**WTW 310 Project 1**

# 1 Preamble

**Definition 1** *A* contraction mapping *on* $\mathbb{R}^p$ *is a function* $f : \mathbb{R}^p \to \mathbb{R}^p$ *for which there exists a constant* $K \in (0, 1)$ *so that* $\|f(\bar{x}) - f(\bar{y})\|_2 \leq K\|\bar{x} - \bar{y}\|_2$ *for all* $\bar{x}, \bar{y} \in \mathbb{R}^p$. *The constant* $K$ *is called the Lipschitz constant for the function* $f$.

The aim of this project is to prove and apply the following result.

**Theorem 2** *Suppose that* $f : \mathbb{R}^p \to \mathbb{R}^p$ *is a contraction mapping. Fix a point* $\bar{x}_1 \in \mathbb{R}^p$, *and let* $\bar{x}_{n+1} = f(\bar{x}_n)$ *for all* $n \in \mathbb{N}$. *Then the following statements are true.*

*(1) There exists a vector* $\bar{a} \in \mathbb{R}^p$ *so that* $S = (\bar{x}_n)$ *converges to* $\bar{a}$.

*(2) If* $\bar{x} \in \mathbb{R}^p$, *then* $f(\bar{x}) = \bar{x}$ *if and only if* $\bar{x} = \bar{a}$.

*(3)* $\|\bar{a} - \bar{x}_n\|_2 \leq \frac{K^{n-1}}{1-K}\|\bar{x}_1 - \bar{x}_2\|_2$ *for all* $n \in \mathbb{N}$.

# 2 Problems

**Problem 1** *The aim of this problem is to prove Theorem 2. Let* $f : \mathbb{R}^p \to \mathbb{R}^p$ *be a contraction mapping with Lipschitz constant* $K \in (0, 1)$. *Select a vector* $\bar{x}_1 \in \mathbb{R}^p$, *and let* $\bar{x}_2 = f(\bar{x}_1)$. *Inductively, set* $\bar{x}_{n+1} = f(\bar{x}_n)$ *for all* $n \in \mathbb{N}$.

*1.1) Show that*

$$\|\bar{x}_{n+1} - \bar{x}_n\|_2 \leq K^{n-1}\|\bar{x}_2 - \bar{x}_1\|_2$$

*for all* $n \in \mathbb{N}$.

*1.2) Prove that*

$$\|\bar{x}_m - \bar{x}_n\|_2 \leq \sum_{i=0}^{m-n-1} \|\bar{x}_{m-i} - \bar{x}_{m-i-1}\|_2 \leq \frac{K^{n-1}}{1-K}\|\bar{x}_2 - \bar{x}_1\|_2$$

*for all* $m, n \in \mathbb{N}$ *with* $m \geq n$.

*HINT: Recall that if* $r \neq 1$, *then* $\displaystyle\sum_{k=0}^{N-1} ar^k = a\frac{1 - r^N}{1 - r}$.

*1.3) Use (1.1) and (1.2) to prove that the sequence* $S = (\bar{x}_n)$ *converges to some vector* $\bar{a} \in \mathbb{R}^p$.

*1.3) Prove that* $f(\bar{a}) = \bar{a}$.
*HINT: Show that* $\|f(\bar{a}) - \bar{a}\|_2 < \epsilon$ *for every* $\epsilon > 0$.

*1.4) Suppose that* $f(\bar{c}) = \bar{c}$ *for some vector* $\bar{c} \in \mathbb{R}^p$. *Prove that* $\bar{a} = \bar{c}$.

*1.5) Prove that* $\|\bar{a} - \bar{x}_n\|_2 \leq \frac{K^{n-1}}{1-K}\|\bar{x}_1 - \bar{x}_2\|_2$ *for all* $n \in \mathbb{N}$.

**Problem 2** *Suppose that* $f : \mathbb{R}^2 \to \mathbb{R}$ *is differentiable on* $\mathbb{R}^2$. *Also assume that there exists a real number* $K \geq 0$ *so that* $\|\nabla f(\bar{x})\|_2 \leq K$ *for all* $\bar{x} \in \mathbb{R}^2$. *Prove that* $|f(\bar{x}) - f(\bar{y})| \leq K\|\bar{x} - \bar{y}\|_2$ *for all* $\bar{x}, \bar{y} \in \mathbb{R}^2$.
*HINT: Apply the Mean Value Theorem to the function* $g(t) = f((1 - t)\bar{x} + t\bar{y})$, $t \in [0, 1]$.

**Problem 3** *Consider a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as*

$$f(\bar{x}) = \langle f_1(\bar{x}), f_2(\bar{x}) \rangle \quad \textit{for all} \quad \bar{x} \in \mathbb{R}^2,$$

*where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$ are differentiable on $\mathbb{R}^2$. Assume that there exist real numbers $K_1 \geq 0$ and $K_2 \geq 0$ so that $\|\nabla f_1(\bar{x})\|_2 \leq K_1$ and $\|\nabla f_2(\bar{x})\|_2 \leq K_2$ for all $\bar{x} \in \mathbb{R}^2$. Prove that $\|f(\bar{x}) - f(\bar{y})\|_2 \leq \sqrt{K_1^2 + K_2^2}\ \|\bar{x} - \bar{y}\|_2$ for all $\bar{x}, \bar{y} \in \mathbb{R}^2$.*

**Problem 4** *The aim of this problem is to apply the results in Problems 1 to 3 to a particular function. Your group has been assigned a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ and an initial point $\bar{x}_1 \in \mathbb{R}^2$, see the document "Project 1, Problem 4: Functions and initial points per group". You must answer the following questions, based on the function and initial point assigned to your group.*

*4.1) Use the results in Problems 2 and 3 to prove that $f$ is a contraction mapping.*

*4.2) What is the value of the Lipschitz constant for $f$?*

*4.3) The sequence $S = (\bar{x}_n)$, with $\bar{x}_2 = f(\bar{x}_1)$ and, in general, $\bar{x}_{n+1} = f(\bar{x}_n)$ for $n \in \mathbb{N}$, gives approximations $\bar{a} \approx \bar{x}_n$ for the solution $\bar{a}$ of the equation $f(\bar{x}) = \bar{x}$. The error in this approximation is given by $\|\bar{a} - \bar{x}_n\|_2$. How many terms must be calculated for the error to be less than $10^{-4}$? Explain how you obtain your answer.*

*4.4) Using the sequence $S = (\bar{x}_n)$ given in (4.3), find an approximation for the solution of the equation $f(\bar{x}) = \bar{x}$ with error less than $10^{-4}$. Give your answer rounded to six decimal places. Include a printout of your calculations.*