But since

$$\left|\frac{\Delta x}{\Delta z}\right| = \frac{\Delta x}{[(\Delta x)^2 + (\Delta y)^2]^{1/2}} \leq 1$$

$$\left|\frac{\Delta y}{\Delta z}\right| = \frac{\Delta y}{[(\Delta x)^2 + (\Delta y)^2]^{1/2}} \leq 1$$

we obtain from (4.9), on taking the limit $\Delta z \to 0$,

$$\frac{df}{dz} = \frac{\partial u(x,y)}{\partial x} + i\,\frac{\partial v(x,y)}{\theta x} \quad (4.$$

which shows that $f(z)$ is differentiable.

To give a more intuitive meaning to the Cauchy-Riemann conditions, suppose that the two real functions $\operatorname{Re} f(z) = u(x,y)$ and $\operatorname{Im} f(z) = v(x,y)$ can be expanded in a double Taylor series about a point with coordinates x_0 and y_0:

$$u(x,y) + iv(x,y)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (x - x_0)^{n-k}(y - y_0)^k \times \frac{\partial^n}{\partial^{n-k}x_0\, \partial^k y_0} [u(x_0,y_0) + iv(x_0,y_0)] \quad (4.11)$$

According to Eq. 4.5

$$\frac{\partial}{\partial y_0}[u(x_0,y_0) + iv(x_0,y_0)] = i\,\frac{\partial}{\partial x_0}[u(x_0,y_0) + iv(x_0,y_0)]$$

Hence

$$\frac{\partial^k}{\partial y_0^k}[u(x_0,y_0) + iv(x_0,y_0)] = i^k\,\frac{\partial^k}{\partial x_0^k}[u(x_0,y_0) + iv(x_0,y_0)]$$

Inserting this in Eq. 4.11 we obtain

$$u(x,y) + iv(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} (x - x_0)^{n-k}(y - y_0)^k \;\; i^k \frac{\partial^n}{\partial x_0^n}[u(x_0,y_0) + iv(x_0,y_0)]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} [(x + iy) - (x_0 + iy_0)]^n \frac{\partial^n}{\partial x_0^n}[u(x_0,y_0) + iv(x_0,y_0)] \quad (4.$$

We see that because of the Cauchy-Riemann conditions, the two real variables x and y enter into the function in the unique combination $x + iy$. Thus, these conditions have as a consequence that a mathematical expression defining a differentiable function can depend explic

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