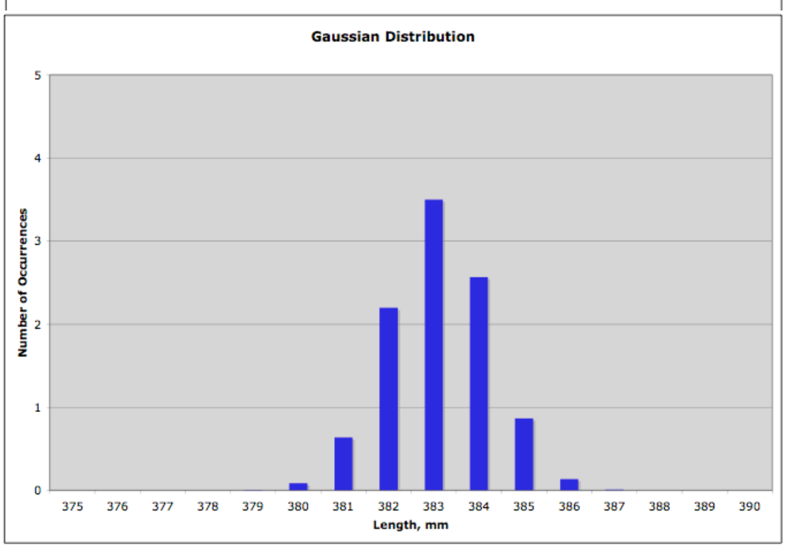
$$2\sigma_{L_i} = 384 - 382 = 2 \text{ mm}$$
$$\sigma_{L_i} = 1 \text{ mm}$$

The uncertainty estimated this way, 1 mm, is a good approximation to the rms uncertainty, 1.2 mm. When asked to calculate the uncertainty, you may use either method. (If you have only a few measurements, 2 or 3, please be careful to use n-1 instead of n if you are calculating the rms uncertainty.)

The top graph is a "histogram", the number of times each value for the length occurred. Estimating the interval that contains 2/3 of the measurements is a bit of an art: +/- 2 mm is too big, it includes all 10. +/-1 is too small, includes only 6, but it's close. So maybe +/-1.2 mm? That's not bad.

The lower graph is what you would see if you repeated this experiment of 10 measurements a very large number of times. The expected values for the number of occurrences follow a Gaussian (bell-shaped) curve with mean 383.1 mm and standard deviation 1.2 mm.





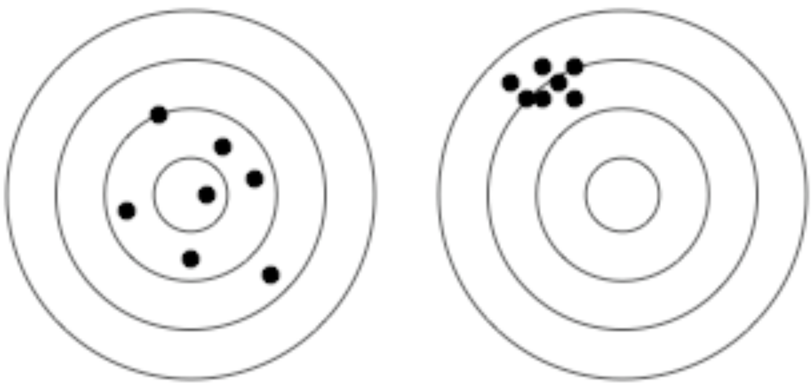
4. Digital Instruments

Repeated measurements with a digital instrument may give the same number over and over again. In that case, there is no variation to give you an idea of the uncertainty. The uncertainty (i.e. the lack of knowledge) is at least as large as ± ½ of the least significant digit of the instrument.

5. Systematic Uncertainties

It may be considerably worse if the instrument is just wrong, and no amount of repetition will reveal that. That is called a "systematic uncertainty" and it is not confined to digital instruments. Estimating such errors is difficult and usually calls for comparison with a different instrument or technique.

Systematic errors (uncertainties) are the main reason to distinguish between the words "accuracy" and "precision". The figure illustrates the difference.



Left: Accurate but not precise. Right: Precise but not accurate. A systematic error (such as a strong wind blowing) might be responsible for the situation on the right.

6. Writing the Answer Down

It is customary to write the measurement with three parts: the measured value, the uncertainty, and the units. For example, a measured length would be written:

Measurement No. 6:

Length $L_6 = 382 \pm 1$ mm

The ± sign reminds us that the true value might deviate from the mean in either direction and informs us that the "1" is an uncertainty, not something to be added or subtracted.

For an uncertainty, do not use more than 2 significant figures. Also, do not write the measurement itself with significant figures that are, in fact, made insignificant by the size of the uncertainty.

OK: $L_6 = 382 \pm 1$
$L_6 = 382.4 \pm 1.2$

NOT OK:
$L_6 = 382 \pm 1.2$ (too many sig. fig. in the uncertainty)
$L_6 = 382.4 \pm 1$ (too many sig. fig. in the number)
$L_6 = 382000 \pm 1000$ (too many sig. fig. in the uncertainty)

To avoid the last bad example, switch to scientific notation:

$$L_6 = (3.82 \pm 0.01) \times 10^5$$

7. Calculating with an Uncertain Quantity

Suppose we had used the string to measure the circumference of a wheel. What could we say about the diameter of the wheel and its uncertainty? This situation comes up all the time in physics – we usually need to use the quantity in a formula. The diameter is:

$$D = \frac{L}{\pi}$$

There is an easy and general method to handle this, as long as the uncertainties are small compared to the mean. If you have not had calculus, simply do the following:

a) calculate D_i with L_i, a measured value.
b) calculate again with L_i increased by 1 standard deviation (σ).
c) take the difference between the two – that's the uncertainty in the calculated quantity (D_i).

Example: Suppose we have one of the measurements of L being 384 mm, and we have determined the uncertainty to be 1 mm. The corresponding value of D for that measurement would be

$$D = \frac{384}{\pi} = 122.2 \text{ mm}$$

Increase L by its uncertainty (1 mm) to 385 mm, recalculate:

$$D + \sigma_D = \frac{L + \sigma_L}{\pi} = 122.5 \text{ mm}$$

The difference between the two gives us $\sigma_D = 0.3$ mm, which is the uncertainty in the diameter for that measurement. You will observe that:

When a quantity is multiplied or divided by a constant, its uncertainty is also multiplied or divided by the same constant.

[U4]

In this case: $\sigma_D = \frac{\sigma_L}{\pi}$. (If the uncertainty comes out negative this way, simply take the absolute value: -- uncertainties are generally reported with the ± sign anyway.)

8. Calculating with More Than 1 Uncertain Quantity

Another very common situation is a formula containing 2 or more quantities, each of which is uncertain. Use the same approach as in 7, holding each quantity at its measured value while you increase a different quantity, ONE at a TIME, by its uncertainty. This yields the effect on the answer of the uncertainty in each quantity by itself. Suppose we are interested in the moment of inertia *I* of the wheel, and we measured the mass *m* to be 1.27 ± 0.01 kg. The formula for *I* is,

$$I = \frac{1}{2}mr^2 = \frac{1}{8}mD^2.$$

The calculation gets long and, to avoid making mistakes, it is wise to make a table:

	D (m)	*m* (kg)	*I* (kg-m^2)	Change in I
Central Value	0.384	1.27	0.02341	-
D + σ_D Value	0.385	1.27	0.02353	0.00012 $\equiv \sigma_I(Diam)$
m + σ_m Value	0.384	1.29	0.02359	0.00018 $\equiv \sigma_I(mass)$

The table gives the individual effects of uncertainties in *D* and in *m* on *I*, namely 0.00012 and 0.00018 kg-m^2, respectively. What is the total uncertainty in *I*? It might be thought to be just the sum, i.e. 0.00030 kg-m^2. However, it would be unlucky for both the measured mass and the diameter to be off from their true values in the same direction. That could happen, but so could it happen that they were off in opposite directions and produced a *better* result by chance. Uncertainties do not simply add together, for this reason. Instead, it can be shown that:

The best estimate for the combined uncertainty is the square root of the sum of the squares.

[U5]

In fact, it is like our old friend the rms error again, in a different situation. So, the uncertainty in *I*, σ_I, is

$$\sigma_I = \sqrt{\sigma_I^2(Diam) + \sigma_I^2(mass)}$$

The uncertainty works out to be 0.00021 kg-m^2. Note that $\sigma_I(Diam)$ is not the <u>uncertainty in the diameter</u>, but the <u>uncertainty in the moment of inertia</u> *caused* by the uncertainty in the diameter. The same applies to $\sigma_I(mass)$.

This procedure is known as "combining the uncertainties in quadrature" [meaning, as squares].

Building a table like this always works, even for complicated formulas with functions like sine, exp, log, etc. However, it is a bit cumbersome for use in common situations with products, quotients, and powers that arise frequently.

Powers (exponents) are like products, but there is a big difference: Suppose $Z = X^n$, and X has a fractional (percentage) uncertainty σ. We could write $Z = X \bullet X \bullet X \ldots$ as a product, and it might be tempting to say the uncertainty in Z is the quadrature of the uncertainties in the n X's. But that would not be right, because the uncertainties in the X's are all in the same direction. It's the same X. In the spirit of building a table to find the uncertainty in Z, increase X by its uncertainty to $X(1+\sigma)$. Then

$$Z(1+\sigma_Z) = X(1+\sigma)X(1+\sigma)X(1+\sigma)...$$
$$\sigma_Z \approx n\sigma$$

using the Binomial theorem and neglecting powers of σ larger than 1.

The fractional uncertainty is multiplied by an exponent: $\sigma_Z = n\sigma$ **[U6a]**

If the answer Z depends on measurements X and Y as:

$$Z = cX^nY^m$$

then the uncertainty in Z, σ_Z, is given by:

$$\sigma_Z = Z\sqrt{\left(n\frac{\sigma_X}{X}\right)^2 + \left(m\frac{\sigma_Y}{Y}\right)^2} \quad \textbf{[U6b]}$$

We first take the powers, then combine in quadrature because X and Y are different quantities. In this formula, the uncertainty in Z is expressed directly in terms of the uncertainties in X and Y, σ_X and σ_Y respectively, instead of in terms of the uncertainties in Z *caused* by X and Y.

It is usable also for quotients if you remember that

$$\frac{1}{X^n} = X^{-n}.$$

It cannot be used for sums and differences. For those cases, for example,

$$Z = AX + BY,$$

then

$$\sigma_Z = \sqrt{(A\sigma_X)^2 + (B\sigma_Y)^2}. \quad \textbf{[U7]}$$

If you look more closely at these formulas, you will see that **to get the uncertainties in products and quotients, you are combining in quadrature the *percentage* or *fractional* uncertainties, such as $\frac{\sigma_X}{X}$. To get the uncertainties in sums and differences, you are combining in quadrature the *absolute* uncertainties, such as σ_X.**

9. Agreement Between Two Measurements

Suppose we used the string to measure the circumference of the wheel, and someone else took a ruler and measured the diameter directly. We then have two different and independent determinations of the same thing (the diameter). If our "theory" (that the circumference is π times the diameter) is correct, then those two determinations should agree. Specifically, if we subtract them, the answer should be zero. However, generally it will not be exactly zero because of the uncertainties. Suppose we call the two measurements

$$D_1 \pm \sigma_1$$

and

$$D_2 \pm \sigma_2.$$

then the difference between them is

$$D_1 - D_2 \pm \sqrt{\sigma_1^2 + \sigma_2^2},$$

where we used the rule that we combine the absolute uncertainties in quadrature for sums and differences. In plain English, we are asking, "Are the two numbers closer together than their combined uncertainties?"

If $|D_1 - D_2| \le \sqrt{\sigma_1^2 + \sigma_2^2}$ then the two measurements "agree". If $|D_1 - D_2| \ge 2\sqrt{\sigma_1^2 + \sigma_2^2}$ then they "disagree".

[U8]

In-between cases crop up about 25% of the time, and then one is not really sure if they agree or disagree.

10. Important Special Case – Uncertainty in the Mean

The above rule can also be used to calculate the uncertainty in the mean. Up until now we have been discussing only the uncertainty in a *single measurement*, even though we determined what it was by making many measurements. But it is probably obvious that the mean is not only the best estimate for the true value, it is also the least uncertain. It is less uncertain than any single measurement. The uncertainties in individual measurements tend to "average out" when taking the mean, something everyone is familiar with. Now we are able to derive quantitatively what the uncertainty in the mean is by using the quadrature rule in section 8. We also need to use the rule in section 7 to handle the constant 10 in the mean. The mean is:

$$\bar{L} = \frac{L_1 + L_2 + ... + L_{10}}{10}$$

Then the uncertainty in the mean is (from building a table and using the quadrature rule):

$$\sigma_{\bar{L}} = \frac{1}{10}\sqrt{\sigma_{L_1}^2 + \sigma_{L_2}^2 + \ldots + \sigma_{L_{10}}^2}$$

But all the individual uncertainties σ_{Li} are the same. Hence,

$$\sigma_{\bar{L}} = \frac{1}{10}\sqrt{10\sigma_{L_1}^2} = \frac{1}{\sqrt{10}}\sigma_{L_1}$$

In other words, more generally,

The uncertainty in the mean of n quantities that have the same uncertainty is √n times *smaller* than the uncertainty in one of the quantities.

[U9]

This is one of the most important results in uncertainty analysis and it should be remembered. (Again, if we used the same data to get both the mean and the uncertainty, we should replace n by n-1.)

The uncertainty in a single quantity is called the "sample uncertainty" and the uncertainty in the mean is, well, the "uncertainty in the mean."

The *systematic* uncertainty (see Digital Instruments) is not improved by repeating measurements and is the same for both the sample and the mean.